Notes on Lebesgue Integration Papers PhL 1.28.05

The **Riemann integral** (1862) we are used to is based on something called the **Jordan Measure**. You can define the R-integral of any function that is **continuous**. You so this with a thin rectangle picture in the usual way.

Here is a famous counterexample that shows up a problem. Let X = reals and $R \subset X$ = rationals:

for $x \in X$:	$f(x) = 1$	if $x \in R$
	$f(x) = 0$	if $x \notin R$

This function f(x) may be visualized as a set of spikes as you go along the real axis. Each time you hit a rational, you spike up to 1. Obviously, such an f(x) is not very continuous or differentiable. In fact, it would seem to lack these two properties everywhere, and integrating such a function is just not defined in the world of Riemann which requires the function being integrated to be at least continuous.

In general, if you have some set R in some other set X, $R \subset X$, the above function which "picks out" the elements of R is called the "**indicator function**" or the "**characteristic function**". The particular example above with sets R and X as noted is called the **Dirichlet function**.

The Lebesgue measure and integral (1902) allows you do define an integral of a function like the Dirichlet function. So it enlarges the class of functions that are integrable in a well defined way.

In recent times, new measures and integrals have been defined, such as the **Henstock** integral also known as the K-H integral (**Kurzweil and Henstock** in the 1950's). There is some question as to whether the Lebesgue theory is made obsolete by the KH method.

When you read about things like Lebesgue integration, you must start with the notion of **measure** and its application to sets and subsets. The "measure" is in some sense the "volume" of a set.

As you read, you run into more and more notational conventions that have to be relearned or learned for the first time.

Definition: A **power set** of a set is the set of all subsets of that set. If the order of the starting set is N, then the order of the power set is 2^N because, for each element of the starting set, you "either take it or you don't" as you construct a subset. Example: set = { 1,2,3} power set = {0}, {1}, {2}, {3), {12}, {13},{23}, {123} . This set has N =3 and the power set has $2^N = 8$. For a set S, the power set is usually indicated by the notation 2^S.

Definition: A **sigma algebra** of a set S is a set of subsets F which has certain properties listed below. You could denote such an algebra as **(S, F)**, and this is thing (S,F) is called a **measurable space** and the sets in F are called the **measurable sets**. The power set $F = 2^S$ discussed above satisfies the properties and always forms a sigma algebra. Here are the properties:

1. F must contain the null set
2. If set A is in F, then complement(A) must also be in F. (complement(A) is written S \ A)
3. The union of any sets A_i taken from F must be in F, even if you take a countable number of A_i.

Remember that A is a set, and $A \in F$ makes perfect sense, since F is a set of sets.

Example of rule 2: Suppose S = {1,2,3,4}. Then if {1} is in F, {2,3,4} must be in F.

Example of rule 3: If {1} and {2,3} are in F, then {1,2,3} must also be in F.

Any F that you find will be a subset of the power set 2^S, because this set has ALL subsets.

Definition: The **Borel sigma algebra** is (X,T) where X is a set comprising a "topological space" (something with a metric) and T is **the set of all open sets** defined on X. This Borel algebra is B(X). Recall that an open set is one where you can put a little neighborhood ball around any point in the set, and the entire neighborhood will be in the set. so an open set excludes endpoints and boundaries.

Comments: We want to define some kind of measure μ on our space X. To do this, we really need to define the measure on all our selected subsets of X.

Example: Consider the measurable space (X,A) where A is a σ-algebra set of sets. For each A, if the set is finite you could define μ = number of elements in A, known as |A| , and we allow ∞ for an infinite sized subset A. This is called **the counting measure** , one of many possible μ's (page 5 of Cheng notes).

The **Lebesgue measure** for E^n is defined by Cheng on his page 6 as just the usual geometric volume.

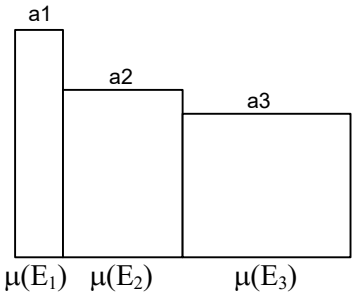
He then provides lots of theorems relating to the measures of sets. If E is a set, then $\mu(E)$ refers to the measure of that set. One example of a theorem: for finite measures, $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$, which seems very reasonable to me. The last term corrects for over-counting the measure.

Definition of a **Measurable Function**: Imagine $f: X \rightarrow Y$. Any region B in Y has a "pre-image" in X which you can write as $f^{-1}(B)$. Now assume that B is some measurable set in Y (which implies the existence in Y of some σ-algebra, that is, a subset structure for Y). Then if the pre-image $f^{-1}(B)$ in X of *every* such measurable B in Y is also measurable in X (which implies some σ-algebra in X!) then the function f is measurable.

So this means that the function f does not "destroy" measure. It seems intuitive that a continuous function would be a measurable function as one example. (Corollary on page 9 Cheng).

The Lebesgue Integral. As I drew on page 11, the idea seems to be this: break up your integration domain into a bunch of sets E_i each of which has a well-defined measure $\mu(E_i)$. Secondly, imagine a function of the form $f(x) = \Sigma_i \, a_i \, \chi_{Ei}(x)$ where the sum is over a finite number of these domain sets E_i, and where this function has a constant value a_i over the domain set E_i. Remember that χ_{Ei} is the characteristic function which is 1 for every element x in E_i and zero otherwise. A function of this form is called a **simple function**. Then the Lebesgue integral is $\mu(\Sigma_i \, a_i \, \chi_{Ei}(x)) = \Sigma_i \, a_i \, \mu \, [\, \chi_{Ei}(x) \,] = \Sigma_i \, a_i \, \mu(E_i)$. What we mean by $\mu(E_i)$ is some notion of the volume of the set E_i.

Comments: I have to visualize this in the following manner



which suggests that we have a generalization of the Riemann integration idea because the domain (x-axis) no longer has a uniform measure. We replace equally spaced intervals with measurable sets.

Definition: If you have a measurable space (X,A) and you define a measure μ(D) for any element of A, then you have **(X,A,μ)** which is called a **measure space**. It is just (X,A) plus the definition of the measure.

Often you find that the underlying space X is called **Ω** , and elements are $\omega \in \mathbf{\Omega}$. The characteristic function might we written as 1_S for a set S in Ω, and as before it is = 1 for any ω in S, else 0. I think the notation R^+ means reals along with $+\infty$.

Now if you want the Lebesgue integral of some function that is not just a simple function, you try to represent your function f(x) as the limit of a sequence of simple functions, perhaps.

Notation: $x_{min} = \inf(X)$ "infimum" lower bound $x_{max} = \sup(X)$. "supremum" upper bnd

Cantor Set = keep cutting out middle thirds, end up with a set of all boundary points.