The four inequalities problem PhL 6.6.15

Assume,

$\varepsilon_3 > \varepsilon_2 > \varepsilon_1$.

From (6.3.1),

$$\begin{aligned} N_1 &= +(\varepsilon_2-\varepsilon_1)^{-1}[\,\varepsilon_2 M - U + (\varepsilon_3-\varepsilon_2)N_3] &&= N_1(N_3) \\ N_2 &= -(\varepsilon_2-\varepsilon_1)^{-1}[\,\varepsilon_1 M - U + (\varepsilon_3-\varepsilon_1)N_3] &&= N_2(N_3)\ . \end{aligned} \qquad (6.3.1)$$

Rewrite these as

$$\begin{aligned} N_1 &= +(\varepsilon_2-\varepsilon_1)^{-1}[(\varepsilon_2-u)M + (\varepsilon_3-\varepsilon_2)N_3] &&= N_1(N_3) \\ N_2 &= -(\varepsilon_2-\varepsilon_1)^{-1}[\,(\varepsilon_1-u)M + (\varepsilon_3-\varepsilon_1)N_3] &&= N_2(N_3)\ . \end{aligned} \qquad (6.3.1)$$

Rewrite again as

$$\begin{aligned} N_1 &= +(\varepsilon_2-\varepsilon_1)^{-1}[(\varepsilon_2-u)M + (\varepsilon_3-\varepsilon_2)N_3] &&= N_1(N_3) \\ N_2 &= +(\varepsilon_2-\varepsilon_1)^{-1}[\,(u-\varepsilon_1)M - (\varepsilon_3-\varepsilon_1)N_3] &&= N_2(N_3)\ . \end{aligned} \qquad (6.3.1)$$

There are four inequalities

$0 \le N_1 \le M$
$0 \le N_2 \le M$

Examine the four inequalities one at a time.

Inequality #1:

$$N_1 \ge 0 \quad \Rightarrow \quad (\varepsilon_2-u)M + (\varepsilon_3-\varepsilon_2)N_3 \ge 0 \quad \Rightarrow \quad N_3 \ge \frac{u-\varepsilon_2}{\varepsilon_3-\varepsilon_2} M$$

This is only a constraint if $u > \varepsilon_2$,

Inequality #2:

$$N_2 \ge 0 \quad \Rightarrow \quad (u-\varepsilon_1)M - (\varepsilon_3-\varepsilon_1)N_3 \ge 0 \quad \Rightarrow \quad N_3 \le \frac{u-\varepsilon_1}{\varepsilon_3-\varepsilon_1} M$$

Since $\varepsilon_1 \le u \le \varepsilon_3$, we always have $u \ge \varepsilon_1$ so in general this is always a constraint.

Inequality #3:

$N_1 \le M \quad \Rightarrow \quad (\varepsilon_2-u)M + (\varepsilon_3-\varepsilon_2)N_3 \le (\varepsilon_2-\varepsilon_1)M$

$\Rightarrow \quad (\varepsilon_3-\varepsilon_2)N_3 \le (\varepsilon_2-\varepsilon_1)M + (u-\varepsilon_2)M = (u-\varepsilon_1)M$

$$\Rightarrow \quad N_3 \le \frac{u-\varepsilon_1}{\varepsilon_3-\varepsilon_2} M$$

Since $\varepsilon_1 \le u \le \varepsilon_3$, we always have $u \ge \varepsilon_1$ so in general this is always a constraint.

Inequality #4:

$N_2 \le M \quad \Rightarrow \quad (u-\varepsilon_1)M - (\varepsilon_3-\varepsilon_1)N_3 \le (\varepsilon_2-\varepsilon_1)M$

$\Rightarrow \quad (\varepsilon_3-\varepsilon_1)N_3 \ge (u-\varepsilon_1)M + (\varepsilon_1-\varepsilon_2)M = (u-\varepsilon_2)M$

$$\Rightarrow \quad N_3 \ge \frac{u-\varepsilon_2}{\varepsilon_3-\varepsilon_1} M$$

This is only a constraint if $u > \varepsilon_2$.

Summary

$$N_3 \ge \frac{u-\varepsilon_2}{\varepsilon_3-\varepsilon_2} M \qquad N_3 \ge \frac{u-\varepsilon_2}{\varepsilon_3-\varepsilon_1} M$$

$$N_3 \le \frac{u-\varepsilon_1}{\varepsilon_3-\varepsilon_1} M \qquad N_3 \le \frac{u-\varepsilon_1}{\varepsilon_3-\varepsilon_2} M$$

Combine these to get

$$N_3 \ge \max\left(\frac{u-\varepsilon_2}{\varepsilon_3-\varepsilon_2} M, \frac{u-\varepsilon_2}{\varepsilon_3-\varepsilon_1} M\right) \qquad \text{only a constraint if } u > \varepsilon_2$$

$$N_3 \le \min\left(\frac{u-\varepsilon_1}{\varepsilon_3-\varepsilon_1} M, \frac{u-\varepsilon_1}{\varepsilon_3-\varepsilon_2} M\right)$$

In the first line, the second denominator is larger, so this second term is smaller, and we get

$$N_3 \ge \frac{u-\varepsilon_2}{\varepsilon_3-\varepsilon_2} M \qquad \text{as the summary of both constraints}$$

In the second line, the second denominator is larger, so this second term is smaller, so

$$N_3 \leq \frac{u-\varepsilon_1}{\varepsilon_3-\varepsilon_1} M$$

So this entire mess can be summarized in this simple way

$$\frac{u-\varepsilon_2}{\varepsilon_3-\varepsilon_2} M \leq N_3 \leq \frac{u-\varepsilon_1}{\varepsilon_3-\varepsilon_1} M$$

In my application this says

$$(u-2)\, M \leq N_3 < (1/2)\,(u-1)\, M$$

If I choose u = 1.5 for the example, we get

$$-(1/2)\, M \leq N_3 < (1/2)\,(1/2)\, M$$

and this boils down to

$$0 \leq N_3 \leq M/4$$