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I recently read through all the old items, they are at least reasonable. The only date I see here is 1978, but probably most are from 1974-7 when I was at Berkeley. These hand-written and typed notes are from the pre-computer era, and I had not yet learned to put dates on things!

Of much interest is the formula for the Gaussian weights A_i. Recently I verified (against Erdelyi) the formula given on the first page for the Gaussian weights A_i , and I verified the derivation presented here of that formula. I did this after studying carefully (for an unrelated reason) the entire Erdelyi section on orthogonal polynomials, and the derivation done there (and verified by me) agrees with this formula.

In these notes, I use the notation of A&S rather than Erdelyi, but they do agree on certain items like k_n and h_n. A&S have $a_n = B_n$ and $b_n = A_n$ but at least they have $c_n = C_n$. A&S don't really do much in a general way, most of the results are specific.

So with that introduction, here is what we have:

1. Gaussian Quadrature (1 page, 2 sides). Here I simply state the n-point Gaussian integration formula and give the formula for the A_i. This formula uses the g_2/g_0 specific ratios shown on A&S page 783. One must be careful because A&S reuse symbols like g_0 in other tables with totally unrelated meanings. The ratios are the same as what is called X/β_n in Erdelyi. Halfway through the first page I comment that the integration formula is exact through degree 2n-1 if you use zeros of $\phi_n(x)$ as your x_i values. I then comment on a "proof" of that fact which appears in (2) below. The comments on the back side of this first page are not very useful.

Comments on the A_i formula. The form shown in (1.) only applies to the classical ortho polys, but the alternative formula involving $\phi_n'(x)\phi_{n+1}(x)$ [see Erdelyi page 162 (8)] applies to all ortho polys. There are several different versions of the formula for the classical polys, but I like the one given here because it is the simplest. It has the lowest possible degree $\phi_n(x)$ in it, which is $\phi_{n-1}(x_i)$, and no derivatives. So at worst you have to compute the zeros x_i of your ortho poly $\phi_n(x)$, and then you have to compute $\phi_{n-1}(x_i)$. For classical polys, you just look up or numerically compute the zeros and plug them into a closed form expression for ϕ_{n-1}. For the general case, we do need $\phi_n'(x_i)$ and $\phi_{n+1}(x_i)$

2. Derivation of the fact that Gaussian integration is not just accurate for degree n, but up to 2n-1. I had lots of trouble understanding my own proof, wondering why I was using the $\otimes$ symbol and talking about "angular momentum". And I could not see why you could not continue the iteration forever. But now I understand, and a clearer proof is given in document *Accuracy of Gaussian Quadrature* below.

3. Derivation of the weight formula (3 pages). Everything is now correct and the final result is the one that appears on the page (1.) above. I checked off each step today and erased a bogus $\phi_0(1)$ factor on the last page. On the last page I was intending to build a table of key things, but I gave up, it is all clearly stated in A&S.

4. Summary of the Chebyshev results. (1p) Here, the A_i have the trivial form $A_i = \pi/n$. This means that you basically just add up the $f(x_i)$. Of course only works for w(x) as shown.

5. Summary of the Legendre results (1p)

6. A "program" to do 4-point Legendre integration for an arbitrary integral. This just reminds you how fast this method is. You add up four numbers and you are done, and result is exact for a polynomial of degree 7 or less, and very good for worse.

7. A repeat of the ideas presented in item (1) above. I see that I learned this from friend Myklos Gyulassy who is now doing particle physics at Columbia, pushing QGP = quark gluon plasma, 235 publications to date. He stayed on the boat. I remember him being quite good in explaining things to me.

8. How to "accelerate" an infinite series sum. More notes from Miklos. He looks at a very slowly converging series and shows how to write it a contour integral which can be deformed to be a nice integral. You can then evaluate that integral doing a 20 point Gaussian quadrature. In this special case we have a closed form answer from Scheid, but I think it is the idea here that is important, you maybe have a complex sum that you don't have an exact answer for.

9. Accuracy of Gaussian Quadrature (Jan 2005). Here I show first why the Gaussian n+1 point formula is exact for polys of degree $\leq$ n for arbitrary distinct $\{ x_i \}$, then I show why the result is exact for polys of degree $\leq 2n+1$ if you select x_i as the zeros of $p_{n+1}(x_i)$. By this time I am doing everything using Erdelyi's constants and general notation. The second part involves that "angular momentum addition" idea alluded to in (2.) above.

See also Scheid chapters and my notes on Scheid for other information on Gaussian Quadrature.