Lebesgue Integration Comments (Luh-BAYG) (STEEL-ches) PhL 1.28.05

One who reads in this area is expected to know certain facts which someone like me does not know.

Fact: A countable set has zero Lebesgue measure.

The *Lebesgue* measure in E^1 is the length of an interval. An isolated point has zero measure, and even if you have a countable number of isolated points, their total measure is still 0.

There are other measures we can define, but we are interested in the Lebesgue measure.

Example: The measure of all the rationals in the interval [0,5] is zero. They are **a set of zero measure**. Thus we can write $m(Q \cap [0,5]) = 0$, where m(X) gives the Lebesgue measure of set X. The letter Q is typically used to represent the rationals, as in rationals are Quotients of integers. Since the rationals make no contribution to the measure, it all comes from the reals that are not rationals. And $m([0,5]) = 5$.

In Lebesgue integration, we get to ignore sets of zero measure. Thus, the Lebesque integral of the rationals in [0,5] is 0. The Riemann integral is undefined. If you consider the outer and inner integrals somehow, they are not the same because the inner one is 0 and the outer one is 5, so to speak.

Lebesgue integration extends Riemann integration inasmuch as you are allowed to ignore any countable set of zero measure in the domain of the integration.

In Riemann integration, there is always fuzziness about interchanging the order of integration and limit, but this becomes well-defined with Lebesgue integration.

Theorem: If you have a (bounded) sequence of (measurable) functions $f_n(x) \rightarrow f(x)$, then you are allowed to interchange the limit with the integral when you do a Lebesgue integration.

The Lebesgue Measure is a special case of **Lebesgue-Stieltjes measure**. With L-S, instead of using a measure function x (which is where we get our naive intervals as measures), you can use any monotonic function F(x) where $F(x) = \mu((-\infty, x])$ and μ is some "Borel measure". As x increases to the right, you can see that F(x) increases monotonically, according to general rules of measures. The term **Borel** is connected with measures on subsets which are "open subsets", meaning any neighborhood ball round a point is in the subset, so endpoints are not allowed.

This thing F(x) is often called $\alpha(x)$ and you see the corresponding **Lebesgue-Stieltjes integral** written in the following way (on the left below) : [where we write $d\mu$ as $dF = d\alpha$]

$$\int_a^b f\, d\alpha = \int_a^b f(x)d\alpha(x) = \int_a^b f(x)\alpha'(x)dx$$

The form on the right is not always valid, but when it is, you end up with a Riemann integral. Note that the L-S integral is sometimes just called a **Stieltjes integral.** You can do the form on the right if $\alpha(x)$ is continuous I think (page 153 Erdelyi). If $\alpha(x)$ has steps of height w_i then you can think of $\alpha'(x)$as being a

sum of weighed delta functions, and you have a sum instead of an integral. The Stieltjes integral thus accounts for both the continuous and discrete "spectrum" parts of an integral.

There is also the **Riemann-Stieltjes integral** which is the same idea as above with $\alpha(x)$, but you follow the usual path of developing a Riemann integral based on intervals, none of the set stuff.

The **Jordan Measure** is the Riemann integration idea where you take a limit and the inner and outer measures converge to the area under your curve.

Comparison of the two methods.

In Riemann integration, we divide the domain up into equally spaced intervals and we do two "coverings" loosely speaking and we take the limit and we find that they approach each other, and this limit is the same from above or below, and this limit is the Riemann integral. In the limit, the intervals becomes very small.

In Lebesgue integration, we instead create "bins" in the range of a function f(x) -- bins on the y axis. For example, one bin might be where $4 < f(x) < 5$. We then define a set E_4 which includes all domain x values such that f(x) is in this bin. We might as well take equally spaced y-axis bins. The set E_4 might be a simple interval on the x-axis (this would be the case if f(x) were monotonic and continuous), or E_4 might be the union of several simple intervals (the case if f(x) goes up and down several times passing through our bin), or E_4 might be a very nasty region of the domain such as the rationals in the range [0,5]. The main idea is that we don't care if our bin domain E_4 is contiguous in R, it can be any set you want I think.

Now, what do we do with these bins? Suppose there are 3 y-axis bins. Then we bound our function from above by creating S(x), and from below creating s(x), both as follows:

$$S(x) = 5\ \mu(E_4) + 4\ \mu(E_3) + 3\ \mu(E_2) + 2\ \mu(E_1) + 1\ \mu(E_0)$$
$$s(x) = 4\ \mu(E_4) + 3\ \mu(E_3) + 2\ \mu(E_2) + 1\ \mu(E_1) + 0\ \mu(E_0)$$

This is very similar to the Riemann idea, but instead of a set of adjacent intervals on the x-axis, we have a set of measurable sets on the X axis, and we are using the Lebesgue measure for each set. The functions shown above are called "simple functions". Notice that the set $\{E_i\}$ partitions the domain.

As we increase the number of y-axis bins, the number of sets in the partition increases, so more terms in the above sums, and they start to approach each other. They sandwich the Lebesgue integral in between, just as in the Riemann case.

Theorem (Monotone Convergence). If $f_n(x) \to f(x)$ from above (in a nested manner, meaning $f_4(x) \leq f_5(x)$, etc), and if all these f(x)'s are positive-valued, then again you can change order of limit and integral. No bounded requirement here. This is a famous theorem.

Fact: When the Riemann integral exists, it always equals the Lebesgue integral.

The terms **"nearly"** and **"almost everywhere"** refer to the idea that you are allowed to knock out any countable set of domain points which have zero measure. Thus, you can Lebesgue-integrate a function which is "nearly continuous".

Not all functions are Lebesgue-integrable, only those that are "measurable", but that is all we really need.

It is easy to decompose a non-positive function as follows:

$f^+(x) = \max(f(x), 0)$ $\quad\quad$ $f^-(x) = \max(-f(x), 0)$

Then you can write $f(x) = f^+(x) - f^-(x)$ and apply Lebesgue integration separately to the two positive pieces.

Comment: You can't just willy-nilly take any old subsets of a space X when you talk about measure. You have to take a set of subsets to work with which have certain properties, and this is where the σ-algebra stuff comes in. You have to have the countable union property, for example.