## Appendix D: Lagrange Multipliers for problems with Constraints

Our main focus in this document is the use of Lagrange Multipliers to find candidates $\mathbf{r}_i$ for which a scalar function f($\mathbf{r}$) is stationary, df = 0, subject to constraints. In this Appendix, however, we consider a different application of Lagrange Multipliers that relates to classical mechanics for a system of N particles in the presence of constraints. A certain amount of background information is required to put this application into context, which we now present.

Constraints and virtual displacements

Let $\mathbf{r}_k(t)$ be the position of the $k^{th}$ particle. At any fixed time t, if the particles are unencumbered with constraints, the variables ($\mathbf{r}_1$, $\mathbf{r}_2$..... $\mathbf{r}_k$) can be regarded as a single point in the space $E^{3N}$ and the "configuration space" of the system indicated by ($\mathbf{r}_1$, $\mathbf{r}_2$..... $\mathbf{r}_k$) is all of this space $E^{3N}$. All points in $E^{3N}$ are "legal" points for the system.

A **holonomic** constraint has the form

$$a(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_N, t) = 0. \qquad (D.1)$$

At a given time t, this equation represents a surface of dimension 3N-1 in $E^N$. If there are s such holonomic constraints

$$f_i(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_N, t) = 0 \qquad i = 1,2...s \qquad (D.2)$$

then at time t the system vector ($\mathbf{r}_1$, $\mathbf{r}_2$..... $\mathbf{r}_N$) is constrained to lie on a surface of dimension 3N-s in $E^N$, as discussed in Appendix C, so each extra constraint lowers the operating surface dimension by 1. One can think of the term holonomic as meaning that the constraints are functions of "whole" coordinates like $\mathbf{r}_2$ as opposed to differential coordinates like $d\mathbf{r}_2$.

A **non-holonomic** constraint is one that is not holonomic. An inequality constraint like $a(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_N, t) < 0$ which might bound the particles to one side of a surface is an example of a nonholonomic constraint, but our interest lies more in nonholonomic constraints of the form

$$\Sigma_{k=1}^{N} \mathbf{b}_k \bullet d\mathbf{r}_k + b_t dt = 0 \qquad (D.3)$$

in which differentials rather than "whole" coordinates appear. The "coefficients" $\mathbf{b}_k$ and $b_t$ can in general be functions of ($\mathbf{r}_1$, $\mathbf{r}_2$..... $\mathbf{r}_k$, t). If it happens that such a constraint can be integrated by some hook or crook to give an equation of the form $a(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_k, t) = 0$, then it is regarded as being a holonomic constraint, so a non-holonomic constraint must be non-integrable. If there are m such non-holonomic constraints imposed on a system, one could write them as

$$\Sigma_{k=1}^{N} \mathbf{A}_{ik} \bullet d\mathbf{r}_k + A_{it} dt = 0 \qquad i = 1,2...m$$

or (D.4)

$$\Sigma_{k=1}^{N} \mathbf{A_{ik}} \bullet \dot{\mathbf{r}}_k + A_{it} = 0 \qquad i = 1,2...m \ .$$

Here bolded $\mathbf{A_{ik}}$ represents 3 matrices each of which is m x N (rows x columns). Since these constraints are non-integrable, one cannot really described them in terms of surfaces in $E^{3N}$.

If one differentiates the holonomic constraints (D.2) one gets

$$\Sigma_{k=1}^{N} [\nabla^{(k)} f_i] \bullet d\mathbf{r}_k + [\partial_t f_i]\, dt = 0 \qquad i = 1,2...s$$

or (D.5)

$$\Sigma_{k=1}^{N} [\nabla^{(k)} f_i] \bullet \dot{\mathbf{r}}_k + [\partial_t f_i] = 0 \qquad i = 1,2...s$$

where $\nabla^{(k)} \equiv (\partial/\partial(\mathbf{r}_k)_1, \partial/\partial(\mathbf{r}_k)_2, \partial/\partial(\mathbf{r}_k)_3)$ is a gradient with respect to the components of $\mathbf{r}_k$. These equations have the same form as those in (D.4), but these are holonomic because they are integrable to give $f_i(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_N, t) = 0$.

For convenience below, we shall now combine (D.4) and (D.5) into a single set of equations:

$$\Sigma_{k=1}^{N} \mathbf{B_{ik}} \bullet d\mathbf{r}_k + B_{it} dt = 0 \qquad i = 1,2....C \qquad // \ C = s+m$$

or (D.6)

$$\Sigma_{k=1}^{N} \mathbf{B_{ik}} \bullet \dot{\mathbf{r}}_k + B_{it} = 0 \qquad i = 1,2....C$$

where

$$\begin{array}{lllll} \mathbf{B_{ik}} = \mathbf{A_{ik}} & B_{it} = \mathbf{A_{it}} & \text{for } i = 1,2...s & \text{non-holonomic} & \\ B_{it} = \nabla^{(k)} f_i & B_{it} = \partial_t f_i & \text{for } i = s+1, s+2....C & \text{holonomic} \ . & (D.7) \end{array}$$

In general, the functions $f_i$, $\mathbf{B_{ik}}$ and $B_{it}$ are all functions of $(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_k, t)$.

In the above we have followed the notation of Ray and Shamanna (R&S 2006) and we continue our path in the manner they propose.

Suppose the set of N differentials $\{d\mathbf{r}_k\}$ satisfies all of equations (D.6) for some time interval dt. Such a set $\{d\mathbf{r}_k\}$ is then called a set of **allowed displacements**. Imagine that $\{d\mathbf{r'}_k\}$ is some other distinct allowed displacement set. Elements of the difference set defined as

$$\{\delta\mathbf{r}_k\} = \{d\mathbf{r}_k\} - \{d\mathbf{r'}_k\} \qquad k = 1,2...N \qquad (D.8)$$

are then called **virtual displacements**. This is how R&S define the vague term virtual displacement which appears in many texts (about which they have much to say in their Section I A). It follows then that for any virtual displacement set $\{\delta\mathbf{r}_k\}$ equations (D.6) take this form

$$\Sigma_{k=1}^{N} \mathbf{B_{ik}} \bullet \delta\mathbf{r}_k = 0 \qquad (D.9)$$

simply because the last term in (D.6) cancels out when one writes $\delta\mathbf{r}_k = d\mathbf{r}_k - d\mathbf{r'}_k$. Notice that the $\delta\mathbf{r}_k$ are not arbitrary differential displacements. They are differences of allowed displacements.

Meanwhile, Newton's Law for a system of N particles subject to constraints has the form

$$m\ddot{\mathbf{r}}_k = \mathbf{F}_k + \mathbf{R}_k \qquad k = 1,2...N \qquad \text{(D.10)}$$

where $\mathbf{R}_k$ is the sum of all constraint forces applied to particle k and $\mathbf{F}_k$ is the sum of all *other* forces applied to particle k.

Assumption that constraints do no virtual work

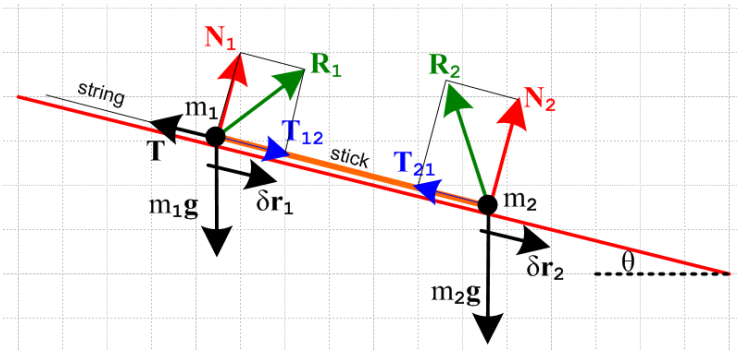
Next, *assume* that ( R&S refer to this as being an "ideal constraint" situation),

$$0 = \Sigma_{k=1}^{N} \mathbf{R}_k \bullet \delta\mathbf{r}_k \ . \qquad \text{(D.11)}$$

In Goldstein this equation corresponds to setting the second term in (1-40) to zero. The main justification for making this assumption is that it is valid for particles which form a rigid body and for many other constraint situations. There seems to be no general justification of this equation for *all* constraint situations, and it is certainly not valid if friction is present. The equation can be interpreted as saying that the **total virtual work** done by all the constraint forces on all the particles of the system vanishes.

As a first simple example, consider two particles 1 and 2 independently sliding down a frictionless static ramp. In this case for particle 1 the constraint force $\mathbf{R}_1$ is the normal force of the ramp acting on the particle, and this is clearly perpendicular to $d\mathbf{r}_1$ for any allowed displacement set $\{d\mathbf{r}_k\}$, and thus $\mathbf{R}_1$ is also perpendicular to any $\delta\mathbf{r}_1$ which is part of any virtual displacement set $\{\delta\mathbf{r}_k\}$. In this case we have $\mathbf{R}_1 \bullet \delta\mathbf{r}_1 = 0$ and $\mathbf{R}_2 \bullet \delta\mathbf{r}_2 = 0$ so the terms in (D.11) are separately zero.

A second example is more interesting. Particles 1 and 2 of arbitrary masses slide down a one dimensional ramp but are glued to the two ends of a massless stick so they comprise a rigid body. The upper mass is attached to a string which provides a constant tension T. Here is a picture,




(D.12)

Particle 2 exerts a constraint force $\mathbf{T}_{12}$ on particle 1 through the stick, and Particle 1 exerts a constraint force $\mathbf{T}_{21}$ on particle 2. Since there is only one force of tension (or compression) in such a stick, $T_{12} = T_{21}$. All allowed displacement sets $\{d\mathbf{r}_1, d\mathbf{r}_2\}$ are of the form $\{d\mathbf{r}, d\mathbf{r}\}$, and therefore all *virtual* displacement sets are of the form $\{\delta\mathbf{r}_1, \delta\mathbf{r}_2\} = \{\delta\mathbf{r}, \delta\mathbf{r}\}$. In (D.12) we have drawn the two normal forces $\mathbf{N}_i$ and the two *total* constraint forces $\mathbf{R}_i$. It is no longer true that $\mathbf{R}_1 \bullet \delta\mathbf{r}_1 = 0$ and $\mathbf{R}_2 \bullet \delta\mathbf{r}_2 = 0$ as the

picture shows. However, it is true that $\Sigma_{k=1}^{2}\ \mathbf{R}_k \bullet \delta\mathbf{r}_k = [\ \Sigma_{k=1}^{2}\ \mathbf{R}_k\ ] \bullet \delta\mathbf{r} = 0$ because $\mathbf{R}_1$ and $\mathbf{R}_2$ have equal and opposite components along the ramp, so equation (D.11) is valid for this example.

Now suppose the ramp is translating in some direction not along the ramp. The displacements $d\mathbf{r}_1 = d\mathbf{r}_2$ are then no longer along the ramp so that $\Sigma_{k=1}^{2}\ \mathbf{R}_k \bullet d\mathbf{r}_k \neq 0$. However, any virtual displacement set would still have $\delta\mathbf{r}_1 = \delta\mathbf{r}_2$ along the ramp and then again $\Sigma_{k=1}^{2}\ \mathbf{R}_k \bullet \delta\mathbf{r}_k = 0$. Remember that a virtual displacement set is the difference of any two allowed displacement sets, and in this subtraction the effect of the ramp translation cancels out. So equation (D.11) is still valid.

Other examples are given in R&S Section III.

Reader Exercise: show in the last example that $T_{12} = T\ m_2 / (m_1+m_2)$ and that this makes sense if $m_1 >> m_2$ and vice versa.

We proceed then assuming our system of interest respects equation (D.11).

Comment on internal forces: In (D.10) we have classified the forces acting on particle k into two groups and we write $\mathbf{R}_k + \mathbf{F}_k$, where $\mathbf{R}_k$ are "constraint" forces and the $\mathbf{F}_k$ are any "other" forces. Goldstein refers to the $\mathbf{F}_k$ forces as "applied" forces in (1-39), while R&S call them "external" in Sec IV. Consider a system consisting of N masses all connected by a network of massless springs. The force acting on particle k due to all the springs to which it is attached would be best classified in the $\mathbf{F}_k$ "other" category. Were we to classify this force into the $\mathbf{R}_k$ group, we could not use the no-constraint-work property (D.12) since spring forces do work as the springs compress and expand. An example of such a system is the Solar System where the spring forces are represented by gravity. It would seem a misnomer to classify the internal forces in this case as "applied" or "external". However, if all the springs are slowly adjusted until they become infinitely stiff, in that limit it would be better to classify the internal spring forces into the $\mathbf{R}_k$ constraint forces group, since these springs do no work and (D.11) applies. The system is then a "rigid body" and in this case, with the internal forces classified as $\mathbf{R}_k$ constraint forces, one benefits from the many simplifications which arise for rigid body mechanics. The internal forces in our spring example fall into the category where $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ and $\mathbf{F}_{ij}$ is in the direction $\mathbf{r}_i - \mathbf{r}_k$ . When this is not the case, the rigid body limit does not apply, but the internal forces can still be classified as "constraint" plus "other".

An Application of Lagrange Multipliers

Recall the C = m+s constraint equations (D.9),

$$\Sigma_{k=1}^{N}\ \mathbf{B}_{ik} \bullet \delta\mathbf{r}_k = 0 \qquad i = 1,2...C \qquad C \equiv m+s = \text{total number of constraints} \qquad \textit{(D.9)}$$

Multiply each of these C equations by a Lagrange Multiplier $\lambda_i$ and then add the equations to get

$$0 = \Sigma_{i=1}^{C}\ \lambda_i\ [\Sigma_{k=1}^{N}\ \mathbf{B}_{ik} \bullet \delta\mathbf{r}_k] = 0$$

$$= \Sigma_{k=1}^{N}\ [\ \Sigma_{i=1}^{C}\ \lambda_i\ \mathbf{B}_{ik}\ ] \bullet \delta\mathbf{r}_k\ . \qquad \text{(D.13)}$$

Now subtract equation (D.13) from (D.11) to get

$$0 = \Sigma_{k=1}^{N}\ [\ \mathbf{R}_k - \Sigma_{i=1}^{C}\ \lambda_i\ \mathbf{B}_{ik}] \bullet \delta\mathbf{r}_k\ . \qquad \text{(D.14)}$$

We know that the N vectors in the set $\{\delta \mathbf{r}_k\}$ are not linearly independent because they must satisfy the equations (D.9), so we cannot claim from (D.14) that $[\ \mathbf{R}_k - \Sigma_{i=1}^{C} \lambda_i\ \mathbf{B}_{ik}]\ = 0$ for each k. But now write out equation (D.14) in full detail, showing all vector components,

$$0 = \Sigma_{k=1}^{N}\ \Sigma_{j=1}^{3}\ [\ (\mathbf{R}_k)_j - \Sigma_{i=1}^{C} \lambda_i\ \ (\mathbf{B}_{ik})_j]\ (\delta \mathbf{r}_k)_j\ . \qquad \text{(D.15)}$$

Next, relabel the 3N terms in this sum using a single index n defined in this way

$$\begin{array}{l} (k,j)\ = (1,1),\ (1,2),\ (1,3),\ (2,1),\ (2,2),\ (2,3),.....(N,1),\ \ (N,2),\ \ (N,3) \\ \quad n\ \ =\ \ 1,\ \ \ \ 2,\ \ \ \ 3,\ \ \ \ 4,\ \ \ \ 5,\ \ \ 6\ , \qquad\qquad\qquad 3N\text{-}2,\ \ 3N\text{-}1,\ \ 3N \end{array}$$

so

$$n = 3(k-1)+j \qquad k = 1+ \mathrm{Int}[(n-1)/3] \qquad j = 1 + \mathrm{Rem}[(n-1)/3]\ . \qquad \text{(D.16)}$$

Then define (here the n value is implied by the k,j labels as per above)

$$x_n \equiv (\mathbf{r}_k)_j \qquad R_n \equiv (\mathbf{R}_k)_j \qquad B_{in} \equiv\ (\mathbf{B}_{ik})_j\ \ . \qquad \text{(D.17)}$$

Equation (D.15) can then be expressed as

$$0 = \Sigma_{n=1}^{3N}\ [\ R_n - \Sigma_{i=1}^{C} \lambda_i\ B_{in}]\ \delta x_n \qquad \text{(D.18)}$$

and (D.9) becomes $0 = \Sigma_{k=1}^{N}\ \Sigma_{j=1}^{3}\ (\mathbf{B}_{ik})_j\ (\delta \mathbf{r}_k)_j$ or

$$0 =\ \Sigma_{n=1}^{3N}\ B_{in}\ \delta x_n\ \ . \qquad\qquad i = 1,2...C\ . \qquad \text{(D.19)}$$

Once again, we cannot set the square brackets in (D.18) to 0 because the 3N coordinates $\delta x_n$ are not linearly independent due to the C equations (D.19).

At any given time t, due to these equations (D.19), there must exist at least one subset of the 3N variables $\delta x_n$ which are independent and the remaining C $\delta x_n$ are dependent. Assume that the dependent variables have indices $n = n_1, n_2,...n_C$. Consider then the following set of C equations involving the dependent-term square brackets in (D.18),

$$[\ R_n - \Sigma_{i=1}^{C} \lambda_i\ B_{in}]\ = 0 \qquad\qquad n = n_1, n_2,...n_C\ \ . \qquad \text{(D.20)}$$

This represents C linear equations in the C parameters $\lambda_i$ so, when all is said and done, there must exist a set of values $\{\lambda_i\}$ which make these C equations be true. We assume these values will be $\lambda_i(t)$ and this is justified below when the full set of equations is considered.

Now the terms in (D.18) indicated by $n = n_1, n_2,...n_C$ all vanish due to (D.20), so (D.18) can be written,

$$0 = \Sigma_{n \neq n_1, n_2, \ldots . n_C}\ [\ R_n - \Sigma_{i=1}^{C} \lambda_i\ B_{in}]\ \delta x_n\ . \qquad \text{(D.21)}$$

But all the $\delta x_n$ appearing in (D.21) are independent, so the square brackets here must also vanish. We then end up with *all* the square brackets in (D.18) being 0 (assuming the correct solution values of $\lambda_i$),

$$[\, R_n - \Sigma_{i=1}^{C} \lambda_i B_{in}] = 0 \qquad n = 1,2....3N$$

or

$$R_n = \Sigma_{i=1}^{C} \lambda_i \, B_{in} \qquad n = 1,2....3N$$

or

$$(\mathbf{R_k})_j = \Sigma_{i=1}^{C} \lambda_i \, (\mathbf{B_{ik}})_j \qquad k = 1,2..N \qquad j = 1,2,3$$

or

$$\mathbf{R_k} = \Sigma_{i=1}^{C} \lambda_i \, \mathbf{B_{ik}} \qquad k = 1,2...N \ . \qquad \text{(D.22)}$$

Thus the constraint forces are certain linear combinations of the $\mathbf{B_{ik}}$ constraint functions shown in (D.7).

Meanwhile, Newton's Law (D.10) says that $m_k \mathbf{a_k} = \mathbf{F_k} + \mathbf{R_k}$ where the $\mathbf{F_k}$ are the non-constraint forces for the problem. Inserting (D.22) gives

$$m_k \ddot{\mathbf{r}}_k = \mathbf{F_k} + \Sigma_{i=1}^{C} \lambda_i \mathbf{B_{ik}} \qquad \mathbf{B_{ik}} = \begin{cases} \mathbf{A_{ik}} & i = 1 \text{ to m (non-holonomic)} \\ [\nabla^{(k)} f_i] & i = m+1 \text{ to s (holonomic)} \end{cases} \qquad \text{(D.23)}$$

The problem is then summarized in the following set of equations from (D.23) and (D.6),

$$m_k \dot{\mathbf{v}}_k(t) = \mathbf{F_k}(\mathbf{r_1}, \mathbf{r_2}..... \mathbf{r_N}, t) + \Sigma_{i=1}^{C} \lambda_i(t) \mathbf{B_{ik}}(\mathbf{r_1}, \mathbf{r_2}..... \mathbf{r_N}, t) \qquad k = 1..N \qquad 3N \text{ equations}$$

$$\dot{\mathbf{r}}_k(t) = \mathbf{v_k}(t) \qquad k = 1..N \qquad 3N \text{ equations}$$

$$\Sigma_{k=1}^{N} \mathbf{B_{ik}}(\mathbf{r_1}, \mathbf{r_2}..... \mathbf{r_N}, t) \bullet \dot{\mathbf{r}}_k(t) + B_{it}(\mathbf{r_1}, \mathbf{r_2}..... \mathbf{r_N}, t) = 0 \qquad i = 1,2....C \ . \qquad C \text{ equations}$$

(D.24)

This can be regarded as a set of 6N+C scalar first-order ODEs in the single variable t for the 6N+C unknown scalar functions $\mathbf{r_k}(t)$, $\mathbf{v_k}(t)$ and $\lambda_i(t)$. Although the first 3N equations happen to be linear in the $\lambda_i(t)$, the equations are in general non-linear in the $\mathbf{r_k}(t)$ so this is then a *non-linear* system of ODE's. Nevertheless, a well known existence theorem (e.g., Ince Section 3.3) states that, provided the $\mathbf{F_k}$, $\mathbf{B_{ik}}$ and $B_{it}$ functions are continuous and differentiable in all their arguments, a unique solution exists for any set of initial values $\mathbf{r_k}(0)$, $\mathbf{v_k}(0)$ and $\lambda_i(0)$. As (D.22) shows, the $\lambda_i(0)$ are related to the constraint forces at time 0.

The uniqueness of the solution agrees with our intuition that a physical system will evolve in a unique manner. Finding an analytic form for the solution might not be possible, but a numeric form can always be obtained.

Summary of the above Lagrange Multiplier Application

We have used a set of Lagrange Multipliers as an aid to solving a mechanics problem involving N particles with C constraints. As already noted, the above presentation is based on Section IV of R&S. The steps were as follows, considered more generically:

1. One has a sum $\Sigma_{n=1}^{N} R_n \delta x_n = 0$. (D.25)

2. One also has a set of sums $\Sigma_{n=1}^{N} B_{in} \delta x_n = 0$ for i = 1,2...C with C < N (constraints in the above case). This implies that one can find C of the $\delta x_n$ which are linearly dependent on the other $\delta x_n$.

3. If $\lambda_i$ are C arbitrary factors, certainly $\Sigma_{i=1}^{C} \lambda_i (\Sigma_{n=1}^{N} B_{in} \delta x_n) = 0$ based on 2. These $\lambda_i$ are "Undetermined Lagrange Multipliers".

4. One constructs from steps 1 and 3 the new sum $\Sigma_{n=1}^{N} [ R_n - \Sigma_{i=1}^{C} \lambda_i B_{in}] \delta x_n = 0$.

5. The C $\lambda_i$ are selected so that [..] = 0 in the C dependent $\delta x_n$ terms of this sum. The remaining sum has only independent $\delta x_n$ terms so [..] = 0 for all those terms as well. Then *all* [...] = 0 in the sum of step 4.

6. The conclusion is that $[ R_n - \Sigma_i \lambda_i B_{in}] = 0$ for all n, assuming the solution $\lambda_i$ values of step 5.

In contrast with our main topic of finding stationary points df = 0 of a function f(**r**), in the Lagrange Multiplier application just presented there is no explicit function f(**r**) we are making stationary. On the other hand, one can regard (D.11) that $\Sigma_k \mathbf{R}_k \bullet \delta \mathbf{r}_k = 0$ as the statement $\delta W = 0$ where $\delta W = \Sigma_k \mathbf{R}_k \bullet \delta \mathbf{r}_k$ is the total differential "virtual work" done by the constraint forces during a differential virtual displacement $\{\delta \mathbf{r}_k\}$. Using Newton's law (D.10) this equation can then be written

$$0 = \delta W = \Sigma_k [m_k \mathbf{a}_k - \mathbf{F}_k] \bullet \delta \mathbf{r}_k \qquad \text{(D.26)}$$

in which form it is known as **D'Alembert's Principle** (1743, Goldstein (1-42)). The analog of f(**r**) is then an action integral W of the differential virtual work $\delta W$ which integral one renders stationary to get $\delta W = 0$. The notion of "the action" is mentioned at the end of this appendix.)

We now present another application of the Lagrange Multiplier technique outlined in (D.25). As before, a certain amount of background is needed before the application can be demonstrated.

Generalized coordinates, generalized forces, and Lagrange's Equations

Usually the Cartesian components of the particle positions $\mathbf{r}_k$ are not the most convenient variables to use in solving a problem. Angles are often more useful. With C holonomic constraints one can replace the 3N variables $\{x_n\}$ in (D.17) with a set of 3N-C *independent* variables $\{q_n\}$ where

$$x_n = x_n(q_1, q_2, ....q_{3N-C}, t) \qquad \text{for n=1 to 3N} \qquad \text{(D.27)}$$

and one can then rework the above presentation in these new "generalized coordinates" $q_j$ which then won't always have the dimensions of length. For example, one then has, for virtual displacements,

$$\delta x_n = \Sigma_{j=1}^{3N-C}(\partial x_n/\partial q_j)\,\delta q_j \qquad n = 1,2..3N$$

or

$$\delta \mathbf{r}_k = \Sigma_{j=1}^{3N-C}(\partial \mathbf{r}_k/\partial q_j)\,\delta q_j \qquad k = 1,2..N \qquad \text{// as in Goldstein (1-44).} \qquad \text{(D.28)}$$

The usual presentation of Lagrangian dynamics with holonomic constraints *avoids* dealing with the forces of constraint and also avoids the need for using a set of Lagrange Multipliers. That presentation appears in Section V of the R&S and Goldstein p 17 (and many other places) and proceeds like this:

1. Start with $\Sigma_k \mathbf{R}_k \bullet \delta \mathbf{r}_k = 0$ in (D.11) which we write in the form of D'Alembert's Principle (D.26) already mentioned above,

$$0 = \Sigma_{k=1}^{M} [m_k \mathbf{a}_k - \mathbf{F}_k] \bullet \delta \mathbf{r}_k \qquad \textit{(D.26)}$$

where $\mathbf{F}_k$ is the total non-constraint force acting on particle k.

2. Install (D.28) into D'Alembert's Principle to get

$$0 = \Sigma_k [m_k \mathbf{a}_k - \mathbf{F}_k] \bullet \Sigma_j \left(\frac{\partial \mathbf{r}_k}{\partial q_j}\right)\delta q_j$$

$$= \Sigma_j \{ \Sigma_k m_k \mathbf{a}_k \bullet \frac{\partial \mathbf{r}_k}{\partial q_j} - \Sigma_k \mathbf{F}_k \bullet \left(\frac{\partial \mathbf{r}_k}{\partial q_j}\right) \}\ \delta q_j$$

$$= \Sigma_j \{ \Sigma_k m_k \mathbf{a}_k \bullet \frac{\partial \mathbf{r}_k}{\partial q_j} - Q_j \}\ \delta q_j \qquad \text{(D.29)}$$

where

$$Q_j \equiv \Sigma_{k=1}^{M} \mathbf{F}_k \bullet \left(\frac{\partial \mathbf{r}_k}{\partial q_j}\right) . \qquad \text{(D.30)}$$

$Q_j$ defined above is the "generalized force" associated with the "generalized coordinate" $q_j$ .

3. Make use of the following non-obvious result (derived below),

$$\Sigma_k m_k \mathbf{a}_k \bullet \frac{\partial \mathbf{r}_k}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \qquad \text{(D.31)}$$

where T is the total kinetic energy of the N particles,

$$T \equiv (1/2)\Sigma_k m_k (\mathbf{v}_k \bullet \mathbf{v}_k) . \qquad \text{(D.32)}$$

Then (D.29) reads,

$$0 = \Sigma_j \{ \frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j}) - \frac{\partial T}{\partial q_j} - Q_j \} \delta q_j . \qquad (D.33)$$

Since the $\delta q_j$ are independent variables, {..} = 0 and we end up with one form of "Lagrange's Equations"

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j}) - \frac{\partial T}{\partial q_j} = Q_j \qquad j = 1,2....3N-C \quad . \qquad \text{// as in Goldstein (1-50)} \qquad (D.34)$$

If the applied forces can be derived from a potential $\mathcal{V}(\mathbf{r}_1, \mathbf{r}_2, ..... \mathbf{r}_N)$ such that

$$\mathbf{F}_k = - \nabla^{(k)} \mathcal{V}(\mathbf{r}_1, \mathbf{r}_2, ..... \mathbf{r}_N) \qquad \text{// no t or } \dot{\mathbf{r}}_i \text{ dependence in } \mathcal{V} \qquad (D.35)$$

then from (D.30),

$$Q_j \equiv \Sigma_k \mathbf{F}_k \bullet (\frac{\partial \mathbf{r}_k}{\partial q_j}) = - \Sigma_k (\frac{\partial \mathcal{V}}{\partial \mathbf{r}_k}) \bullet (\frac{\partial \mathbf{r}_k}{\partial q_j}) = - \frac{\partial V}{\partial q_j} \qquad (D.36)$$

where $V(q_i) \equiv \mathcal{V}(\mathbf{r}_k(q_i))$ is a function of the generalized coordinates $q_i$. Since V is not a function of the $\dot{q}_i$ we know that $\frac{\partial V}{\partial \dot{q}_j} = 0$. Using this fact, and (D.36) that $Q_j = - \frac{\partial V}{\partial q_j}$, (D.34) can be written

$$\frac{d}{dt}(\frac{\partial (T-V)}{\partial \dot{q}_j}) - \frac{\partial (T-V)}{\partial q_j} = 0 . \qquad (D.37)$$

The last step is to define the classical **Lagrangian**

$$L \equiv T - V \qquad (D.38)$$

to obtain

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j}) - \frac{\partial L}{\partial q_j} = 0 \qquad j = 1,2....3N-C \quad . \qquad \text{// as in Goldstein (1-53)} \qquad (D.39)$$

which is the more conventional form of "Lagrange's Equations" with C holonomic constraints.
ok to here

Footnote: Derivation of the result (D.31) used above

Define the 3N-C independent generalized coordinates $q_i$ as in (D.27) according to

$$\mathbf{r}_k = \mathbf{r}_k(q_1, q_2....q_{3N-C}, t) \equiv \mathbf{r}_k(q_i, t) . \qquad (D.40)$$

Compute the total time derivative of $\mathbf{r_k}$ ,

$$\mathbf{v_k} = \dot{\mathbf{r}}_\mathbf{k} = \frac{d\mathbf{r_k}}{dt} = \Sigma_j \frac{\partial \mathbf{r_k}}{\partial q_j} \frac{\partial q_j}{\partial t} + \frac{\partial \mathbf{r_k}}{\partial t} = \Sigma_j \frac{\partial \mathbf{r_k}}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r_k}}{\partial t} \qquad // = \mathbf{v_k}(q_i, \dot{q}_i, t) \; . \qquad (D.41)$$

Then based on this result compute these two partial derivatives of $\mathbf{v_k}$ ,

$$\frac{\partial \mathbf{v_k}}{\partial q_i} = \Sigma_j \frac{\partial^2 \mathbf{r_k}}{\partial q_i \partial q_j} \dot{q}_j + \frac{\partial^2 \mathbf{r_k}}{\partial q_i \partial t} \qquad (D.42)$$

$$\frac{\partial \mathbf{v_k}}{\partial \dot{q}_i} = \frac{\partial \mathbf{r_k}}{\partial q_i} \; . \qquad (D.43)$$

Next, the total kinetic energy of all k particles is given by (D.32),

$$T \equiv (1/2) \Sigma_k \; m_k \; (\mathbf{v_k} \bullet \mathbf{v_k}) \qquad \textit{(D.32)}$$

so that

$$\frac{\partial T}{\partial \dot{q}_i} = \Sigma_k \, m_k \, \mathbf{v_k} \bullet \frac{\partial \mathbf{v_k}}{\partial \dot{q}_i} = \Sigma_k \, m_k \, \mathbf{v_k} \bullet \frac{\partial \mathbf{r_k}}{\partial q_i} \; . \qquad // \text{ using (D.43)} \qquad (D.44)$$

Apply d/dt to the above, with $\mathbf{a_k} = \dot{\mathbf{r}}_\mathbf{k}$ :

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \Sigma_k \, m_k \, \mathbf{a_k} \bullet \frac{\partial \mathbf{r_k}}{\partial q_i} + \Sigma_k \; m_k \; \mathbf{v_k} \bullet \frac{d}{dt}\left(\frac{\partial \mathbf{r_k}}{\partial q_i}\right) \; . \qquad (D.45)$$

Compute the total time derivative appearing in the second term,

$$\frac{d}{dt}\left(\frac{\partial \mathbf{r_k}}{\partial q_i}\right) = \Sigma_j \frac{\partial^2 \mathbf{r_k}}{\partial q_j \partial q_i} \dot{q}_j + \frac{\partial^2 \mathbf{r_k}}{\partial t \, \partial q_i} \; . \qquad (D.46)$$

Meanwhile,

$$\frac{\partial T}{\partial q_i} = \Sigma_k \, m_k \, \mathbf{v_k} \bullet \frac{\partial \mathbf{v_k}}{\partial q_i}$$

$$= \Sigma_k \, m_k \, \mathbf{v_k} \bullet \left[ \Sigma_j \frac{\partial^2 \mathbf{r_k}}{\partial q_i \partial q_j} \dot{q}_j + \frac{\partial^2 \mathbf{r_k}}{\partial q_i \partial t} \right] \qquad // \text{ (D.42)}$$

$$= \Sigma_k \, m_k \, \mathbf{v_k} \bullet \frac{d}{dt}\left(\frac{\partial \mathbf{r_k}}{\partial q_i}\right) \; . \qquad // \text{ (D.46)} \qquad (D.47)$$

Subtract (D.47) from (D.45) to get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = \Sigma_k\, m_k\, \mathbf{a}_k \bullet \frac{\partial \mathbf{r}_k}{\partial q_i} \qquad \text{(D.48)}$$

which is the result quoted above in (D.31).

Lagrange Equations with non-holonomic constraints

This is our final Lagrange Multiplier example, and it exactly follows the generic prescription of (D.25). The general topic is well described in Goldstein Chapter 2.

As the starting point we invoke **Hamilton's Principle** which is this :

$$\delta S = 0 \qquad \text{where} \qquad S = \int_{t1}^{t2} dt\, L(q_n(t), \dot{q}_n(t), t) \qquad \text{(D.49)}$$

where there are M generalized coordinates $q_n$. Here L = T - V is the Lagrangian (D.38) with $T(\dot{q}_i)$ and $V(q_i)$ being the kinetic and potential energies associated with the M generalized coordinates. We shall assume that M = N-D where there were D holonomic constraints on N initial problem coordinates. As shown in Goldstein p 41 (2-20') based on p 37 (2-15), the equation $\delta S = 0$ results in the following sum being 0,

$$1.\quad \Sigma_{n=1}^{M} \left[ \int_{t1}^{t2} dt \left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) - \frac{\partial L}{\partial q_n} \right\} \right] \delta q_n(t) = 0 \ . \qquad \text{(D.50)}$$

The number 1. on the left refers to the first step outlined in (D.25). In this equation the $\delta q_n(t)$ are M independent functions of t ("path variations"), restricted only by $\delta q_n(t_1) = \delta q_n(t_2) = 0$. In order for (D.50) to be true for M arbitrary independent *functions* $\delta q_n(t)$, one must have {} = 0,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) - \frac{\partial L}{\partial q_n} = 0 \qquad \text{(D.51)}$$

which are the Lagrange Equations (D.39). Thus it is that the Lagrange Equations can be derived from Hamilton's Principle in the case of only holonomic constraints.

But suppose that in addition to the D holonomic constraints there are C non-holonomic constraints, so only M-C of the $\delta q_n$ are independent. In this case the above conclusion is incorrect.

2. The C non-holonomic constraints result in C sums $\Sigma_n A_{ni}\, \delta q_n = 0$ i = 1,2...C analogous to (D.9) above.

3. If $\lambda_i$ are C arbitrary factors, certainly $\Sigma_i \lambda_i (\Sigma_n A_{in} \delta q_n) = 0$ based on 2. These $\lambda_i$ are "Undetermined Lagrange Multipliers". Then $\int_{t1}^{t2} dt\ (\Sigma_i \lambda_i \Sigma_n A_{in} \delta q_n) = 0$ as well.

4. Construct from 1 and 3 the new sum,

$$\Sigma_{n=1}^{M} \left[ \int_{t1}^{t2} dt \left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) - \frac{\partial L}{\partial q_n} - \Sigma_i \lambda_i A_{in} \right\} \right] \delta q_n(t) = 0 \quad \text{// as in Goldstein (2-26)} \qquad (D.52)$$

5. The C $\lambda_i$ are selected so that {..} = 0 in the C dependent $\delta q_n(t)$ terms of this sum. The remaining sum has only independent $\delta q_n(t)$ terms so {..} = 0 for all those terms as well. Then *all* {...} = 0 in the sum of 4.

6. The conclusion is that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) - \frac{\partial L}{\partial q_n} = \Sigma_i \lambda_i A_{in} \ . \qquad (D.53)$$

This then is the corrected form of the Lagrange Equations in the presence of non-holonomic constraints, as in Goldstein (2-30). Comparing (D.53) with (D.34), one may *interpret* $Q'_n \equiv \Sigma_i \lambda_i A_{in}$ as the generalized forces that would have to be present to simulate the effect of the non-holonomic constraints were they not present. The forces for any holonomic constraints are already accounted for by L as in (D.39).

The integral S shown in (D.49) is called "the action" and Hamilton's Principle is the classical mechanics incarnation of the **Principle of Least Action**. In relativistic classical and quantum field theory the action is a spacetime integral of the Lagrangian density $\mathcal{L}(\varphi_n(x^\mu), \partial_\mu \varphi_n(x^\mu), x^\mu)$ where the coordinates $q_n$ of L are replaced by fields $\varphi_n(x^\mu)$, and t is replaced by $x^\mu$, a point in spacetime.

END END END END END END END END END END END END END END

E. L. Ince, *Ordinary Differential Equations* (Longmans, Green & Co. (Ltd.), London, 1927). This classic text became a Dover book in 1956 and is available online used for $4, shipping included (576p).

S. Ray and J. Shamanna, "On Virtual Displacement and Virtual Work in Lagrangian Dynamics", European Journal of Physics, 27 (2006) 311-329. Also https://arxiv.org/abs/physics/0510204v2 .

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**Comments:**

This is from Greenwood Google books.

> If the Lagrange multiplier method is applied to this system and the resulting differential equations are completely solved, the result will be that the $q$'s and $\lambda$'s are expressed as explicit functions of time. Then, with the aid of Eq. (2-47), the generalized constraint forces $C_i$ can also be obtained as explicit functions of time.

# On virtual displacement and virtual work in Lagrangian dynamics

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So I have just found a newer (and published) version of the R&S document, and I printed it from the archiv and read it. It is basically the same except for several items:

1. The Intro and Conclusion sections are much longer than in the 2003 paper.

2. They have generalized their paper to handle s holo and m non-holo constraints. But all conclusions and sections are basically exactly the same. It's just that in certain places they have to state both forms of constraint equations. I am glad I found this later version.

**********

**Questions on the above**: We know for sure that $[\nabla^{(k)} f_i]$ is a function of $\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_N$, t so we might as well assume as well that $\mathbf{A}_{ik}$ and $A_{it}$ are all functions of $\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_N$, t. The equations we then want to solve are these

$$m_k \ddot{\mathbf{r}}_k(t) = \mathbf{F}_k(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_N, t) + \Sigma_{i=1}^{C} \lambda_i \mathbf{B}_{ik}(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_N, t) \qquad k = 1,2...N$$

$$f_i(\mathbf{r}_1, \mathbf{r}_2, \ ...\mathbf{r}_M, t) = 0 \qquad i = 1,2...s$$

$$\Sigma_{k=1}^{N} \mathbf{A}_{ik}(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_N, t) \bullet \dot{\mathbf{r}}_k(t) + A_{it}(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_N, t) = 0 . \qquad i = s+1 ...C$$

Are we supposed to think of $\lambda_i$ as numbers, or are they $\lambda_i(t)$, or are they $\lambda_i(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_k, t)$ ???

My guess is that we want them to be $\lambda_i(t)$. So how would one approach solving the above generic set of equations. I would replace the second group of equations perhaps with (D.5) which says

$$\Sigma_{k=1}^{N} [\nabla^{(k)} f_i(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_k, t) ] \bullet \dot{\mathbf{r}}_k(t) + [\partial_t f_i(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_k, t)] = 0 \qquad i = 1,2...s \qquad (D.5)$$

We could then define

$H_i = A_{it}$ i = 1,2...m

$H_i = \partial_t f_i$ i = m+1,2...C

Then we have this more uniform set of set of equations to deal with,

$$m_k \ddot{\mathbf{r}}_k(t) = \mathbf{F}_k(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_N, t) + \Sigma_{i=1}^{C} \lambda_i(t)\, \mathbf{B}_{ik}(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_N, t) \qquad k = 1....N$$

$$\Sigma_{k=1}^{N} \mathbf{B}_{ik}(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_N, t) \bullet \dot{\mathbf{r}}_k(t) + H_i(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_N, t) = 0 . \qquad i = s+1 ...C \qquad (**)$$

Remember that we *know* the $\mathbf{F}_k$ and the $\mathbf{B}_{ij}$ and the $H_i$ functions of all their variables. Above we still have a set of 3N+C equations in 3N+C unknowns $\mathbf{r}_k(t)$ and $\lambda_i(t)$ . Note that

$$\dot{\mathbf{B}}_{ik}(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_N, t) = \Sigma_{m=1}^{N} (\nabla^{(m)} \mathbf{B}_{ik}) \bullet \dot{\mathbf{r}}_m(t) + (\partial_t \mathbf{B}_{ik})$$

$$\dot{H}_i(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_N, t) = \Sigma_{m=1}^{N} (\nabla^{(m)} H_i) \bullet \dot{\mathbf{r}}_m(t) + (\partial_t H_i)$$

Suppose we apply d/dt to the second equation set (**) above. We would get

$$\Sigma_{k=1}^{N} \mathbf{B}_{ik}(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_k, t) \bullet \ddot{\mathbf{r}}_k(t) + \Sigma_{k=1}^{N} \dot{\mathbf{B}}_{ik}(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_k, t) \bullet \dot{\mathbf{r}}_k(t) + \dot{H}_i(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_k, t) = 0$$

and then making the replacements,

$$\Sigma_{k=1}^{N} \mathbf{B}_{ik}(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_k, t) \bullet \ddot{\mathbf{r}}_k(t) + \Sigma_{k=1}^{N} \{ \Sigma_{m=1}^{N} (\nabla^{(m)} \mathbf{B}_{ik}) \bullet \dot{\mathbf{r}}_m(t) + (\partial_t \mathbf{B}_{ik}) \} \bullet \dot{\mathbf{r}}_k(t)$$

$$+ \Sigma_{m=1}^{N} (\nabla^{(m)} H_i) \bullet \dot{\mathbf{r}}_m(t) + (\partial_t H_i) = 0$$

But now we can use $\ddot{\mathbf{r}}_k(t) = [\mathbf{F}_k + \Sigma_{i=1}^{C} \lambda_i(t)\, \mathbf{B}_{ik}]/m_k$ to rewrite the above as

$$\Sigma_{k=1}^{N} \mathbf{B}_{ik} \bullet [\mathbf{F}_k + \Sigma_{i=1}^{C} \lambda_i(t)\, \mathbf{B}_{ik}]/m_k$$

$$+ \Sigma_{k=1}^{N} \{ \Sigma_{m=1}^{N} (\nabla^{(m)} \mathbf{B}_{ik}) \bullet \dot{\mathbf{r}}_m(t) + (\partial_t \mathbf{B}_{ik}) \} \bullet \dot{\mathbf{r}}_k(t)$$

$$+ \Sigma_{m=1}^{N} (\nabla^{(m)} H_i) \bullet \dot{\mathbf{r}}_m(t) + (\partial_t H_i) = 0$$

This equation has the following form

$$\Sigma_{kmj} K_{ikmj} (\mathbf{v}_m)_j (\mathbf{v}_k)_j + [\Sigma_{kmj} L_{ik} + \Sigma_{mj} M_{mi}] (\mathbf{v}_k)_j = - \Sigma_{k=1}^{N} \mathbf{B}_{ik} \bullet [\mathbf{F}_k + \Sigma_{i=1}^{C} \lambda_i(t)\, \mathbf{B}_{ik}]/m_k$$

Rewrite showing all r's,

$$\Sigma_{kmj}\, K_{ikmj}(\mathbf{r}_i..)(\ (\dot{\mathbf{r}}_m)_j(\dot{\mathbf{r}}_k)_j + [\Sigma_{kmj}\, L_{ik}(\mathbf{r}_i..)\ + \Sigma_{mj} M_{mi}(\mathbf{r}_i..)]\ (\dot{\mathbf{r}}_k)_j\ \ = \ -\ \Sigma_{k=1}^{N}\, \mathbf{B}_{ik}(\mathbf{r}_i..) \bullet$$
$$[\mathbf{F}_k(\mathbf{r}_i..)\ + \Sigma_{i=1}^{C}\, \lambda_i(t)\ \mathbf{B}_{ik}(\mathbf{r}_i..)]/m_k$$

This is a *very* non-linear equation. The $\mathbf{r}_i$ could appear in any functional form inside the various factors, and you can see that the $\dot{\mathbf{r}}_i$ have quadratic times functions of $\mathbf{r}_i$ , I would call this "hopeless".

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That paper comments (at great length) on the often vague term "virtual displacement" as it appears in Goldstein and other standard mechanics texts, and we have followed their suggestion for defining this term as a difference of "allowed displacements" which then removes the dt time term in (D.1).

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In this last case, we can assume solutions $\mathbf{r}_k(t)$ and then we have in effect $\lambda_i(\mathbf{r}_1(t), \mathbf{r}_2(t)..... \mathbf{r}_N(t), t) = \lambda'_i(t)$, so we can rule out the third situation. So in my opinion the $\lambda_i$ are supposed to be $\lambda_i(t)$.

2. For each of these possibilities, how do we know that a solution to the above equations exists?

$$\Sigma_{k=1}^{N}\, [\nabla^{(k)} f_i(\mathbf{r}_1, \mathbf{r}_2..... \mathbf{r}_k, t)\ ] \bullet\ \dot{\mathbf{r}}_k(t) + [\partial_t f_i(\mathbf{r}_1, \mathbf{r}_2..... \mathbf{r}_k, t)] = 0 \qquad \text{derivative of the above}$$

(D.5)

$$\Sigma_{k=1}^{N}\, \mathbf{A}_{ik}(\mathbf{r}_1, \mathbf{r}_2..... \mathbf{r}_N, t) \bullet \dot{\mathbf{r}}_k(t) + A_{it}(\mathbf{r}_1, \mathbf{r}_2..... \mathbf{r}_N, t)\ = 0\ . \qquad i = s+1\ ...C$$

In the Comment 4 at the end of Section 2 we note that sometimes the constraints on a system with "generalized coordinates" $q_k$ can be written $\Sigma_k a_{ik} dq_k + a_{it} dt = 0$. Such constraints written in terms of differentials of the coordinates are examples of **non-holonomic** constraints. This situation arises typically when constraints act on velocities rather than positions. An example is a vertical disk rolling on a horizontal plane as shown in Goldstein p 13. Such constraints are difficult to handle because the coordinates $q_k$ are not generally independent, meaning that the values of all the $q_k$ cannot be set independently. There is no constraint equation of the form $a(q_1,q_2.....) = 0$ which is a function of the whole coordinates $q_i$; differential constraints are "non-whole".

Inequality constraints are also non-holonomic, such as for particles bounded inside a box, but we shall use the non-holonomic term only in the sense just discussed.

As shown in our Comment 4, there is a role to be played by "Lagrange Multipliers". There is one such multiplier $\lambda_i$ for each non-holonomic constraint and the values of the $\lambda_i$ are determined as part of the problem solution.

Our main focus in this document is the use of Lagrange Multipliers to find candidates $\mathbf{r}_i$ for which a scalar function $f(\mathbf{r})$ is stationary, $df = 0$, subject to constraints. This is application of Lagrange multipliers is distinct from that outlined in Comment 4 where there is no obvious function $f(\mathbf{r})$ which we are trying make stationary.

However, by extending the notation of stationarity one can first define a certain function of the possible solution trajectories $q_k(t)$ for a problem. That function is called "the action" S which is the time integral of the Lagrangian $L(q_k, \dot{q}_k, t)$. The Lagrange equations of motion (including constraints) can be generated by setting $\delta S = 0$, a fact known as **Hamilton's Principle** (Goldstein Chapter 2 or the web). This notion is also used in the path integral formulations of quantum mechanics and quantum field theory. But these action applications are really not the same as the relatively simple problem addressed in this document concerning finding extrema of some scalar function $f(\mathbf{r})$ .

A constraint of the form $a(q_1,q_2....q_N, t) = 0$ is referred to as a **holonomic** constraint, and the main text of our document deals only with such constraints.

We shall now outline another mechanics application of Lagrange multipliers, again unrelated to making some obvious function $f(\mathbf{r})$ be stationary. The presentation below follows Section IV of a 2006 paper by Ray and Shamanna (R&S) and tries to stick to their notation. Usually one wants to remove constraint forces from an analysis, but here they are instead the focus of the approach. In any event, this is just another example of the use of Lagrange Multipliers.

Note: In this appendix subscripts are just labels, they do not indicate differentiation as we used elsewhere in this document. For example, $f_i$ does not mean $\partial f/\partial x_i$, it just refers to the $i^{th}$ constraint function. Similarly $\mathbf{r}_k$ is the 3D coordinate of the $k^{th}$ particle.

An application of Lagrange Multipliers

Consider a system of N particles each having a 3D coordinate $\mathbf{r}_k$, so there are 3N scalar variables. Assume there is a set of s holonomic constraints $f_i(\mathbf{r}_1, \mathbf{r}_2,,,,\mathbf{r}_M, t) = 0$ for i = 1,2...s. Assume in addition that there is a set of m non-holonomic constraints $\Sigma_{k=1}^{M} \mathbf{A}_{ik}(t) \bullet d\mathbf{r}_k + A_{it}(t)dt$ for i = 1,2...m .

These C constraint surfaces each of dimension 3M-1 in $E^{3M}$ intersect in some surface of dimension 3M-C in $E^{3M}$ which then restricts the positions of the particles, as outlined in Appendix C: only certain $\{\mathbf{r}_1, \mathbf{r}_2 ... \mathbf{r}_M\}$ are allowed in any solution of the problem.

For example, suppose there are M=3 particles constrained to lie at the vertices of an equilateral triangle of edge 3, and in addition particle 1 is only allowed to be on a sphere of radius 5. The four constraints then are $|\mathbf{r}_1 - \mathbf{r}_2| - 3 = 0$, $|\mathbf{r}_2 - \mathbf{r}_3| - 3 = 0$, $|\mathbf{r}_3 - \mathbf{r}_1| - 3 = 0$, and $(\mathbf{r}_1)_1^2 + (\mathbf{r}_1)_2^2 + (\mathbf{r}_1)_3^2 - 5^2 = 0$. Each such constraint fits the holonomic template form $f_i(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, t) = 0$. Each constraint is a surface of dimension 8 in $E^9$ (there are many extrusions as discussed in Appendix C), and the intersection surface has dimension 5 in $E^9$. One could count these 5 degrees of freedom as follows: location of particle 1 on the sphere (2) [perhaps θ.φ] and orientation of the triangle in space (3) [three Euler angles].

In Newtonian mechanics, the equation of motion of a particle may be written $\mathbf{F}_k + \mathbf{G}_k = m_k \mathbf{a}_k$ where $\mathbf{G}_k$ is the total force of the constraints acting on particle k and $\mathbf{F}_k$ is the total of all *other* forces acting on this particle. Parameter $m_k$ is the particle's mass and $\mathbf{a}_k$ is its vector acceleration.

A simple example of a holonomic constraint for one particle (particle 1) has that particle sliding down a frictionless curvy water-slide type surface (but no water) in the presence of gravity. In that case, $\mathbf{F}_1$ is the force of gravity acting on the particle, while $\mathbf{G}_1$ is the force of the slide surface acting on the particle. This force is normal to the instantaneous path of the particle so then $\mathbf{G}_1 \bullet d\mathbf{r}_1 = 0$, which can also be written in terms of the particle's velocity $\mathbf{G}_1 \bullet \mathbf{v}_1 = 0$ .

Now differentiate the C constraint equations $f_i(\mathbf{r}_1, \mathbf{r}_2,,,,\mathbf{r}_M, t) = 0$ to get

$$df_i = 0 = \Sigma_{k=1}^{M} \nabla^{(k)} f_i \bullet d\mathbf{r}_k + (\partial_t f_i)\, dt \qquad i = 1,2...C \qquad (D.1)$$

where $\nabla^{(k)} \equiv ( \partial/\partial[(\mathbf{r}_k)_1], \partial/\partial[(\mathbf{r}_k)_2], \partial/\partial[(\mathbf{r}_k)_3])$ is the 3D gradient with respect to the vector coordinate $\mathbf{r}_k$. Let $\{d\mathbf{r}_k\}$ for k = 1,,M be a set of differential displacements which satisfy equations (D.1), and call this a set of **allowed displacements**. Let $\{d\mathbf{r'}_k\}$ be some other set of allowed displacements so that

$$df_i = 0 = \Sigma_{k=1}^{M} \nabla^{(k)} f_i \bullet d\mathbf{r'}_k + (\partial_t f_i)\, dt \qquad i = 1,2...C \ . \qquad (D.2)$$

Now define $\delta\mathbf{r}_k \equiv d\mathbf{r}_k - d\mathbf{r'}_k$ to obtain a set $\{\delta\mathbf{r}_k\}$ of so-called **virtual displacements**. Such a set is the difference of any two different sets of allowed displacements. Then subtracting the above two equations gives

$$0 = \Sigma_{k=1}^{M} \nabla^{(k)} f_i \bullet \delta\mathbf{r}_k \qquad i = 1,2...C \ . \qquad (D.3)$$

This equation is true only as a sum, the individual terms in general do not vanish. Now multiply each of the C sums in (D.3) by a separate (and arbitrary) Lagrange Multiplier $\lambda_i$ and add the sums together to get

$$0 = \Sigma_{i=1}^{C} \lambda_i \left[\Sigma_{k=1}^{M} \nabla^{(k)} f_i \bullet \delta\mathbf{r}_k\right]$$

or

$$0 = \Sigma_{k=1}^{M} \; [\Sigma_{i=1}^{C} \lambda_i \; \nabla^{(k)} f_i] \bullet \delta \mathbf{r}_k \; . \qquad \text{(D.4)}$$

This method of defining the meaning of a virtual displacement set follows the prescription (5) of R&S.

Assumption that constraints do no virtual work

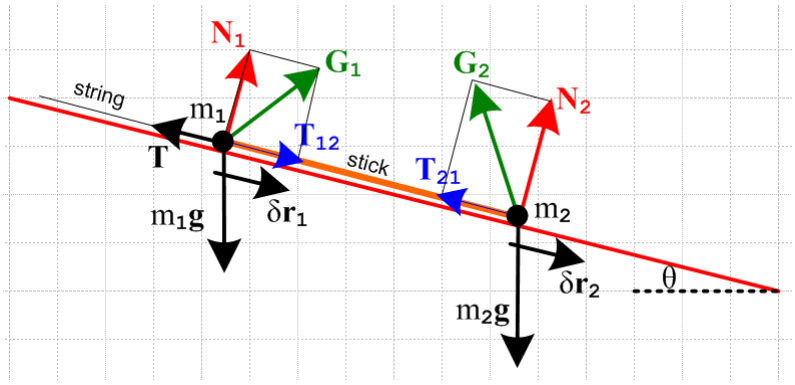
Next, *assume* that

$$0 = \Sigma_{k=1}^{M} \mathbf{G}_k \bullet \delta \mathbf{r}_k \; . \qquad \text{(D.5)}$$

In Goldstein this equation corresponds to setting the second term in (1-40) to zero. The main justification for making this assumption is that it is valid for particles which form a rigid body and for many other constraint situations. There seems to be no general justification of this equation for all constraint situations, and it is certainly not valid if friction is present. The equation can be interpreted as saying that the "total virtual work" done by the constraints on all the particles of the system vanishes.

As a first simple example, consider two particles 1 and 2 independently sliding down a frictionless static ramp. In this case for particle 1 the constraint force $\mathbf{G}_1$ is the normal force of the ramp acting on the particle, and this is clearly perpendicular to $d\mathbf{r}_1$ for any allowed displacement set $\{d\mathbf{r}_k\}$, and thus $\mathbf{G}_1$ is also perpendicular to any $\delta\mathbf{r}_1$ which is part of any virtual displacement set $\{\delta\mathbf{r}_k\}$. In this case we have $\mathbf{G}_1 \bullet \delta\mathbf{r}_1 = 0$ and $\mathbf{G}_2 \bullet \delta\mathbf{r}_2 = 0$ so the terms in (D.5) are separately zero.

A second example is more interesting. Particles 1 and 2 of arbitrary masses slide down a one dimensional ramp but are glued to the two ends of a massless stick so they comprise a rigid body. The upper mass is attached to a string which provides a constant tension T. Here is a picture,




(D.6)

Particle 2 exerts a constraint force $\mathbf{T}_{12}$ on particle 1 through the stick, and Particle 1 exerts a constraint force $\mathbf{T}_{21}$ on particle 2. Since there is only one force of tension (or compression) in such a stick, $T_{12} = T_{21}$. All allowed displacement sets $\{d\mathbf{r}_1, d\mathbf{r}_2\}$ are of the form $\{d\mathbf{r}, d\mathbf{r}\}$, and therefore all *virtual* displacement sets are of the form $\{\delta\mathbf{r}_1, \delta\mathbf{r}_2\} = \{\delta\mathbf{r}, \delta\mathbf{r}\}$. In (D.6) we have drawn the two normal forces $\mathbf{N}_i$ and the two total constraint forces $\mathbf{G}_i$. It is no longer true that $\mathbf{G}_1 \bullet \delta\mathbf{r}_1 = 0$ and $\mathbf{G}_2 \bullet \delta\mathbf{r}_2 = 0$ as the picture shows. However, it is true that $\Sigma_{k=1}^{2} \mathbf{G}_k \bullet \delta\mathbf{r}_k = [\Sigma_{k=1}^{2} \mathbf{G}_k] \bullet \delta\mathbf{r} = 0$ because $\mathbf{G}_1$ and $\mathbf{G}_2$ have equal and opposite components along the ramp, so equation (D.5) is valid for this example.

Now suppose the ramp is translating in some direction not along the ramp. The displacements $d\mathbf{r}_1 = d\mathbf{r}_2$ are then no longer along the ramp so that $\Sigma_{k=1}^{2} \mathbf{G}_k \bullet d\mathbf{r}_k \neq 0$. However, any virtual displacement set

would still have $\delta\mathbf{r}_1 = \delta\mathbf{r}_2$ along the ramp and then $\Sigma_{k=1}^{2}\ \mathbf{G}_k \bullet \delta\mathbf{r}_k = 0$. Remember that a virtual displacement set is the difference of any two allowed displacement sets, and in this subtraction the effect of the ramp translation cancels out. So equation (D.5) is still valid.

Other examples are given in R&S Section III.

Reader Exercise: show in the last example that $T_{12} = T\ m_2 / (m_1+m_2)$ and that this makes sense if $m_1 >> m_2$ and vice versa.

We proceed then assuming our system of interest respects equation (D.5).

Resume Lagrange Mulitpliers application

Now subtract equation (D.4) from (D.5) to get,

$$0 = \Sigma_{k=1}^{M} [\ \mathbf{G}_k - \Sigma_{i=1}^{C} \lambda_i\ \nabla^{(k)} f_i] \bullet \delta\mathbf{r}_k\ . \qquad \text{(D.7)}$$

We know that the M vectors in the set $\{\delta\mathbf{r}_k\}$ are not linearly independent because they must satisfy the equations (D.3), so we cannot claim from (D.7) that $[\ \mathbf{G}_k - \Sigma_{i=1}^{C} \lambda_i\ \nabla^{(k)} f_i] = 0$ for each k. But now write out equation (D.7) in full detail, showing all vector components,

$$0 = \Sigma_{k=1}^{M}\ \Sigma_{j=1}^{3} [\ (\mathbf{G}_k)_j - \Sigma_{i=1}^{C} \lambda_i\ \frac{\partial f_i}{\partial(\mathbf{r}_k)_j}\ ]\ (\delta\mathbf{r}_k)_j\ . \qquad \text{(D.8)}$$

Next, relabel the 3M terms in this sum using a single index n defined in this way

$$\begin{array}{llllllllll} (k,j) & = (1,1), & (1,2), & (1,3), & (2,1), & (2,2), & (2,3), & \ldots\ldots\ldots (M,1), & (M,2), & (M,3) \\ n & = 1, & 2, & 3, & 4, & 5, & 6\ , & 3M-2, & 3M-1, & 3M \end{array} \qquad \text{(D.9)}$$

so that n = 3(k-1)+j. Then define

$$G_n \equiv (\mathbf{G}_k)_j \qquad x_n \equiv (\mathbf{r}_k)_j\ . \qquad // \ n = 3(k-1)+j \qquad \text{(D.10)}$$

Equation (D.8) can then be expressed as

$$0 = \Sigma_{n=1}^{3M} [\ G_n - \Sigma_{i=1}^{C} \lambda_i \frac{\partial f_i}{\partial x_n}\ ]\ \delta x_n \qquad \text{(D.11)}$$

and (D.3) becomes

$$0 = \Sigma_{n=1}^{3M} \frac{\partial f_i}{\partial x_n} \delta x_n \qquad i = 1,2...C\ . \qquad \text{(D.12)}$$

Once again, we cannot set the square brackets in (D.11) to 0 because the 3M coordinates $\delta x_n$ are not linearly independent as shown by (D.12).

At any given time t, there must exist at least one subset of the 3M-C variables $\delta x_n$ which are independent and the remaining $\delta x_n$ are dependent. Assume that the dependent variables have indices $n = n_1, n_2,...n_C$. Consider then the following set of C equations involving the dependent-term square brackets in (D.11),

$$[\, G_n - \Sigma_{i=1}^{C} \left(\frac{\partial f_i}{\partial x_n}\right) \lambda_i \,] = 0 \qquad n = n_1, n_2,...n_C \quad . \tag{D.13}$$

This represents C linear equations in the C parameters $\lambda_i$ so when all is said and done, there must exist a set of values $\{\lambda_i\}$ which make these C equations be true. At this point, however, we don't know the $G_n$ and we don't know the $x_i$ which appear in $\dfrac{\partial f_i(x_1, x_2....x_{3M},t)}{\partial x_n}$ so we cannot yet compute values for the $\lambda_i$. When the problem is fully solved, the $G_n(t)$, $x_n(t)$ and $\lambda_i(t)$ will be known, where here we momentarily exhibit the fact that all these objects can be functions of time.

Now the terms in (D.11) indicated by $n = n_1, n_2,...n_C$ all vanish due to (D.12), so (D.11) can be written,

$$0 = \Sigma_{n \neq n_1, n_2, \ldots . n_C} [\, G_n - \Sigma_{i=1}^{C} \lambda_i \frac{\partial f_i}{\partial x_n} \,] \, \delta x_n \, . \tag{D.14}$$

But all the $\delta x_n$ appearing in (D.14) are independent, so the square brackets here must also vanish. We then end up

$$[\, G_n - \Sigma_{i=1}^{C} \lambda_i \frac{\partial f_i}{\partial x_n} \,] = 0 \qquad n = 1,2....3M$$

or

$$G_n = \Sigma_{i=1}^{C} \lambda_i \frac{\partial f_i}{\partial x_n} \qquad n = 1,2....3M$$

or

$$(\mathbf{G}_k)_j = \Sigma_{i=1}^{C} \lambda_i \frac{\partial f_i}{\partial (\mathbf{r}_k)_j}$$

or

$$\mathbf{G}_k = \Sigma_{i=1}^{C} \lambda_i (\nabla^{(k)} f_i) \qquad k = 1,2...M \quad . \tag{D.15}$$

Meanwhile, Newton's Law says that $m_k \mathbf{a}_k = \mathbf{F}_k + \mathbf{G}_k$ where the $\mathbf{F}_k$ are the "applied" forces for the problem. Inserting (D.15) gives

$$m_k \ddot{\mathbf{r}}_k = \mathbf{F}_k + \Sigma_{i=1}^{C} \lambda_i [\nabla^{(k)} f_i(\mathbf{r}_1, \,] \tag{D.16}$$

which is 3M differential equations in 3M+C unknowns $\mathbf{r}_k$ and $\lambda_i$. But we also have the C constraint equations $f_i(\mathbf{r}_1, \mathbf{r}_2, \ ...\mathbf{r}_M, t) = 0$, so in total there are 3M+C equations in 3M+C unknowns, which then defines a solvable problem. Once the problem is solved, the $\lambda_i$ are known and the constraint forces are given by (D.15).

To summarize, we have used an application of Lagrange Multipliers as an aid to solving a mechanics problem involving M particles with C holonomic constraints. As already noted, the above presentation is based on Section IV of R&S. That paper comments (at great length) on the often vague term "virtual displacement" as it appears in Goldstein and other standard mechanics texts, and we have followed their suggestion for defining this term as a difference of "allowed displacements" which then removes the dt time term in (D.1).

In contrast with our main topic of finding stationary points df = 0 of a function f(**r**), in the Lagrange Multiplier application just presented there is no explicit function f(**r**) we are making stationary. On the other hand, one can regard (D.5) that $\Sigma_k \mathbf{G_k} \bullet \delta\mathbf{r_k} = 0$ as the statement $\delta W = 0$ where $\delta W = \Sigma_k \mathbf{G_k} \bullet \delta\mathbf{r_k}$ is the differential "virtual work" done by the constraint forces during a differential virtual displacement. This equation can then be written $0 = \delta W = \Sigma_k [m_k \mathbf{a_k} - \mathbf{F_k}] \bullet \delta\mathbf{r_k}$ in which form it is known as **D'Alembert's Principle** (1743, Goldstein (1-42)). The analog of f(**r**) is then presumably some action-like integral W of the virtual work δW.