

1.1.2 Solving systems by finding eigenvalues and eigenvectors

Using the scalar equation as a guide, we assume that the vector equation (2) has a solution of the form

$$\mathbf{x}(t) = e^{\lambda t} \mathbf{u}, \quad (5)$$

where $\mathbf{u}$ is a constant vector.

Substituting the solution (5) in left and right side of equation (2) yields

$$\frac{d}{dt}(e^{\lambda t} \mathbf{u}) = e^{\lambda t} \lambda \mathbf{u}, \quad (6)$$

$$A(e^{\lambda t} \mathbf{u}) = e^{\lambda t} A\mathbf{u}. \quad (7)$$

Evidently, the vector equation (2) is satisfied only, if

$$A\mathbf{u} = \lambda \mathbf{u}, \quad (8)$$

i.e.

$$(A - \lambda I)\mathbf{u} = 0. \quad (9)$$

Therefore, if λ is an eigenvalue of A and $\mathbf{u}$ is a corresponding eigenvector, then $\mathbf{x}(t) = e^{\lambda t} \mathbf{u}$ is a solution to $\mathbf{x}'(t) = A\mathbf{x}(t)$ ($\mathbf{u} = \mathbf{0}$ gives a trivial solution). Obviously, the solutions

$$\mathbf{x}_1(t) = e^{\lambda_1 t} \mathbf{u}_1, \dots, \mathbf{x}_n(t) = e^{\lambda_n t} \mathbf{u}_n, \quad (10)$$

associated with eigenvalues $\lambda_1, \dots, \lambda_n$ and eigenvectors $\mathbf{u}_1, \dots, \mathbf{u}_n$, are linearly independent. It is easy to verify that any linear combination of $\mathbf{x}_1(t), \dots, \mathbf{x}_n(t)$ is also a solution. The general solution to the vector equation (2) is given by

$$\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) + \dots + c_n \mathbf{x}_n(t). \quad (11)$$

Substituting (10) in (11) yields

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{u}_1 + c_2 e^{\lambda_2 t} \mathbf{u}_2 + \dots + c_n e^{\lambda_n t} \mathbf{u}_n. \quad (12)$$

The initial value problem consists in finding a solution to (2) that satisfies an initial condition

$$\mathbf{x}(0) = \mathbf{x}_0, \quad (\text{or } \mathbf{x}(a) = \mathbf{x}_0 \text{ in general}). \quad (13)$$

It follows from (12), (13) that

$$c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \cdots + c_n \mathbf{u}_n = \mathbf{x}_0. \quad (14)$$

Determining the integration constants from linear algebraic system (14) and inserting in (12) one obtains the solution to initial value problem (2), (13).

General solution algorithm using linear algebra software (MATLAB, MAPLE,...) is following:

1. find the eigenvalues and eigenvectors
2. compose the general solution
3. determine the integration constants
4. compose the solution to the initial value problem.

Note: Steps 3-4 must be filled only for initial value problems.

Example 1: Solve the initial value problem

$$x_1' = 6x_1 + 8x_2, \quad (15)$$

$$x_2' = x_1 + 4x_2,$$

$$x_1(0) = 1, \quad x_2(0) = 0. \quad (16)$$

In matrix notation the system (15) reads

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \quad (17)$$

The characteristic polynomial of the matrix A is

$$f(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} 6 - \lambda & 8 \\ 1 & 4 - \lambda \end{pmatrix} = (\lambda - 2)(\lambda - 8). \quad (18)$$

Hence, the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 8$. Substituting the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 8$ in (9), we can determine corresponding eigenvectors

$$\mathbf{u}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}. \quad (19)$$

Now, the general solution of equation (15) can be presented as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{8t} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -2c_1 e^{2t} + 4c_2 e^{8t} \\ c_1 e^{2t} + c_2 e^{8t} \end{pmatrix}. \quad (20)$$

The initial conditions (16) in matrix notation take the form

$$\mathbf{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (21)$$

It follows from (20),(21) that

$$\begin{aligned} -2c_1 + 4c_2 &= 1, \\ c_1 + c_2 &= 0. \end{aligned} \quad (22)$$

Solving system (22) yields: $c_1 = -\frac{1}{6}$ and $c_2 = \frac{1}{6}$. Inserting $c_1 = -\frac{1}{6}$ and $c_2 = \frac{1}{6}$ into general solution (20) gives the solution to the initial value problem (15), (16) as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}e^{2t} + \frac{2}{3}e^{8t} \\ -\frac{1}{6}e^{2t} + \frac{1}{6}e^{8t} \end{pmatrix}. \quad (23)$$

Example 2: Find the general solution to the linear system of differential equations

$$\begin{aligned} x_1' &= x_1, \\ x_2' &= 3x_2 - 2x_3, \\ x_3' &= -2x_2 + 3x_3. \end{aligned} \quad (24)$$

The matrix of system (24)

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & -2 & 3 \end{pmatrix} \quad (25)$$

has the eigenvalues $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 5$. So, $\lambda = 1$ is an eigenvalue of multiplicity 2. Generally, if λ is an eigenvalue of A of multiplicity k , and the rank of the matrix $(A - \lambda I)$ is equal to $n - k$, then we can find k linearly independent eigenvectors of A associated with the eigenvalue λ . For given A with (25)

$$r = \text{rank}(A) = 1, \quad k = 2, \quad n - k = 1. \quad (26)$$

Therefore, we can find two linearly independent eigenvectors associated with $\lambda = 1$ using relation (9)

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \quad (27)$$

Computing the eigenvector $\mathbf{u}_3$ associated with the eigenvalue $\lambda = 5$

$$\mathbf{u}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad (28)$$

we can determine the general solution to system (24) as

$$\begin{aligned} \mathbf{x}(t) &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{5t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \\ &= \begin{pmatrix} c_1 e^t \\ c_2 e^t + c_3 e^{5t} \\ c_2 e^t - c_3 e^{5t} \end{pmatrix}. \end{aligned} \quad (29)$$

Exercises

1. Solve the initial value problem

$$\begin{aligned} x_1' &= x_1 + 12x_2, \\ x_2' &= 3x_1 + x_2, \\ x_1(0) &= 0, \quad x_2(0) = 1. \end{aligned} \quad (30)$$

2. Find the general solution of the following linear system of differential equations:

$$\begin{aligned} x_1' &= 3x_1 - x_2 - x_3, \\ x_2' &= -12x_1 + 5x_3, \\ x_3' &= 4x_1 - 2x_2 - x_3 \end{aligned} \quad (31)$$

3. Solve two point boundary value problem with differential equations (30) and boundary conditions $x_1(1) = 1$ and $x_2(0) = 0$.

1.1.3 Solution by converting a system to diagonal form

We assume, that $n \times n$ matrix A is diagonalisable over K , i.e. it has n linearly independent eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ belonging to eigenvalues $\lambda_1, \dots, \lambda_n$ over K (here $K = \mathbb{R}$ or $K = \mathbb{C}$)

$$A\mathbf{v}_i = \lambda_i\mathbf{v}_i, \quad i = 1, \dots, n. \quad (32)$$

Assembling the eigenvectors into $n \times n$ matrix S

$$S = (\mathbf{v}_1, \dots, \mathbf{v}_n) \quad (33)$$

we can write

$$AS = (\lambda_1\mathbf{v}_1, \lambda_2\mathbf{v}_2, \dots, \lambda_n\mathbf{v}_n) = (\mathbf{v}_1, \dots, \mathbf{v}_n)D = SA \quad (34)$$

where Λ is a $n \times n$ diagonal matrix with entries $\lambda_1, \dots, \lambda_n$. Multiplying (34) on the right by S^{-1} one obtains the matrix A in diagonally factorised form

$$A = S\Lambda S^{-1}. \quad (35)$$

Substituting (35) in (2) and multiplying obtained result on the left by S^{-1} yields

$$S^{-1}\mathbf{x}' = DS^{-1}\mathbf{x}, \quad (36)$$

or

$$\mathbf{y}' = D\mathbf{y}. \quad (37)$$

Here a vector $\mathbf{y}(t)$ is defined as

$$\mathbf{y} = S^{-1}\mathbf{x}. \quad (38)$$

Evidently, the system of differential equations (37) is a diagonal system, because the matrix D is diagonal. In explicit form the system (37) reads

$$\begin{aligned} y_1' &= \lambda_1 y_1, \\ y_2' &= \lambda_2 y_2, \\ &\dots \dots \dots \\ y_n' &= \lambda_n y_n. \end{aligned} \quad (39)$$

Integrating (39), we obtain

$$y_i(t) = C_i e^{\lambda_i t} \quad (40)$$

or in matrix form

$$\mathbf{y} = E(t)\mathbf{c}, \quad (41)$$

where

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \quad E(t) = \begin{pmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{pmatrix}. \quad (42)$$

Substituting (38) in (41) and multiplying obtained result on the left by S one obtains the general solution of system (2) in terms of original variables x

$$\mathbf{x}(t) = SE(t)\mathbf{c}. \quad (43)$$

The vector of integration constants $\mathbf{c}$ can be determined satisfying the initial conditions $\mathbf{x}(a) = \mathbf{x}_0$ (or boundary conditions (4)).

Satisfaction conditions $\mathbf{x}(a) = \mathbf{x}_0$ gives

$$\mathbf{c} = [E(t = a)]^{-1}S^{-1}\mathbf{x}_0, \quad (44)$$

and

$$\mathbf{x}(t) = SE(t)[E(t = a)]^{-1}S^{-1}\mathbf{x}_0. \quad (45)$$

In most practical case ($a = 0$), the formula (45) reduces to

$$\mathbf{x}(t) = SE(t)S^{-1}\mathbf{x}_0. \quad (46)$$

General solution algorithm using linear algebra software is following:

1. find the eigenvalues and eigenvectors
2. compose matrices S and $E(t)$
3. compute the general solution to (2) by (43) or solution to the initial value problem by (46).

Example 1: Consider the system

$$\begin{aligned} x_1' &= x_1 - x_2, \\ x_2' &= 2x_1 + 4x_2, \end{aligned} \quad (47)$$

find the general solution.

In matrix notation the system (47) becomes

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \quad (48)$$

The characteristic polynomial of matrix A

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \quad (49)$$

is

$$\begin{aligned} f(\lambda) &= \det(A - \lambda I) = \det \begin{pmatrix} 1 - \lambda & -1 \\ 2 & 4 - \lambda \end{pmatrix} = \\ &= (1 - \lambda)(4 - \lambda) + 2 = (\lambda - 2)(\lambda - 3). \end{aligned} \quad (50)$$

Therefore the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$. The eigenvector $\mathbf{v}_1$ associated with $\lambda_1 = 2$ is obtained by solving equation $(A - \lambda_1 I)\mathbf{v}_1 = \mathbf{0}$

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (51)$$

Analogically

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad (52)$$

and therefore

$$S = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}, \quad E(t) = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix}. \quad (53)$$

Finally, we can compute the general solution of system (47) using formula (53)

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{2t} + c_2 e^{3t} \\ -c_1 e^{2t} - 2c_2 e^{3t} \end{pmatrix}. \quad (54)$$

Example 2: Solve the following initial value problem

$$x_1' = x_2, \quad (55)$$

$$x_2' = x_3,$$

$$x_3' = 8x_1 - 14x_2 + 7x_3,$$

$$x_1(0) = 4, \quad x_2(0) = 6, \quad x_3(0) = 8. \quad (56)$$

For the system (55) the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -4 & 7 \end{pmatrix} \quad (57)$$

has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 4$. Associated eigenvectors are

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 4 \\ 16 \end{pmatrix}, \quad (58)$$

respectively.

The general solution of system (55) is given by

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{pmatrix} \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{4t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \\ &= \begin{pmatrix} c_1 e^t + c_2 e^{2t} + c_3 e^{4t} \\ c_1 e^t + 2c_2 e^{2t} + 4c_3 e^{4t} \\ c_1 e^t + 4c_2 e^{2t} + 16c_3 e^{4t} \end{pmatrix}. \end{aligned} \quad (59)$$

From (44) the integration constants are

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ 3 \\ -\frac{1}{3} \end{pmatrix} \quad (60)$$

and the solution of the initial value problem (45)-(46) can be presented as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{4}{3}e^t + 3e^{2t} - \frac{1}{3}e^{4t} \\ \frac{4}{3}e^t + 4e^{2t} - \frac{4}{3}e^{4t} \\ \frac{4}{3}e^t + 12e^{2t} - \frac{16}{3}e^{4t} \end{pmatrix}. \quad (61)$$

Exercises

1. Consider a linear system of differential equations

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{pmatrix} \mathbf{x} \quad (62)$$

a) find the general solution

b) find the solution to the initial value problem determined by the initial conditions $x_1(0) = 2$, $x_2(0) = 7$, $x_3(0) = 20$.

2. Find the general solution to the linear system of differential equations.

$$\begin{aligned} x_1' &= x_1 + 2x_2 + 3x_3; \\ x_2' &= x_2; \\ x_3' &= 2x_1 + x_3 + 2x_2. \end{aligned} \quad (63)$$

1.1.4 Principal matrix solution

Theorem 1: For any constant vector $\mathbf{c}$,

$$\mathbf{x}(t) = e^{At} \mathbf{c} \quad (64)$$

is a solution of system (2). Moreover, the solution of (2) given by

$$\mathbf{x}(t) = e^{At} \mathbf{x}_0 \quad (65)$$

satisfies the initial condition $\mathbf{x}(0) = \mathbf{x}_0$.

Proof: Evidently

$$\mathbf{x}(t) = e^{At} \mathbf{c} = [I + At + A^2 \frac{t^2}{2!} + A^3 \frac{t^3}{3!} + \dots] \mathbf{c}, \quad (66)$$

$$\mathbf{x}'(t) = \frac{d}{dt}(e^{At} \mathbf{c}) = A[I + At + A^2 \frac{t^2}{2!} + A^3 \frac{t^3}{3!} + \dots] \mathbf{c} = Ae^{At} \mathbf{c} = A\mathbf{x}(t), \quad (67)$$

and

$$\mathbf{x}(0) = e^{A \cdot 0} \mathbf{x}_0 = I \mathbf{x}_0 = \mathbf{x}_0. \quad (68)$$

Definition 1: The matrix e^{At} is called the principal matrix solution of the system (2).

General solution algorithm using linear algebra software:

1. compute e^{At}
2. compute the general solution of system (2) by (64) or solution to the initial value problem by (65).

Example 1: Consider the system

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad (69)$$

find the general solution.

1. Computing e^{At}

$$\begin{aligned} e^{At} &= I + At + A^2 \frac{t^2}{2!} + A^3 \frac{t^3}{3!} + \dots = \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} t & 0 & 0 \\ 0 & 2t & 0 \\ 0 & 0 & t \end{pmatrix} + \\ &+ \begin{pmatrix} \frac{t^2}{2!} & 0 & 0 \\ 0 & \frac{2^2 t^2}{2!} & 0 \\ 0 & 0 & \frac{t^2}{2!} \end{pmatrix} + \begin{pmatrix} \frac{t^3}{3!} & 0 & 0 \\ 0 & \frac{2^3 t^3}{3!} & 0 \\ 0 & 0 & \frac{t^3}{3!} \end{pmatrix} + \dots = \\ &= \begin{pmatrix} 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots & 0 & 0 \\ 0 & 1 + 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \dots & 0 \\ 0 & 0 & 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \end{pmatrix} = \\ &= \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^t \end{pmatrix} \end{aligned} \quad (70)$$

2. computing the general solution of system (2) by (64)

$$\mathbf{x}(t) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} c_1 e^t \\ c_2 e^{2t} \\ c_3 e^t \end{pmatrix}. \quad (71)$$

Exercises

1. Solve the initial value problem

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}. \quad (72)$$

1.2 Weighted residual methods for solving ODE and PDE

Below we discuss some approximation methods to solve arbitrary linear differential equations. Both, ordinal (ODE) and partial (PDE) differential equations, are considered. With limited scope we consider as example the following one-dimensional differential equation

$$Lu + g = 0, \quad x \in [a, b], \quad (73)$$

where $u(x)$ is the unknown function and $g(x)$ is a known function. The capital L denotes a linear differential operator, which specifies the actual form of the differential equation (73) (for example $L = \frac{d}{dx}$ or $L = \frac{d^2}{dx^2}$). The boundary conditions are given by

$$u(a) = u_a, \quad u(b) = u_b. \quad (74)$$

Multiplying (73) by an arbitrary weight function $v(x)$ and integrating over the interval $[a, b]$ one obtains

$$\int_a^b v(Lu + g)dx = 0. \quad (75)$$

Evidently (73) and (75) are equivalent, because $v(x)$ is an arbitrary function. Now we seek a numerical solution to (75), (74) in the form

$$u^* = a_1\psi_1 + \dots + a_n\psi_n. \quad (76)$$

Here $\psi_1, \dots, \psi_n$ are functions of x and $a_1, \dots, a_n$ are unknown coefficients.

In vector form (76) becomes

$$u^* = \psi \mathbf{a}, \quad (77)$$

where

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad \psi = (\psi_1, \dots, \psi_n). \quad (78)$$

In (75) we may substitute u by u^* to obtain

$$\int_a^b v(Lu^* + g)dx = 0. \quad (79)$$

However, substituting $u(x)$ by its approximation $u^*(x)$ in (73), generally it appears that (73) is not satisfied exactly, e.g.

$$Lu^* + g = e. \quad (80)$$

Here $e(x)$ is a measure for the error.

It follows from (79)-(80) that

$$\int_a^b v e dx = 0. \quad (81)$$

Obviously, the residual, $e(x)$, depends on the unknown parameters given by vector $\mathbf{a}$. Therefore the coefficients $a_1, \dots, a_n$ must be determined so, that expression (81) is satisfied.

Generally

$$v = c_1 V_1 + \dots + c_n V_n, \quad (82)$$

where $V_1, \dots, V_n$ are known functions of x and $c_1, \dots, c_n$ are certain parameters. In terms of vector notation (82) reads

$$v = \mathbf{V}\mathbf{c}, \quad (83)$$

where

$$\mathbf{V} = (V_1, \dots, V_n), \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}. \quad (84)$$

Evidently

$$v = v^T = \mathbf{c}^T \mathbf{V}^T, \quad (85)$$

and therefore (see (81))

$$c^T \int_a^b V^T e \, dx = 0. \quad (86)$$

Relation (86) holds for arbitrary c^T - matrices, i.e.

$$\int_a^b \mathbf{V} e \, dx = \mathbf{0}. \quad (87)$$

or

$$\int_a^b V_1 e \, dx = 0, \dots, \int_a^b V_n e \, dx = 0. \quad (88)$$

Now, we have n equations (88) to determine coefficients $a_1, \dots, a_n$. Inserting (77) in (80) yields

$$e = L(\psi \mathbf{a}) + g = L(\psi) \cdot \mathbf{a} + g, \quad (89)$$

and the condition (86) can be rewritten as

$$\left[\int_a^b \mathbf{V}^T L(\psi) \, dx \right] \cdot \mathbf{a} = - \int_a^b \mathbf{V}^T g \, dx. \quad (90)$$

Introducing the matrix K and the vector $\mathbf{f}$ as

$$K = \int_a^b \mathbf{V}^T L(\psi) \, dx, \quad \mathbf{f} = - \int_a^b \mathbf{V}^T g \, dx, \quad (91)$$

we can write (90) in compact form

$$K \mathbf{a} = \mathbf{f}. \quad (92)$$

Finally, we have n linear equations (92) for determining n coefficients $a_1, \dots, a_n$.

1.2.1 Point collocation method

In the point collocation method, the weight function v (82) is chosen such that

$$\mathbf{V} = [\delta(x - x_1) \delta(x - x_2) \dots \delta(x - x_n)], \quad (93)$$

where the fixed points $x_i \in [a, b]$, ($i = 1, \dots, n$) are called collocation points. Here Dirac's delta function $\delta(x - x_i)$ is defined as

$$\delta(x - x_i) = \begin{cases} \infty, & \text{if } x = x_i \\ 0 & \text{else.} \end{cases} \quad (94)$$

Inserting (93) in (88) gives

$$\begin{aligned} \int_a^b V_i e(x) \, dx &= \int_a^b \delta(x - x_i) e(x) \, dx = \int_{x_i^-}^{x_i^+} \delta(x - x_i) e(x) \, dx = \\ &= e(x_i) \int_{x_i^-}^{x_i^+} \delta(x - x_i) \, dx = e(x_i) = 0, \quad i = 1, \dots, n, \quad (95) \end{aligned}$$

i.e. the residual $e(x)$ is forced to be zero at the collocation points.

The linear system (92) takes the form

$$K = \begin{pmatrix} L(\psi_1(x_1)) & L(\psi_2(x_1)) & \dots & L(\psi_n(x_1)) \\ \vdots & \vdots & \ddots & \vdots \\ L(\psi_1(x_n)) & L(\psi_2(x_n)) & \dots & L(\psi_n(x_n)) \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} =$$

$$= - \begin{pmatrix} g(x_1) \\ \vdots \\ g(x_n) \end{pmatrix}. \quad (96)$$

{indexsubdomain collocation method}

1.2.2 Subdomain collocation method

The weight function v in (82) is defined such that

$$V_i = \begin{cases} 1, & \text{if } x_i \leq x \leq x_{i+1} \\ 0 & \text{else,} \end{cases} \quad i = 1, \dots, n. \quad (97)$$

This means that the average of the residual over each subdomain is forced to be zero

$$\int_a^b V_i e(x) \, dx - \int_{x_i}^{x_{i+1}} e(x) \, dx = 0, \quad i = 1, \dots, n. \quad (98)$$

Instead of system (96) we have

$$K = \begin{pmatrix} \int_{x_1}^{x_2} L(\psi_1) dx & \int_{x_1}^{x_2} L(\psi_2) dx & \dots & \int_{x_1}^{x_2} L(\psi_n) dx \\ \vdots & \vdots & \ddots & \vdots \\ \int_{x_n}^{x_{n+1}} L(\psi_1) dx & \int_{x_n}^{x_{n+1}} L(\psi_2) dx & \dots & \int_{x_n}^{x_{n+1}} L(\psi_n) dx \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} =$$

$$= - \begin{pmatrix} \int_{x_1}^{x_2} g \, dx \\ \vdots \\ \int_{x_n}^{x_{n+1}} g \, dx \end{pmatrix}. \quad (99)$$

1.2.3 Least square method

In the least square method the functions $V_i[x]$ in (82) are defined as

$$V_i = \frac{\partial e}{\partial q_i}, \quad i = 1, \dots, n, \quad (100)$$

and due to (88) we have

$$\int_a^b \frac{\partial e}{\partial a_i} e \, dx = 0, \quad i = 1, \dots, n. \quad (101)$$

Our interest is the square of the error over interval $[a, b]$

$$J = \int_a^b e^2 dx. \quad (102)$$

Next we compute the derivatives

$$\frac{\partial J}{\partial a_i} = 2 \int_a^b e \frac{\partial e}{\partial a_i} dx, \quad i = 1, \dots, n. \quad (103)$$

It implies from (101), (103) that

$$\frac{\partial J}{\partial a_i} = 0, \quad i = 1, \dots, n. \quad (104)$$

Therefore J is stationary and the square of the error $e(x)$ attains its minimum. In explicit form the linear system (92) reads

$$K = \begin{pmatrix} \int_a^b L(\psi_1)L(\psi_1) dx & \int_a^b L(\psi_1)L(\psi_2) dx & ... & \int_a^b L(\psi_1)L(\psi_n) dx \\ \\ \int_a^b L(\psi_n)L(\psi_1) dx & \int_a^b L(\psi_n)L(\psi_2) dx & ... & \int_a^b L(\psi_n)L(\psi_n) dx \end{pmatrix}. \quad (105)$$

$$\cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = - \begin{pmatrix} \int_a^b L(\psi_1)g dx \\ \vdots \\ \int_a^b L(\psi_n)g dx \end{pmatrix}.$$

Evidently, the matrix K is symmetric.

1.2.4 Galerkin's method

In the Galerkin's method the functions $V_i(x)$ in (82) are defined as

$$V_i = \psi_i, \quad i = 1, \dots, n. \quad (106)$$

Inserting (106) in (88) yields

$$\int_a^b \psi_i e \, dx = 0, \quad i = 1, \dots, n. \quad (107)$$

The linear system (92) becomes

$$\begin{aligned}
K &= \begin{pmatrix} \int_a^b \psi_1 L(\psi_1) dx & \int_a^b \psi_1 L(\psi_2) dx & \dots & \int_a^b \psi_1 L(\psi_n) dx \\ \dots & \dots & \dots & \dots \\ \int_a^b \psi_n L(\psi_1) dx & \int_a^b \psi_n L(\psi_2) dx & \dots & \int_a^b \psi_n L(\psi_n) dx \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \\
&= - \begin{pmatrix} \int_a^b \psi_1 g dx \\ \vdots \\ \int_a^b \psi_n g dx \end{pmatrix}.
\end{aligned}
\tag{108}$$

Example 1: We consider the two point boundary value problem

$$\frac{d^2 u}{dx^2} + u + x = 0, \quad x \in [0, 1], \quad (109)$$

$$u(0) = 0, \quad u(1) = 0. \quad (110)$$

The function $g(x)$ and the differential operator L are determined as

$$g(x) = x, \quad L = \frac{d^2}{dx^2} + 1. \quad (111)$$

The two point boundary value problem (109)-(110) has the exact solution

$$u(x) = \frac{\sin x}{\sin 1} - x. \quad (112)$$

Next we approximate the solution of (109)-(110) with trigonometric series

$$\begin{aligned} u^* &= b_0 + (a_1 \sin(px) + b_1 \cos(px)) + (a_2 \sin(2px) + b_2 \sin(2px)) + \dots + \\ &+ (a_n \sin(npx) + b_n \cos(npx)). \end{aligned} \quad (113)$$

The boundary conditions (110) are satisfied if

$$p = \pi, \quad b_0 = 0, \quad b_1 = 0, \dots, b_n = 0. \quad (114)$$

Taking $n = 2$ in (113) we get approximation u^* with two terms as

$$u^* = a_1 \sin \pi x + a_2 \sin 2\pi x. \quad (115)$$

Comparing (76) and (115) yields

$$\psi = (\psi_1 \ \psi_2) = (\sin \pi x \ \sin 2\pi x). \quad (116)$$

Substituting ψ_1 and ψ_2 in (96) gives 2×2 linear system

$$\begin{pmatrix} (1 - \pi^2) \sin \pi x_1 & (1 - 4\pi^2) \sin 2\pi x_1 \\ (1 - \pi^2) \sin \pi x_2 & (1 - 4\pi^2) \sin 2\pi x_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \quad (117)$$

According to the point collocation method we choose the collocation points x_1, x_2 and solve (117) with respect a_1 and a_2 . For $x_1 = \frac{1}{3}$ and $x_2 = \frac{2}{3}$ one obtains $a_1 = 0.0651$, $a_2 = -0.005$ and

$$u^* = 0.0651 \sin \pi x - 0.005 \sin 2\pi x. \quad (118)$$

Subdomain collocation method leads to the following solution

Taking $x_2 = \frac{1}{2}$ (two subdomains with equal length) we can write the system (99) as

$$\begin{pmatrix} \int_0^{\frac{1}{2}} (1 - \pi^2) \sin(\pi x) \, dx & \int_0^{\frac{1}{2}} (1 - 4\pi^2) \sin(2\pi x) \, dx \\ \int_{\frac{1}{2}}^1 (1 - \pi^2) \sin(\pi x) \, dx & \int_{\frac{1}{2}}^1 (1 - 4\pi^2) \sin(2\pi x) \, dx \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = - \begin{pmatrix} \int_0^{\frac{1}{2}} x \, dx \\ \int_{\frac{1}{2}}^1 x \, dx \end{pmatrix}. \quad (119)$$

The system (119) has solutions $a_1 = 0.0885$, $a_2 = -0.0102$ and therefore

$$u^* = 0.0885 \sin \pi x - 0.0102 \sin 2\pi x. \quad (120)$$

Let us use the least square method now

For the present example the system (105) reduces to

$$\begin{pmatrix} \int_0^1 (1 - \pi^2)^2 \sin^2(\pi x) dx & \int_0^1 (1 - \pi^2)(1 - 4\pi^2) \sin(\pi x) \sin(2\pi x) dx \\ \int_0^1 (1 - \pi^2)(1 - 4\pi^2) \sin(\pi x) \sin(2\pi x) dx & \int_0^1 (1 - 4\pi^2)^2 \sin^2(2\pi x) dx \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = - \begin{pmatrix} \int_0^1 (1 - \pi^2) \sin(\pi x) x dx \\ \int_0^1 (1 - 4\pi^2) \sin(2\pi x) x dx \end{pmatrix}. \quad (121)$$

Solving (121) we get $a_1 = 0.0718$ $a_2 = -0.0083$ and

$$u^* = 0.0718 \sin \pi x - 0.0083 \sin 2\pi x. \quad (122)$$

Galerkin's method gives the following solution

For the considered example the system (108) reduces to

$$\begin{pmatrix} \int_0^1 (1 - \pi^2) \sin^2(\pi x) dx & \int_0^1 \sin(\pi x)(1 - 4\pi^2) \sin(2\pi x) dx \\ \int_0^1 \sin(2\pi x)(1 - \pi^2) \sin(\pi x) dx & \int_0^1 (1 - 4\pi^2) \sin^2(2\pi x) dx \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = - \begin{pmatrix} \int_0^1 \sin(\pi x) x dx \\ \int_0^1 \sin(2\pi x) x dx \end{pmatrix}. \quad (123)$$

Solving system (123) yields $a_1 = 0.0718$, $a_2 = -0.0083$ and therefore

$$u^* = 0.0718 \sin \pi x - 0.0083 \sin 2\pi x. \quad (124)$$

Comparison of solutions:

It is seen in Fig. 1., that the numerical results, obtained by four weighted residual methods are quite close to the exact solution. However, in present example only two terms in trigonometric series are considered.

Exercises

1. Solve the two point boundary value problem (109)-(110) taking $n = 3$ and $n = 4$. Compare results.