B.2 The permutation group and the permutation tensor ε

Definition. A set of elements $\{g_i\}$ form a **group** G if :

- $g_i g_j$ is also in the group (closure)
- g_i^{-1} exists for each g_i in G, where g_i^{-1} is also in G (inverse)
- $(g_i g_j)g_k = g_i(g_j g_k)$ (associative) (B.2.1)

Comment: It is implicit in the above definition that the group has some "operation" which gives meaning to $g_i g_j$, which we shall just think of as "multiplication". When group elements are represented by matrices, that operation is multiplication of those matrices, and that will apply to the permutation group below.

Fact: $g_i G = G$ (the rearrangement theorem) (B.2.2)

This says that multiplication of all the elements of a group by an element g_i in the group creates a reordering of the group elements. Here G is the set of group elements.

Proof. Consider $g_i G = g_i [g_1, g_2....g_n] = [g_i g_1, g_i g_2....g_i g_n]$ = set of n elements. Unless two elements are the same, this must exhaust the entire group. How do we know that $g_i g_1$ and $g_i g_2$ might not be the same? Apply g_i^{-1} from the left and that would say $g_1 = g_2$ which is not the case.

Fact: $\Sigma_g f(g) = \Sigma_g f(g_1 g)$ if Σ_g runs over the entire group G (B.2.3)

Proof: In $\Sigma_g f(g_1 g)$, as g runs over G, the argument $g' \equiv g_1 g$ runs over G by the above rearrangement theorem. Thus, the sum $\Sigma_g f(g_1 g)$ is just a reordering of the terms in the sum $\Sigma_g f(g)$.

It is easy to show that the set of permutations of $\mathbf{z}_0$ forms a group and therefore the above facts can be used. For example, the product of two permutations is a permutation, and every permutation clearly has an inverse, and (AB)C = A(BC) for any matrices. Fact (B.2.3) can be written

$\Sigma_B f(B\mathbf{z}_o) = \Sigma_B f(CB\mathbf{z}_0)$ // $\mathbf{b} = B\mathbf{z}_0$

Here the sum Σ_B is over all permutation matrices which correspond to permutations **b** of $\mathbf{z}_0$, and C is some arbitrary permutation matrix associated with some permutation **c** of $\mathbf{z}_0$. An equivalent way of stating the above involves a direct summation Σ_b over all permutation vectors **b** of $\mathbf{z}_0$,

$\Sigma_b f(\mathbf{b}) = \Sigma_b f(C\mathbf{b})$. // permutation sum rearrangement theorem (B.2.4)

Throwing in the arbitrary permutation C merely causes a reordering of the sum. This is an extremely powerful and useful fact. Consider then :

$$\det(M) \equiv \frac{1}{n!}\,\Sigma_{\mathbf{a}}\,[\Sigma_{\mathbf{b}}\,(-1)^{S_{\mathbf{a}}+S_{\mathbf{b}}}\;\Pi(\mathbf{a},\mathbf{b};M)\,] \qquad \text{// (B.1.8)}$$

$$= \frac{1}{n!}\,\Sigma_{\mathbf{a}}\;[\Sigma_{\mathbf{b}}\,(-1)^{S_{\mathbf{a}}+S_{\mathbf{b}}}\;\;\Pi(A^{-1}\mathbf{a},A^{-1}\mathbf{b};M)] \qquad \text{// (B.1.6) with } C = A^{-1};\; A^{-1}\mathbf{a} = A^{-1}A\mathbf{z}_0 = \mathbf{z}_0$$

$$= \frac{1}{n!}\,\Sigma_{\mathbf{a}}\,[\,\Sigma_{\mathbf{b}}\;\text{Parity}(A^{-1}\mathbf{b})\;\;\Pi(\mathbf{z}_0,A^{-1}\mathbf{b};M)\,] \qquad \text{// (B.1.4) with } C = A^{-1} \text{ and } \mathbf{a} = \mathbf{b}$$

$$= \frac{1}{n!}\,\Sigma_{\mathbf{a}}\,[\,\Sigma_{\mathbf{b}}\;\text{Parity}(\mathbf{b})\;\;\Pi(\mathbf{z}_0,\mathbf{b};M)\,] \qquad \text{// (B.2.4) with } C = A^{-1} \text{ (key step)}$$

$$= \Sigma_{\mathbf{b}}\;\text{Parity}(\mathbf{b})\;\;\Pi(\mathbf{z}_0,\mathbf{b};M) \qquad \text{// n! identical terms in } \Sigma_{\mathbf{a}}$$

$$= \Sigma_{\mathbf{a}}\;\text{Parity}(\mathbf{a})\;\;\Pi(\mathbf{z}_0,\mathbf{a};M) \qquad \text{// rename dummy sum variable}$$

$$= \Sigma_{\mathbf{a}}\;\text{Parity}(\mathbf{a})\;\;M_{1a_1}M_{2a_2}\,.....\,M_{na_n}\,. \qquad \text{// (B.1.5)} \qquad \text{(B.2.5)}$$

Since $\det(M^{\mathbf{T}}) = \det(M)$ from Theorem 1 (B.1.10), we can also write this result as

$$\det(M) = \Sigma_{\mathbf{a}}\;\text{Parity}(\mathbf{a})\;\;M_{a_1 1}M_{a_2 2}\,.....\,M_{a_n n} \qquad \text{(B.2.6)}$$

Proof: $\det(M) = \det(M^{\mathbf{T}}) = \Sigma_{\mathbf{a}}\;\text{Parity}(\mathbf{a})\;\;M^{\mathbf{T}}_{1a_1}M^{\mathbf{T}}_{2a_2}\,.....\,M^{\mathbf{T}}_{na_n} = \Sigma_{\mathbf{a}}\;\text{Parity}(\mathbf{a})\;\;M_{a_1 1}M_{a_2 2}\,.....\,M_{a_n n}$

Here then are the last two results: (recall that $S_{\mathbf{a}}$ is the number of pairwise swaps to get from $\mathbf{z}_0$ to $\mathbf{a}$)

$$\det(M) \equiv \Sigma_{\mathbf{a}}\;\text{Parity}(\mathbf{a})\;\;M_{1a_1}M_{2a_2}\,.....\,M_{na_n} \qquad \text{Parity}(\mathbf{a}) = (-1)^{S_{\mathbf{a}}} \qquad \text{(B.2.7a)}$$

$$\det(M) \equiv \Sigma_{\mathbf{a}}\;\text{Parity}(\mathbf{a})\;\;M_{a_1 1}M_{a_2 2}\,.....\,M_{a_n n} \qquad \text{Parity}(\mathbf{a}) = (-1)^{S_{\mathbf{a}}}\;. \qquad \text{(B.2.7b)}$$

Definition: The **permutation tensor** $\varepsilon_{ijk..}$ (n subscripts) :

- $\varepsilon_{12...n} = 1$
- for any index swap, ε changes sign: $\varepsilon_{..i..j..} = -\varepsilon_{..j..i..}$
- therefore, if any two indices are the same, ε = 0 $\varepsilon_{..i..i..} = -\varepsilon_{..i..i..} = 0$ (B.2.8)

Fact: If $\mathbf{a}$ is a permutation of $\mathbf{z}_0$, then $\varepsilon_{a_1a_2\cdots a_n} = \text{Parity}(\mathbf{a}) = (-1)^{S_{\mathbf{a}}} \equiv \varepsilon_{\mathbf{a}}$. (B.2.9)

Proof: Since indices a_i represent a permutation of $\mathbf{z}_0$, it takes $S_{\mathbf{a}}$ swaps to get from $\varepsilon_{a_1a_2\cdots a_n}$ to $\varepsilon_{123...n}$ by the definition of ε. In dense notation, one could say $\varepsilon_{\mathbf{a}} = (-1)^{S_{\mathbf{a}}}\,\varepsilon_{\mathbf{z}_0}$.

Using (B.2.9) in (B.2.7) then gives these two classic det(M) expressions :

Theorem 2: det(M) can be represented in these two ways:

$$\det(M) = \Sigma_{a_1a_2\cdots a_n}\,\varepsilon_{a_1a_2\cdots a_n}\,M_{1a_1}M_{2a_2}\,.....\,M_{na_n} \qquad \text{(B.2.10a)}$$

$$\det(M) = \Sigma_{a_1a_2\cdots a_n}\,\varepsilon_{a_1a_2\cdots a_n}\,M_{a_1 1}M_{a_2 2}\,.....\,M_{a_n n}\;. \qquad \text{(B.2.10b)}$$

In dense notation, one could write the above as (**a** is now a vector),

$$\det(M) = \Sigma_{\mathbf{a}} \varepsilon_{\mathbf{a}} M_{\mathbf{z_0 a}} \qquad \text{(B.2.10a)'}$$

$$\det(M) = \Sigma_{\mathbf{a}} \varepsilon_{\mathbf{a}} M_{\mathbf{a z_0}} \; . \qquad \text{(B.2.10b)'}$$

In most applications, the following simpler notation suffices:

$$\det(M) = \Sigma_{\mathbf{abc..q}} \, \varepsilon_{\mathbf{abc..q}} \, M_{\mathbf{1a}} M_{\mathbf{2b}} M_{\mathbf{3c}} .... M_{\mathbf{nq}} \qquad \text{(B.2.11a)}$$

$$\det(M) = \Sigma_{\mathbf{abc..q}} \, \varepsilon_{\mathbf{abc..q}} \, M_{\mathbf{a1}} M_{\mathbf{b2}} M_{\mathbf{c3}} .... M_{\mathbf{qn}} \; . \qquad \text{(B.2.11b)}$$

Theorem 3: For a square matrix M, adding a multiple of one row to another does not change det(M). The same is true for adding a multiple of one column to another. (B.2.12)

Proof: (rows) Suppose we replace $\mathbf{r_3} \rightarrow \mathbf{r_3} + \alpha \, \mathbf{r_2}$. This says $M_{\mathbf{3i}} \rightarrow M_{\mathbf{3i}} + \alpha \, M_{\mathbf{2i}}$. Eq (B.2.11a) says :

$$\det(M') = \Sigma_{\mathbf{abc..q}} \, \varepsilon_{\mathbf{abc..q}} \, M_{\mathbf{1a}} M_{\mathbf{2b}} (M_{\mathbf{3c}} + \alpha \, M_{\mathbf{2c}}) .... M_{\mathbf{nq}}$$

$$= \det(M) + \alpha \, \Sigma_{\mathbf{abc..q}} \, \varepsilon_{\mathbf{abc..q}} \, M_{\mathbf{1a}} M_{\mathbf{2b}} M_{\mathbf{2c}} .... M_{\mathbf{nq}} \; .$$

Since $M_{\mathbf{1a}} M_{\mathbf{2b}} M_{\mathbf{2c}} .... M_{\mathbf{nq}}$ is symmetric under b↔c while $\varepsilon_{\mathbf{abc..q}}$ is anti-symmetric, the extra α term vanishes. That is to say, sum = $\Sigma_{\mathbf{bc}} \, A_{\mathbf{bc}} S_{\mathbf{bc}} = \Sigma_{\mathbf{cb}} \, A_{\mathbf{cb}} S_{\mathbf{cb}} = \Sigma_{\mathbf{cb}} \, (-A_{\mathbf{bc}})(S_{\mathbf{bc}})$ = - sum = 0. Using (B.2.11b) this argument shows that $\mathbf{c_3} \rightarrow \mathbf{c_3} + \alpha \, \mathbf{c_2}$ similarly does not alter det(M).

Theorem 4: Swapping two rows (columns) of square matrix M causes det(M) → - det(M). (B.2.13)

Proof: Let's swap *rows* 1 and 3 in M to get M'. Then from (B.2.11a),

$$\det(M') = \Sigma_{\mathbf{abc..q}} \, \varepsilon_{\mathbf{abc..q}} \, M_{\mathbf{3a}} M_{\mathbf{2b}} M_{\mathbf{1c}} .... M_{\mathbf{nq}}$$

$$= \Sigma_{\mathbf{abc..q}} \, [- \varepsilon_{\mathbf{cba..q}}] \, M_{\mathbf{3a}} M_{\mathbf{2b}} M_{\mathbf{1c}} .... M_{\mathbf{nq}} \qquad \text{// swap a↔c on } \varepsilon$$

$$= - \Sigma_{\mathbf{abc..q}} \, \varepsilon_{\mathbf{abc..q}} \, M_{\mathbf{3c}} M_{\mathbf{2b}} M_{\mathbf{1a}} .... M_{\mathbf{nq}} \qquad \text{// dummy rename a↔c}$$

$$= - \Sigma_{\mathbf{abc..q}} \, \varepsilon_{\mathbf{abc..q}} \, M_{\mathbf{1a}} M_{\mathbf{2b}} M_{\mathbf{3c}} .... M_{\mathbf{nq}} \qquad \text{// reorder product}$$

$$= - \det(M) \; .$$

Using Theorem 1 that det(A) = det($A^{\mathbf{T}}$), we then know that det($M'^{\mathbf{T}}$) = - det($M^{\mathbf{T}}$), so swapping *columns* 1 and 3 negates det(M). An obvious corollary follows if the two swapped columns have identical data :

Corollary 4: If two rows (columns) of square matrix A are the same, then det(M) = 0. (B.2.14)

Here is one more well-known determinant theorem which again demonstrates the power of the rearrangement theorem (B.2.4).

Theorem 5: det(AB) = det(A)det(B) (B.2.15)

Proof: Since we already use A in $\mathbf{a} = A\mathbf{z}_0$, we shall prove det(XY) = det(X)det(Y) to avoid overloading symbols. Note that Parity(**a**) Parity(**b**) = $(-1)^{s_a}(-1)^{s_b} = (-1)^{s_a+s_b}$ = Parity(A**b**) from (B.1.4). Then :

$\det(X)\det(Y) = [\,\Sigma_{\mathbf{a}}\ \text{Parity}(\mathbf{a})\ \Pi(\mathbf{z}_0,\mathbf{a};X)]\,[\,\Sigma_{\mathbf{b}}\ \text{Parity}(\mathbf{b})\ \Pi(\mathbf{z}_0,\mathbf{b};Y)]$ // (B.2.5) twice

$= \Sigma_{\mathbf{a}}\,[\,\Sigma_{\mathbf{b}}\ \text{Parity}(\mathbf{a})\ \text{Parity}(\mathbf{b})\ \Pi(\mathbf{z}_0,\mathbf{a};X)\ \ \Pi(\mathbf{z}_0,\mathbf{b};Y)\,]$

$= \Sigma_{\mathbf{a}}\,[\,\Sigma_{\mathbf{b}}\ \ \text{Parity}(A\mathbf{b})\ \ \Pi(\mathbf{z}_0,\mathbf{a};X)\ \ \Pi(A\mathbf{z}_0,A\mathbf{b};Y)\,]$ // (B.1.6) with C = A

$= \Sigma_{\mathbf{a}}\,[\,\Sigma_{\mathbf{b}}\ \ \text{Parity}(A\mathbf{b})\ \ \Pi(\mathbf{z}_0,\mathbf{a};X)\ \ \Pi(\mathbf{a},A\mathbf{b};Y)\,]$ // $A\mathbf{z}_0 = \mathbf{a}$

$= \Sigma_{\mathbf{a}}\,[\,\Sigma_{\mathbf{b}}\ \ \text{Parity}(\mathbf{b})\ \ \Pi(\mathbf{z}_0,\mathbf{a};X)\ \ \Pi(\mathbf{a},\mathbf{b};Y)\,]$ // (B.2.4) with C = A

$= \Sigma_{\mathbf{b}}\ \ \text{Parity}(\mathbf{b})\ \Sigma_{\mathbf{a}}\ \Pi(\mathbf{z}_0,\mathbf{a};X)\,\Pi(\mathbf{a},\mathbf{b};Y)$ // move $\Sigma_{\mathbf{a}}$

$= \Sigma_{\mathbf{b}}\ \ \text{Parity}(\mathbf{b})\ \Sigma_{\mathbf{a}}\ \Pi(\mathbf{z}_0,\mathbf{b};XY)$ // see below

$= \det(XY)$. // (B.2.5)

The idea here is to get **a** in the right place on both Π's so that

$\Sigma_{\mathbf{a}}\ \Pi(\mathbf{z}_0,\mathbf{a};X)\,\Pi(\mathbf{a},\mathbf{b};Y) = \Sigma_{\mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_n}\ X_{1\mathbf{a}_1}X_{2\mathbf{a}_2}\ldots\ldots X_{n\mathbf{a}_n}\ \ Y_{\mathbf{a}_1\mathbf{b}_1}Y_{\mathbf{a}_2\mathbf{b}_2}\ldots\ldots Y_{\mathbf{a}_n\mathbf{b}_n}$

$= \Sigma_{\mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_n}(X_{1\mathbf{a}_1}Y_{\mathbf{a}_1\mathbf{b}_1})(X_{2\mathbf{a}_{21}}Y_{\mathbf{a}_2\mathbf{b}_2})\ldots\ldots(X_{n\mathbf{a}_n}Y_{\mathbf{a}_n\mathbf{b}_n})$

$= (XY)_{1\mathbf{b}_1}\,(XY)_{2\mathbf{b}_2}\ldots(XY)_{n\mathbf{b}_n} = \Pi(\mathbf{z}_0,\mathbf{b};XY)$

which in dense notation one would write as

$\Sigma_{\mathbf{a}}\,X_{\mathbf{z}_0\mathbf{a}}Y_{\mathbf{a}\mathbf{b}} = (XY)_{\mathbf{z}_0\mathbf{b}}$.

Corollary 5: det(ABC) = det(A)det(B)det(C) and so on. (B.2.16)

Proof: det(ABC) = det(A[BC]) = det(A)det(BC) = det(A)det(B)det(C) .

It is not hard to generalize the results of Theorem 2 to obtain (proof follows) :

Theorem 2A: det(M) can be represented in these two ways:

$$\varepsilon_{b_1 b_2 \cdots b_n} \det(M) = \Sigma_{a_1 a_2 \cdots a_n} \varepsilon_{a_1 a_2 \cdots a_n} M_{b_1 a_1} M_{b_2 a_2} \ldots\ldots M_{b_n a_n} \quad \text{(B.2.17a)}$$

$$\varepsilon_{b_1 b_2 \cdots b_n} \det(M) = \Sigma_{a_1 a_2 \cdots a_n} \varepsilon_{a_1 a_2 \cdots a_n} M_{a_1 b_1} M_{a_2 b_2} \ldots\ldots M_{a_n b_n} \ . \quad \text{(B.2.17b)}$$

In dense notation, one could write the above as (**a** and **b** are now vectors),

$$\varepsilon_b \det(M) = \Sigma_a \varepsilon_a M_{ba} \quad \text{(B.2.18a)}$$

$$\varepsilon_b \det(M) = \Sigma_a \varepsilon_a M_{ab} \ . \quad \text{(B.2.18a)}$$

For the special case $\mathbf{b} = \mathbf{z}_0$ Theorem 2A reduces to Theorem 2.

Proof : (we prove only the first lines, the proof for the second lines is similar)

$\det(M) = \Sigma_a$ Parity(**a**) $\Pi(\mathbf{z}_0, \mathbf{a}; M)$ // line 6 of (B.2.5)

$= \Sigma_a$ Parity(**a**) $\Pi(B\mathbf{z}_0, B\mathbf{a}; M)$ // (B.1.6) with C = B

$= \Sigma_a$ Parity(B**a**) Parity(**b**) $\Pi(B\mathbf{z}_0, B\mathbf{a}; M)$ // Parity(B**a**) Parity(**b**) = Parity(**a**)

$= \Sigma_a$ Parity(**a**) Parity(**b**) $\Pi(B\mathbf{z}_0, \mathbf{a}; M)$ // (B.2.4) that $\Sigma_a f(B\mathbf{a}) = \Sigma_a f(\mathbf{a})$

$=$ Parity(**b**) Σ_a Parity(**a**) $\Pi(\mathbf{b}, \mathbf{a}; M)$ // $\mathbf{b} = B\mathbf{z}_0$

$= \varepsilon_{b_1 b_2 \cdots b_n} \Sigma_{a_1 a_2 \cdots a_n} \varepsilon_{a_1 a_2 \cdots a_n} M_{b_1 a_1} M_{b_2 a_2} \ldots M_{b_n a_n}$ // (B.1.5) and (B.2.9)

$= \varepsilon_b \Sigma_a \varepsilon_a M_{ba}$ // previous line in dense notation (B.2.19)

Here it is assumed that B and hence **b** is associated with a permutation of $\mathbf{z}_0$, In this case, $(\varepsilon_b)^2 = 1$ so we can move ε_b to the left side to get

$$\varepsilon_b \det(M) = \Sigma_a \varepsilon_a M_{ba} \ . \quad \text{(B.2.20)}$$

In this form, the equation is valid whether or not **b** is a permutation of $\mathbf{z}_0$. If in $\mathbf{b} = (b_1, b_2 \ldots b_n)$ the b_i are arbitrary elements of the set {1,2,...n} where two or more b_i are the same (that is, **b** is not a permutation of $\mathbf{z}_0$), then $\Sigma_a \varepsilon_{a_1 a_2 \cdots a_n} M_{b_1 a_1} M_{b_2 a_2} \ldots M_{b_n a_n} = 0$ by symmetry. For example, if $b_1 = b_2$ then

$S \equiv \Sigma_a \varepsilon_{a_1 a_2 \cdots a_n} M_{b_1 a_1} M_{b_1 a_2} \ldots M_{b_n a_n}$

$= \Sigma_a \varepsilon_{a_2 a_1 \cdots a_n} M_{b_1 a_2} M_{b_1 a_1} \ldots M_{b_n a_n}$ // rename dummy indices $a_1 \leftrightarrow a_2$

$= \Sigma_a [-\varepsilon_{a_1 a_2 \cdots a_n}] M_{b_1 a_1} M_{b_1 a_2} \ldots M_{b_n a}$ // (B.2.8) for ε and slide $M_{b_1 a_2}$ to the right

$= -\Sigma_a \varepsilon_{a_1 a_2 \cdots a_n} M_{b_1 a_1} M_{b_1 a_2} \ldots M_{b_n a} = -S \quad \Rightarrow \quad S = 0 \quad \text{(B.2.21)}$

So in this case the right side of (B.2.20) is zero. But from (B.2.8) $\varepsilon_{\mathbf{b}} = 0$ so the left side is also zero.