Binomial and Poisson PhL 3.5.03

1. The 1D drunk. This is outlined in Reif's brown book on stat mech. The drunk makes N steps, R of them to the right and L of them to the left. What is the probability that he does R steps to the right, assuming that the probability of a step to the right is r and to the left is ℓ = 1-r.

First, here is the answer:

$$P(R) = \binom{N}{R} r^R \ell^{N-R}$$

The binomial coefficient factor is the number of ways of doing R steps to the right out of N, that certainly seems clear. For each step to the right, we get a factor of r. But for each step to the left, we get a probability of ℓ. Each step is a random independent event, so we are supposed to multiply probabilities in this way. So this result seems pretty reasonable.

By the way, if he takes R steps to the right, his distance from the start is R*dx - L*dx = dx(R-L) which is dx*(R - (N-R)) = dx*(2R-N) = dx*D. So you could set D = 2R-N and replace R in the formula to get the probability of ending up distance D steps to the right.

2. Trials and successes. We are supposed to think of the above example in a more general way as N trials in which there were R successes where the probability of a single success is r, see M&M page 438.

3. Radiation from an isotope. At first this problem seems quite unrelated. We have a Geiger counter which measures 3 clicks per second on the average from a random radioactive source. What is the probability of having 5 clicks in a 1 second interval?

If we divide time into msec, then we have 1000 intervals for each second, and the probability of a click in one of these intervals is r = .003. The probability of having 5 clicks is

$$P(5) = \binom{1000}{5} (.003)^5 (.997)^{995}$$

Why is this so? In order to have 5 clicks, we assume (with slight inaccuracy) that this means we have to have a click in each of 5 of our 1 msec intervals. That is, we ignore cases of 2 or more clicks in the same interval (for the moment). The first factor is the ways to pick 5 intervals. For each way, the rest is the probability of having 5 clicks and 995 non-clicks. Each click here is like a step of the drunk, an independent event having no correlation to the last click.

With the drunk, the events might occur at some even rate like 1 step per second, so we could add the element of time to the drunk problem. Then the "events" of the problem (the steps) occur one after the other at a fixed rate. This notion of time is completely irrelevant and has nothing to do with time in the radiation problem. For the drunk, there were N trials, and R is the number of successful steps to the right. For radiation, there are 1000 trials, and 5 is the number of successful clicks. Each trial in the second case is the elapse of 1 msec, and it is a successful trial if a click occurs in that 1 msec. For the drunk, each trial is a step, regardless of how long it takes to make that step.

Let's now restate the solution to the radiation problem as follows. Let R_0 be the mean number of particles to radiate in one second, so R_0 is going to be the mean of our distribution. In the above example, R_0 was 3 and R was 5. So we can say

$$P(R) = \binom{N}{R} (R_0/N)^R (1 - R_0/N)^{N-R}$$

where N is the number of divisions in the interval, and R is the number of clicks of interest, and as noted, R_0 is the mean number of clicks during the whole integral. If we can take the limit as $N \to \infty$, then we have corrected the inaccuracy mentioned above. To do this limit, consider these facts:

$$\binom{N}{R} \quad \to \quad N^R/R! \qquad \text{// just write it out!}$$

$$\lim (1 + c/n)^n = e^c \qquad => \qquad (1 - R_0/N)^N \quad \to e^{-Ro}$$

$$\text{and} \qquad (1 - R_0/N)^{-R} \quad \to 1^{-R} = 1$$

So we have

$$P(R) \to N^R/R! \ (R_0/N)^R e^{-Ro} * 1 = (R_o)^R e^{-Ro} / R!$$

Normally the mean number of success events is written as λ and not Ro, so we have

$$P(R) = \lambda^R e^{-\lambda} / R!$$

and this is the famous Poisson Distribution!

4. Confirm that <R> = λ:

This is pretty easy!

$$<R> = \sum_{R=0}^{\infty} R\, P(R) = e^{-\lambda} \sum_{R=0}^{\infty} R\, \lambda^R/ R! = e^{-\lambda} \sum_{R=1}^{\infty} R\, \lambda^R/ R!$$

$$e^{-\lambda} \sum_{R=1}^{\infty} \lambda\, \lambda^{R-1}/ (R-1)! = e^{-\lambda}\, \lambda \sum_{R=1}^{\infty} \lambda^{R-1}/ (R-1)! = e^{-\lambda}\, \lambda \sum_{R=0}^{\infty} \lambda^R/ (R)!$$

$$= e^{-\lambda}\, \lambda\, e^{\lambda} = \lambda$$

5. Restate things in terms of a car traffic problem.

(a) A freeway onramp gets 5 cars per minute average. What is the probability of getting no cars in a given minute?

$$P(R) = \lambda^R e^{-\lambda} / R!$$

λ = the mean = 5 cars/minute.

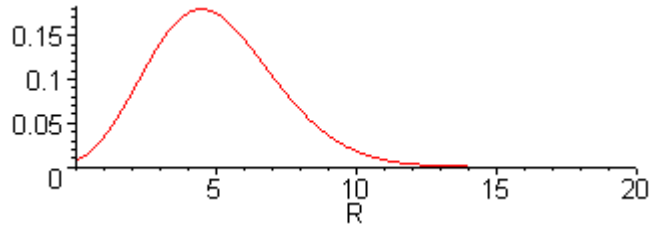
$P(0) = 5^0 e^{-5} / 0! = e^{-5} = 0.7\%$

(b) What is the probability of getting 10 cars in a given minute

$P(10) = 5^{10} e^{-10} / 10! = 9765625 * .00673 / 3628800 = .018 = 1.8\%$

(c) what is the probability of getting the mean in a minute? It is about 17%.

Here is a plot of P(R) versus R for mean being 5:



The chances of 20 cars appearing in a minute is very small.

6. Restate in terms of Sklar's ALOHA discussion.

What is the probability that we will see K messages start in a time interval T, if the mean number of transmitted message starts is λ_t messages/sec. The mean number of message starts in this time interval will be λ_t T. So put this in for the mean in the formula to get

$P(K) = (\lambda_t T)^K e^{-\lambda_t T} / K!$

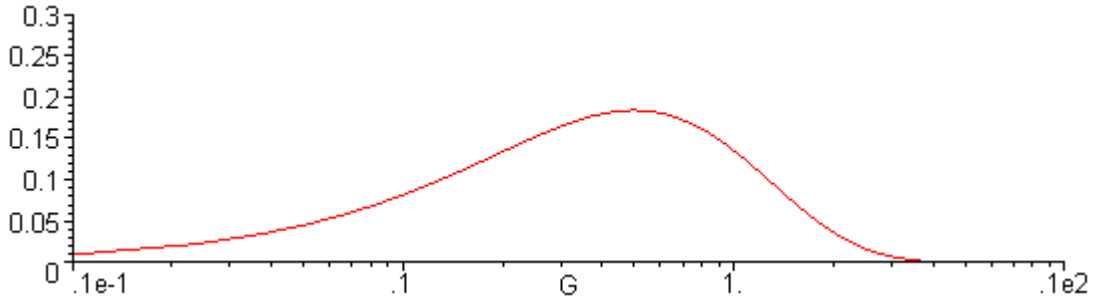
Now when anyone sends a message, it lasts time τ. This message will avoid collision if there are no message starts between τ *before* this message starts and τ *after* this message ends. This is a window of width 2τ. It this window is clean, you get no collision and the message will be a success. What is the probability of getting zero messages in this window?

$P(0) = (\lambda_t 2\tau)^0 e^{-\lambda_t 2\tau} / 0! = e^{-\lambda_t 2\tau}$

This is the probability of success in sending a message and it must equal λ/λ_t where λ is the mean number of messages that are successes per second. This results in

$\lambda\tau = \lambda_t\tau\, e^{-2\lambda_t\tau}$ $\quad$ $y = x\, e^{-2x}$

which you can plot as I just did in Maple. This is a normalized success rate as a function of the attempt rate, and it peaks at a lousy 18% which is pretty surprising. Here it is from maple



So this is the "pure ALOHA" curve, see Sklar and Sklar notes for more.