Discussion of a Set of Equations PhL 10.1.16

In my Lagrange App D.2 I have this set of equations,

$$m_k \ddot{\mathbf{r}}_k = \mathbf{F}_k(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_N, t) + \Sigma_{i=1}^{C} \lambda_i \mathbf{B}_{ik}(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_N, t) \qquad k = 1,2...N \quad \text{Newton}$$

$$\Sigma_{k=1}^{N} \mathbf{A}_{ik}(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_N, t) \bullet \dot{\mathbf{r}}_k + A_{it}(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_N, t) = 0 . \qquad i = 1,2..m \quad \text{non-holonomic}$$

$$f_i(\mathbf{r}_1, \mathbf{r}_2, ...\mathbf{r}_N, t) = 0 . \qquad i = m+1,m+2...C \quad \text{holonomic} \qquad (D.25)$$

and I am pondering how to classify this thing. One approach is to define a large vector of functions of time,

$$\mathbf{R} \equiv (\mathbf{r}_1, \mathbf{r}_2, ...\mathbf{r}_N)$$

Then write the above as [changed notation on A and B]

$$m_k \ddot{\mathbf{r}}_k = \mathbf{F}_k(\mathbf{R}, t) + \Sigma_{i=1}^{C} \lambda_i \mathbf{B}^{(i)}{}_k(\mathbf{R}, t) \qquad k = 1,2...N \quad \text{Newton}$$

$$\Sigma_{k=1}^{N} \mathbf{A}^{(i)}{}_k(\mathbf{R}, t) \bullet \dot{\mathbf{r}}_k + A^{(i)}{}_t(\mathbf{R}, t) = 0 . \qquad i = 1,2..m \quad \text{non-holonomic}$$

$$f_i(\mathbf{R},t) = 0 . \qquad i = m+1,m+2...C \quad \text{holonomic}$$

Define M to be a diagonal matrix of the N masses and $\ddot{\mathbf{R}}(t)$ in the obvious manner. Define

$$\mathbf{B}^{(i)} \equiv (\mathbf{B}^{(i)}{}_1, \mathbf{B}^{(i)}{}_2 ..... \mathbf{B}^{(i)}{}_N) \qquad = \text{a large vector like } \mathbf{R}$$

$$\mathbf{F} = (\mathbf{F}_1, \mathbf{F}_2 ..... \mathbf{F}_N) \qquad = \text{a large vector like } \mathbf{R}$$

$$\mathbf{A}^{(i)} = (\mathbf{A}^{(i)}{}_1, \mathbf{A}^{(i)}{}_2 ..... \mathbf{A}^{(i)}{}_N) \qquad = \text{a large vector like } \mathbf{R}$$

Then can write the above equations as

$$M \ddot{\mathbf{R}}(t) = \mathbf{F}(\mathbf{R}, t) + \Sigma_{i=1}^{C} \lambda_i \mathbf{B}^{(i)}(\mathbf{R}, t) \qquad k = 1,2...N \quad \text{Newton} \qquad (1)$$

$$\mathbf{A}^{(i)} \bullet \dot{\mathbf{R}}(t) + A^{(i)}{}_t(\mathbf{R}, t) = 0 \qquad i = 1,2....m \quad \text{non-holo} \qquad (2)$$

$$f_i(\mathbf{R},t) = 0 . \qquad i = m+1,m+2...C \qquad (3)$$

According

to https://en.wikipedia.org/wiki/Ordinary_differential_equation#General_definition_of_an_ODE

(1) would be a "system" of N 2nd order ODEs.

(2) would be a system of m 1st order ODE's

(3) would be a system of s 0th order ODE's.

These are all ODE and not PDE because there is only one variable for derivatives!

Now let's try this again where I incorporate λ_i into the set of functions. Start over with $\lambda_i(t)$

$$M\ddot{\mathbf{R}}(t) = \mathbf{F}(\mathbf{R}, t) + \Sigma_{i=1}^{C} \lambda_i(t)\mathbf{B}^{(i)}(\mathbf{R}, t) \quad k = 1,2...N \quad \text{Newton} \qquad (1)$$

$$\mathbf{A}^{(i)} \bullet \dot{\mathbf{R}}(t) + A^{(i)}{}_t(\mathbf{R}, t) = 0 \qquad i = 1,2....m \qquad \text{non-holo} \qquad (2)$$

$$f_i(\mathbf{R},t) = 0 \ . \qquad i = m+1,m+2...C \qquad (3)$$

Make a new even larger vector of functions

$$\mathbf{R}'(t) \equiv (\mathbf{r}_1(t), \mathbf{r}_2(t), ...\mathbf{r}_N(t), \lambda_1(t),\lambda_2(t) .....\lambda_C(t)) \quad = \text{set of N+C unknown functions}$$

Now the $M\ddot{\mathbf{R}}(t)$ term is generalized to $M'\ddot{\mathbf{R}}'(t)$ where M' has extra zeros on the bottom part of the diagonal. The FORM of the three equations is then

$$M'\ddot{\mathbf{R}}'(t) - \mathbf{Q}'(\mathbf{R}',t) \bullet \mathbf{R}'(t) - \mathbf{F}'(\mathbf{R}', t) = 0 \qquad \text{// N+C unknown functions in } \mathbf{R}'(t) \text{ , only N equations}$$

$$\mathbf{T}'_i(\mathbf{R}',t) \bullet \dot{\mathbf{R}}'(t) + g_i(\mathbf{R}',t) = 0 \qquad \text{m of these}$$

$$f_i(\mathbf{R},t) = 0 \qquad \text{C-m of these}$$

Taken all together, you could say this is "a system of N+C 2nd order ODE's in N+C unknown functions".

Is this a linear system? What is the definition of a linear system of ODE's? All the unknown functions have to appear linearly, whether differentiated or not! But this is clearly not the general intent of my equations, so they are not linear equations! So we have a system of **non-linear** 2nd order ODE's.

The fact some of the equations are first order and some zeroth order does not make this claim untrue.

So I think I have at least classified the system of equations! But wiki says not much about whether there is always a solution to such a system.

Other sources? After much fiddling, I have found the "Chapter 20" notes of Olver which addresses my questions.

1. My equations above contain arbitrary functions of the functions $r_i(t)$ and are therefore not linear equations. To be linear, every term involving $r_i(t)$ would have to just be proportional to $r_i(t)$, but my general form allows for example for $[r_i(t)]^2$ and in general much worse. It happens that the equations are linear in the $\lambda_i(t)$ functions.

2. My equations are non-linear.

3. My equation set has 3N + C scalar functions f(t) that we want to solve for.

4. The set has 3N + C scalar equations. Of these, the first 3N are second order.
The remaining equations can be regarded as being "first order".

5. I think one could convert the first 3N equation set of 2nd order to a set of 1st order by adding 3N new function names, one for each coordinate. This would add 3N new equations as well. Then the first set of equations would be 6N equations in 6N+C unknowns.

6. Overall then you would have 6N + C non-linear first-order ODE's in 6N + C unknown functions.

7. If everything is continuous and differentiable, Olver says that the equations have a unique solution based on initial conditions for all the variables.

How would you then interpret the $\lambda_i(t=0)$ values? They would be related to the t = 0 generalized and probably also regular forces of all the constraints.

Specifically, here is how you would divide the first set of equations:

$$m_k \ddot{\mathbf{r}}_k = \mathbf{F}_k(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_N, t) + \Sigma_{i=1}^{C} \lambda_i \mathbf{B}_{ik}(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_N, t) \qquad k = 1,2...N \quad \text{Newton}$$

It would be this

$$m_k \dot{\mathbf{v}}_k = \mathbf{F}_k(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_N, t) + \Sigma_{i=1}^{C} \lambda_i \mathbf{B}_{ik}(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_N, t) \qquad \text{3N equations in 6N+C functions}$$

$$\dot{\mathbf{r}}_k = \mathbf{v}_k \qquad \text{3N equations in 6N functions}$$

$$\Sigma_{k=1}^{N} \mathbf{A}_{ik}(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_N, t) \bullet \dot{\mathbf{r}}_k + A_{it}(\mathbf{r}_1, \mathbf{r}_2 ..... \mathbf{r}_N, t) = 0 . \qquad \text{m equations in 3N functions}$$

$$f_i(\mathbf{r}_1, \mathbf{r}_2, \ ...\mathbf{r}_N, t) = 0 . \qquad \text{s equations in 3N functions}$$

The total equation count is 3N+3N + C = 6N+C and this matches the number of unknown functions.

Therefore, the anzats that $\lambda = \lambda(t)$ is justified since it results in a unique solution!!