Show that for lin indep vectors, det(M) = 0 PhL 6.10.15

Plan A: Gauss Elimination Approach

Theorem. For an n x n square matrix A, if the column rank is n, then det(A) ≠ 0. (A.4.2)

Proof #1. If the n columns of A are linearly independent, then each column must contain at least one non-zero number. One cannot have a column of all zeros. We know we can swap rows around without changing the value of |det(A)|. So find a row which has a non-zero number in the left column and swap this with the first row so there is a non-zero number in the upper left corner. Then add multiples of this new first row to all the other rows so as to clear out (to zeros) the rest of the first column. These operations also do not affect det(A).

After

After doing the above, find a non-zero number in the second column (it must exist WHY?) and swap rows so it is in the A_{22} matrix position.

Repeat this process to get a matrix whose lower left triangle is all zeros and whose diagonal has all non-zero values. This matrix then has upper triangular form. The determinant of such a matrix is the product of the diagonal values and thus we have shown that |det(A)| ≠ 0. This process is usually called Gauss elimination. One might wonder why *all* the diagonal elements are non-zero for some general case. This is assured at each step of the elimination. We leave it to the reader to make this proof more rigorous.

If the n columns of M are linearly independent, then each column must contain at least one non-zero element. One cannot have a column of all zeros.

So column 1 must have at least one non-zero element. Pick a row which has such a non-zero element in column 1. Swap this row with the top row without changing |det(M) by Theorem 4. Then add multiples of this new first row to all the other rows so as to clear out (to zeros) the rest of the first column.

At this point, what can be said about column 2?

It seems that somehow column 2 could end up being 0. How can I show this can't happen? I would have to show that adding multiples of rows within a matrix does not alter the fact that the columns are independent. But I am not allowed to use the det(M) ≠ 0 test since that is what I am trying to show

From Theorem 4 we can swap rows around without changing the value of |det(M)|.

So find a row which has a non-zero number in the left column and swap this with the first row so there is a non-zero number in the upper left corner. Then add multiples of this new first row to all the other rows so as to clear out (to zeros) the rest of the first column. These operations also do not affect det(M). After doing the above, find a non-zero number in the second column (it must exist) and swap rows so it is in the A_{22} matrix position.

Plan B: Try relating det(M) to the unit vector matrix

If the rows $\mathbf{r}_i$ are linearly independent in E^n then they form a complete set or basis for E^n. Thus, one can write any of the unit vectors $\mathbf{e}_i$ as a linear combination of the $\mathbf{r}_i$.

$$\mathbf{e}_i = \Sigma_j S_{ij} \mathbf{r}_j = S_{ii}\mathbf{r}_i + \Sigma_{j\neq i} S_{ij} \mathbf{r}_j$$

Similarly one could write

$$\mathbf{r}_i = \Sigma_j Q_{ij} \mathbf{e}_j = Q_{ii}\mathbf{e}_i + \Sigma_{j\neq i} Q_{ij} \mathbf{e}_j$$

Perhaps

$$\begin{aligned} \mathbf{r}_1 &= Q_{11}\mathbf{e}_1 + \Sigma_{j\neq 1} Q_{1j} \mathbf{e}_j && Q_{11} \neq 0 \\ \mathbf{r}_2 &= \Sigma_{j\neq 2} Q_{2j} \mathbf{e}_j \\ \mathbf{r}_3 &= Q_{33}\mathbf{e}_3 + \Sigma_{j\neq 3} Q_{3j} \mathbf{e}_j && Q_{33} \neq 0 \end{aligned}$$

Then

$$\det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \det (Q_{11}\mathbf{e}_1 + \Sigma_{j\neq 1} Q_{1j} \mathbf{e}_j, \Sigma_{j\neq 2} Q_{2j} \mathbf{e}_j, Q_{33}\mathbf{e}_3+\Sigma_{j\neq 3} Q_{3j} \mathbf{e}_j)$$

$$= Q_{11}Q_{33} \det (\mathbf{e}_1 + \Sigma_{j\neq 1} [Q_{1j}/Q_{11}] \mathbf{e}_j, \Sigma_{j\neq 2} Q_{2j} \mathbf{e}_j, \mathbf{e}_3 + \Sigma_{j\neq 3} [Q_{3j}/Q_{33}] \mathbf{e}_j)$$

This is a sum of four determinants. Ignore leading factors and write it as

$$\det_1 = \det (\mathbf{e}_1, \Sigma_{j\neq 2} Q_{2j} \mathbf{e}_j, \mathbf{e}_3) = 0 \text{ since det } (\mathbf{e}_1,\mathbf{e}_1,\mathbf{e}_3) = 0, \text{ etc}$$

$$\det_1 = \det (\Sigma_{j\neq 1} [Q_{1j}/Q_{11}] \mathbf{e}_j, \Sigma_{j\neq 2} Q_{2j} \mathbf{e}_j, \mathbf{e}_3)$$

$$= \Sigma_{j\neq 1} [Q_{1j}/Q_{11}] \Sigma_{k\neq 2} Q_{2k} \det (\mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_3)$$

$$= \Sigma_{j\neq 1} [Q_{1j}/Q_{11}] \Sigma_{k\neq 2} Q_{2k} \varepsilon_{jk3}$$

$$= \Sigma_{j\neq 1} [Q_{1j}/Q_{11}] Q_{21} \varepsilon_{j13}$$

$$= [Q_{12}/Q_{11}] Q_{21} \varepsilon_{213}$$

This is too complicated!!!

We now apply Theorem 3 twice to eliminate "adding multiples of other rows" to get

$$= Q_{11}Q_{33} \det (\mathbf{e}_1, \Sigma_{j\neq 2} Q_{2j} \mathbf{e}_j, \mathbf{e}_3)$$

$$= Q_{11}Q_{33} \det (\mathbf{e}_1, Q_{21} \mathbf{e}_1 + Q_{23}\mathbf{e}_3, \mathbf{e}_3)$$

$= Q_{11}Q_{33}\,[\;\; Q_{21}\det(\mathbf{e}_1,\mathbf{e}_1,\mathbf{e}_3) + Q_{23}\det(\mathbf{e}_1,\mathbf{e}_3,\mathbf{e}_3)\;] = 0 + 0 = 0\;!$

So I have shown exactly the wrong result: $\det(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = 0$ if r_i are linearly independent!

According to Theorem 3 we can ignore mul

In general some S_{ii} will vanish and some will not. So here is a typical mixed case:

$$\mathbf{e}_1 = S_{11}\mathbf{r}_1 + \Sigma_{j\neq i}\, S_{1j}\,\mathbf{r}_j$$
$$\mathbf{e}_2 = \Sigma_{j\neq i}\, S_{2j}\,\mathbf{r}_j$$
$$\mathbf{e}_2 = S_{33}\mathbf{r}_1 \quad \Sigma_{j\neq i}\, S_{3j}\,\mathbf{r}_j$$

First define some new rows

$$\mathbf{r}_i' \equiv \mathbf{r}_i + \Sigma_{j\neq i}\, S_{ij}\,\mathbf{r}_j$$

According to Theorem 3, $\det(\mathbf{r}'_1,\mathbf{r}'_2....\mathbf{r}'_n) = \det(\mathbf{r}_1,\mathbf{r}_2....\mathbf{r}_n)$.

Now define some new rows $\mathbf{r}''_i$,

$$\mathbf{r}''_i =$$

Now define some new rows r_i''

$$\mathbf{r}_i'' = S_{ii}\,\mathbf{r}_i' .$$

Then we know that

$$\det(\mathbf{r}_i'') = (S_{ii})^n \det(\mathbf{r}_i') = (S_{ii})^n \det(\mathbf{r}_i)$$

But $\mathbf{r}_i'' = \mathbf{e}_i$ and $\det(\mathbf{e}_i) = 1$ so

$$1 = (S_{ii})^n \det(\mathbf{r}_i)$$

and therefore $\det(\mathbf{r}_i) \neq 0$

Now suppose $S_{ii} = 0$. Then we start off with

$$\mathbf{e}_i = \Sigma_{j\neq i}\, S_{ij}\,\mathbf{r}_j$$

Plan C: Linear equations argument

If the rows are independent, then we have a set of n linear equations in n unknowns and we know it has a solution so det(M) ≠ 0. But HOW do we know it has a solution if we don't know that det(M) ≠ 0? It could maybe have a solution even if det(M) = 0,

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Plan D: The volume argument

Quote my n-piped thing that if vectors are independent, must have volume ≠ 0 so det(M) ≠ 0. But that volume thing is a real bear. At least this Plan D convinces me the theorem is true.

Plan E: What does Shilov say?

Have to search high and low to find anything. I searched on "linearly dependent" and could find no theorem. Indep search same thing, there is so much I cannot find anything. So no conclusion here. I need someone who exactly states my theorem.

I need to be able to search for linearly near determinant! Witzend can do this.

Plan F: What does Lucht Matrix Binder say?

As I page through tab 1, I see this theorem supposedly appearing as Phil Theorem 9.2 in Section 2. I move to Section 2 page 9 and there is the theorem. Let's read it. It is the wrong theorem. Still:

I say to consider $A\mathbf{x} = 0$ or $\Sigma_k A_{jk} x_k = 0$ or $\Sigma_k (\mathbf{c}_k)_j \, x_k = 0$ true for all j. This is a vector equation

$\Sigma_k \mathbf{c}_k \, x_k = 0$ so lin indep columns $\Rightarrow$ $\{x_k\} = 0$.

So this shows that: $A\mathbf{x} = 0 \quad \Rightarrow \quad \mathbf{x} = 0$ if columns lin indep

This does say something: lin indep $\Rightarrow$ nullspace = 0

How about this then.

(1) columns linearly independent $\Rightarrow$ nullspace is only the trivial nullspace
(2) the mapping is 1 to 1 therefore, think I can show that
(3) therefore inverse of Ax = y must exist and be unique
(4) but A^{-1} exists implies detA ≠ 0.

Maybe that will fly? I think so.

Plan G : Here from http://www.math.lsa.umich.edu/~hochster/419/det.html

Fact 8. The determinant of an n by n matrix A is 0 if and only if the rows are linearly dependent (and not zero if and only if they are linearly independent). That is, the determinant of A is not zero if and only if A is invertible.

Here is why: The issue is not affected by switching rows, adding a multiple of one row to another, or multiplying a row by a nonzero scalar. Thus, we may assume that A is a square matrix in RREF. If the rows are independent, it will then be the identity, while otherwise it has a row of zeros.

So at least they are confirming my theorem. RREF means "reduced row echelon form".