A Note on Sine and Cosine Integrals PhL 1.12.03

I am often interested in an integral of the following form,

$$I2 = \int_a^\infty dx \frac{e^{ikx}}{x^2} \qquad I1 = \int_a^\infty dx \frac{e^{ikx}}{x}$$

mainly because these tend to show up in optical integrals.

Look Up Attempts

Now, if you try to look up the I2 integral in GR, you could try page 317 #3 with $\upsilon = -1$ and $\mu = -ik$ and u = a with the result $\Gamma(-1,-ika)$, where Γ is the incomplete gamma function, discussed on GR940 and AS260. This is not a very enlightening form for the result of my integral!

A second attempt would be in the indefinite GR section as on page 93. The problem here is that you get your result in terms of one of the exponential integral types Ei(ikx), so you have Ei with an imaginary argument and that is also poorly documented. This is also the function that was mislabeled in Schaum, and I made a note there.

A third attempt is to look this up with the cos(kx) + i sin(kx) breakdown. However, the closest GR comes in this case is GR420 where you are back to those incomplete gamma functions.

A fourth attempt is to change variable so that y = a/x to move the lower integration endpoint down to 1. When this is done, we get:

$$I2 = \frac{1}{a}\int_1^\infty dy \frac{e^{ikay}}{y^2}$$

But this too does not appear in GR. Maple, by the way, gives messy answers for our two integrals I1 and I2 in terms of the Ei(x) function, again not very useful from a practical sense.

Manual Parts Integration to simplify I2

Let

u = 1/x du/dx = $-1/x^2$
v = exp(ibx) dv/dx = ib exp(ibx)

d(uv)/dx = u dv/dx + v du/dx

I did this by hand, but later found that **Maple** can do this as follows:

```
with(student):intparts(Int( i*b*exp(i*b*x)/x,x), 1/x);
with(student):intparts(Int(       dv/dx     *  u, x),   u );          // how to interpret previous line!
```

Integrate both sides from 1 to infinity to get

$$\left.\frac{e^{ibx}}{x}\right|_1^\infty = ib\int_1^\infty dx\,\frac{e^{ibx}}{x} - \int_1^\infty dx\,\frac{e^{ibx}}{x^2}$$

The LHS is - e^{ib} so we can arrange to get:

$$\int_1^\infty dx\,\frac{e^{ibx}}{x^2} = e^{ib} + ib\int_1^\infty dx\,\frac{e^{ibx}}{x}$$

Now we have at least made some progress. Now look at this integral:

$$I1' = \int_1^\infty dx\,\frac{e^{ibx}}{x}$$

We have the same problems noted above trying to look this up in GR. However, now if we do the sine and cosine breakdown, we can use GR405 2 and 3 to get:

$$I1' = -\,[ci(b) + i\,si(b)]$$

where the si(z) and ci(z) functions are those of GR. The lower si(z) is also in AS. However, AS tend to use capitalized functions instead (see AS p 231)

$$Ci(b) = ci(b) \qquad\qquad Si(b) = si(b) + \pi/2$$

There is lots of information on these two functions, known as the Sine and Cosine Integral Functions. So, let's summarize our results:

$$\int_1^\infty dx\,\frac{e^{ibx}}{x} = -[ci(b) + i\,si(b)]$$

$$\int_1^\infty dx\,\frac{e^{ibx}}{x^2} = e^{ib} + b[si(b) - i\,ci(b)]$$

Now finally we have some results in terms of earlier desired integrals:

$$I2 = \int_a^\infty dx\,\frac{e^{ikx}}{x^2} = \frac{1}{a}\int_1^\infty dy\,\frac{e^{ikay}}{y^2} = (1/a) * \{e^{ika} + ka*[si(ka) - i\,ci(ka)]\}$$

$$I1 = \int_a^\infty dx\,\frac{e^{ikx}}{x} = \int_1^\infty dy\,\frac{e^{ikay}}{y} = -[\,ci(ka) + i\,si(ka)\,]$$

Comments:

1. About the si(x) and ci(x) functions;

Notice that we can only do these as definite integrals with ∞ upper end point. Finally, we note these facts:

$si(0) = -\pi/2 \quad ci(0) = \infty \qquad\qquad si(\infty) = ci(\infty) = 0$

To learn these facts, we have to study AS page 232. In particular, we know that

$si(x) = - f(z)\cos(z) - g(z)\sin(z)$
$ci(x) = f(z)\sin(z) - g(z)\cos(z)$

This tells us both functions in terms of the two "auxiliary functions" f(z) and g(z). Large z expansions are given for these two and we find that $f(z) \sim 1/z$ and $g(z) \sim 1/z^2$ so we know that $f(\infty) = g(\infty) = 0$, and that is why we know $si(\infty) = ci(\infty) = 0$. Also, for use below, note these facts:

$\text{limit } x*si(x) = -\cos(x)$ large x
$\text{limit } x*ci(x) = \sin(x)$ large x

2. Large k limits.

Qualitatively, if we look at the I1 and I2 integrals, we would be tempted to say that as k increases, the phase winds faster and faster, and with a fixed lower endpoint a, the integrand is more and more chopped up and ends up being zero. This is in fact true for both I1 and I2. It is obvious looking at the I1 result since we know $si(\infty) = ci(\infty) = 0$. Showing it for I2 is less obvious. Using the limits shown above for large x, the second term in fact exactly cancels the first term!

3. Convergence.

We have shown that the phasor over either x or over x^2 both converge at the *upper* integral endpoint, otherwise we would not have obtained our nice closed form answers! For $1/x^2$ we are not surprised, but for 1/x we see that the phasor has tamed a logarithmic divergence for us!

Neither integral converges if we take a to 0, as one would expect. Function ci(z) has a log divergence there, and the first term in I2 goes worse, as 1/a. Just as you would expect with no phasor.

4. Problems with Equation Editor.

Cannot save to disk sometimes, see MathType link for how to fix this problem, saved as bookmark.