Levi-Civita Symbol PhL 2.11.03

See the "two attached short papers" on this subject, I will refer to equations in these papers. Authors of the papers are Guio (G) and Littlejohn (L).

[I could not find these on 9.1.11, but found the Guio one on line and made it a PDF]

Guio (G) http://folk.uio.no/patricg/teaching/a112/levi-civita/
Littlejohn (L) ??

Summary of Results (all proven below)

$$\varepsilon_{ijk} = \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix} \tag{1}$$

$$\varepsilon_{ijk}\,\varepsilon_{i'j'k'} = \begin{vmatrix} \delta_{ii'} & \delta_{ij'} & \delta_{ik'} \\ \delta_{ji'} & \delta_{jj'} & \delta_{jk'} \\ \delta_{ki'} & \delta_{kj'} & \delta_{kk'} \end{vmatrix} \quad \text{no sums} \tag{2}$$

$$\varepsilon_{ijk}\,\varepsilon_{ij'k'} = \begin{vmatrix} \delta_{jj'} & \delta_{jk'} \\ \delta_{kj'} & \delta_{kk'} \end{vmatrix} = \delta_{jj'}\,\delta_{kk'} - \delta_{jk'}\,\delta_{kj'} \quad \text{sum on i} \tag{3}$$

$$\varepsilon_{ijk}\,\varepsilon_{ijk'} = 2\,\delta_{kk'} \quad \text{sum on i and j} \tag{4}$$

$$\varepsilon_{ijk}\,\varepsilon_{ijk} = 2\,\delta_{kk} = 6 \quad \text{sum on i and j and k} \tag{5}$$

Each of these results could have been "guessed" because the Kroenecker delta is the only "thing" we have to play with apart from numbers like 1,2,3. The requirement of antisymmetry on any pair of indices of the L-C symbol certainly suggests the second result which is really the more general result -- we can recover the first result setting ijk = 123. For example, in the second result, if you swap i and j, you swap the first two rows. Swap i' and j' and you swap the first two columns. There are 6 required swaps and that exactly matches the number possible with the determinant form, so the result HAS to be proportional to the determinant shown. The scale factor can then be shown to be 1.

A similar argument can be made for the result (3). We have two pairs of swap requirements, and the 2x2 matrix provides exactly this. You can also make an "arm-waving" derivation of (3) that, for a given value of i, the other indices have to match either one way or the other, and the other gives a minus sign.

Who was Levi-Civita?

Tullio Levi-Civita (Italian Jew, 1873-1941) took his degree at the University of **Padua** where one of his teachers was **Ricci** with whom Levi-Civita was to collaborate.

Levi-Civita was appointed to the Chair of Mechanics at Padua in 1898 (age 25), a post which he was to hold for 20 years. In 1918 (age 45) he was appointed to the Chair of Mechanics at Rome where he spent another 20 years until removed from office (1938, age 65) by the discrimination policies of the government (he was of Jewish descent).

Levi-Civita had very great command of pure mathematics, his geometric intuition was particularly strong, which he applied to a variety of problems of applied mathematics. One of his papers in 1895 improved on **Riemann's** contour integral formula for the number of primes in a given interval.

Levi-Civita is best known for his work on the **absolute differential calculus** with its applications to the theory of relativity. In 1887 he published a famous paper in which he developed **the calculus of tensors**, following on the work of Christoffel, including **covariant differentiation**. In **1900** he published, jointly with Ricci, the **theory of tensors** Méthodes de calcul differential absolu et leures applications in a form which was used by **Einstein** 15 years later.

Weyl was to take up Levi-Civita's ideas and make them into a unified theory of gravitation and electromagnetism. Levi-Civita's work was of extreme importance in the theory of **relativity**, and he produced a series of papers treating elegantly the problem of a static gravitational field.

Analytic dynamics was another topic studied by Levi-Civita, many of his papers examining special cases of the Three Body Problem. He also wrote on **hydrodynamics** and the theory of systems of partial differential equations. He added to the theory of **Cauchy** and **Kovalevskaya** and wrote up this work in an excellent book written in 1931.

In 1933 he contributed to **Dirac's** equations of quantum theory.

The Royal Society conferred the Sylvester medal on Levi-Civita in 1922, while in 1930 he was elected a foreign member. He was also an honorary member of the London Mathematical Society, the Royal Society of Edinburgh, and the Edinburgh Mathematical Society. He attended a meeting of the Edinburgh Mathematical Society in St Andrews.

Levi-Civita, like Volterra and many other Italian scientists, were strongly and actively opposed to Fascism. After he was dismissed from his post the blow soon told on his health and he developed severe heart problems. He died of a stroke.

Article by: J J O'Connor and E F Robertson

The Levi-Civita Symbol

Start with Guio's paper. For me, the fundamental appearance of this symbol in any number of dimensions is in the evaluation of a **determinant**. For n=3, this is shown in G (2), where the letters "det" are not needed.

$$\det \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk} a_{1i} a_{2j} a_{3k} = \epsilon_{ijk} a_{1i} a_{2j} a_{3k} \qquad (2)$$

Consider a unit vector in the 0 or x direction. We could write it's components as follows:

$[\hat{\varepsilon}_1]_i = e_{1i} = \delta_{1,i}$ or more generally $[\hat{\varepsilon}_j]_i = e_{ji} = \delta_{j,i}$ for j = 0,1,2

Now we can consider the rows or columns of a determinant to be *any* three vectors we want, so let's take the three rows to be unit vectors $\hat{\varepsilon}_i$, $\hat{\varepsilon}_j$, and $\hat{\varepsilon}_k$ with components as shown above. We need a picture of this thing, a good task for the equation editor:

$$\begin{vmatrix} \varepsilon_{i1} & \varepsilon_{i2} & \varepsilon_{i3} \\ \varepsilon_{j1} & \varepsilon_{j2} & \varepsilon_{j3} \\ \varepsilon_{k1} & \varepsilon_{k2} & \varepsilon_{k3} \end{vmatrix}$$

Now look at the first row, and use the same a_{ij} notation appearing in G (2). We have

$$a_{11} = e_{i1} \quad a_{12} = e_{i2} \quad a_{13} = e_{i3}$$

which we can combine as one equation to get

$$a_{1\ell} = e_{i\ell} = \delta_{i,\ell}$$

Similarly we have

$$a_{2m} = e_{jm} = \delta_{j,m} \qquad \text{and} \qquad a_{3n} = e_{kn} = \delta_{k,n}$$

Now consider the triple sum on the right side of G (2) to be over indices ℓmn. We get

$$= \varepsilon_{\ell mn} \quad a_{1\ell}\, a_{2m}\, a_{3n} = \varepsilon_{\ell mn} \quad \delta_{i,\ell}\, \delta_{j,m}\, \delta_{k,n} = \varepsilon_{ijk}$$

This took me a long time to get right for some reason! So this proves that:

$$\varepsilon_{ijk} = \begin{vmatrix} \varepsilon_{i1} & \varepsilon_{i2} & \varepsilon_{i3} \\ \varepsilon_{j1} & \varepsilon_{j2} & \varepsilon_{j3} \\ \varepsilon_{k1} & \varepsilon_{k2} & \varepsilon_{k3} \end{vmatrix} = \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix} \qquad (1)$$

where we are leaving out the comma in the Kroenecker delta just for simplicity. You can multiply out the delta functions and in doing so you get the six terms which are the 6 non-zero terms of the symbol. The first term, for example, is $\delta_{i,1}\, \delta_{j,2}\, \delta_{k,3}$. Each term is a product of three delta functions and a sign.

Now consider the product of two symbols with all 6 coefficients different. We have

$$\varepsilon_{ijk}\, \varepsilon_{i'j'k'} = \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix} \begin{vmatrix} \delta_{i'1} & \delta_{i'2} & \delta_{i'3} \\ \delta_{j'1} & \delta_{j'2} & \delta_{j'3} \\ \delta_{k'1} & \delta_{k'2} & \delta_{k'3} \end{vmatrix}$$

Now transpose the second matrix without changing its determinant to get

$$\varepsilon_{ijk}\,\varepsilon_{i'j'k'} = \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix} \begin{vmatrix} \delta_{i'1} & \delta_{j'1} & \delta_{k'1} \\ \delta_{i'2} & \delta_{j'2} & \delta_{k'2} \\ \delta_{i'3} & \delta_{j'3} & \delta_{k'3} \end{vmatrix}$$

Next, use rule that product of dets is det of product matrix. Then notice what the terms look like in this product matrix. For example, the top left term has this form:

$$\delta_{i1}\delta_{i'1} + \delta_{i2}\delta_{i'2} + \delta_{i3}\delta_{i'3} \quad = \delta_{ii'}$$

Why this RHS? We can see that in each of the three terms, we need i = i'. For each specific value of i, only one of the three terms hits to give 1. But the RHS gives the same result. Now all terms in the product matrix have this same form, so we arrive at this conclusion:

$$\varepsilon_{ijk}\,\varepsilon_{i'j'k'} = \begin{vmatrix} \delta_{ii'} & \delta_{ij'} & \delta_{ik'} \\ \delta_{ji'} & \delta_{jj'} & \delta_{jk'} \\ \delta_{ki'} & \delta_{kj'} & \delta_{kk'} \end{vmatrix}$$

So this lets us write out the product again as the sum of 6 terms each having the product of three δ's. This result appears as L (3.5).

Now if we "contract" on the first index, meaning we set i' = i in the second symbol and then sum on i, what happens?

$$\varepsilon_{ijk}\,\varepsilon_{ij'k'} = \begin{vmatrix} \delta_{ii} & \delta_{ij'} & \delta_{ik'} \\ \delta_{ji} & \delta_{jj'} & \delta_{jk'} \\ \delta_{ki} & \delta_{kj'} & \delta_{kk'} \end{vmatrix}$$

If you write this out going down the first column say as the usual three sub-determinants, you find that only the first sub-determinant survives, and the second two exactly cancel. The result is then,

$$\varepsilon_{ijk}\,\varepsilon_{ij'k'} = \begin{vmatrix} \delta_{jj'} & \delta_{jk'} \\ \delta_{kj'} & \delta_{kk'} \end{vmatrix} = \delta_{jj'}\,\delta_{kk'} - \delta_{jk'}\quad\delta_{kj'}$$

This is the result I was having a lot of trouble proving, but now we have done it. You see this result as G (7) and as L (3.6).

We can then contract on a second index as well, to get

$$\varepsilon_{ijk}\,\varepsilon_{ijk'} = \begin{vmatrix} \delta_{jj} & \delta_{jk'} \\ \delta_{kj} & \delta_{kk'} \end{vmatrix} = \delta_{jj}\,\delta_{kk'} - \delta_{jk'}\quad\delta_{kj} = 3\delta_{kk'} - \delta_{kk'} \quad = 2\,\delta_{kk'}$$

and this result appears as L (3.7) and G (8). The last step is to contract the last index to get

$$\varepsilon_{ijk}\,\varepsilon_{ijk} = 2\,\delta_{kk} = 6$$