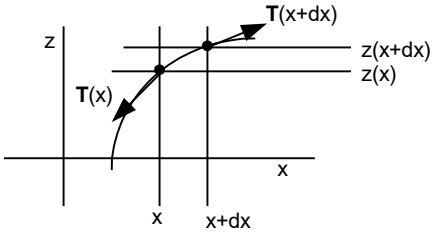
Why is a static rubber sheet described by the Laplace Equation? PhL 12.3.04

In fact, this is only true if the vertical amplitude z(x,y) is small relative to the x and y dimensions. In this situation, you can ignore certain terms in the equations, and only then do you get Laplace. In this writeup, we use lots of different notations for the reader to figure out along the way.

1. The starting point is to look at a string instead of a sheet. Here is our exaggerated picture,



In reality, the two tension arrows are almost perfectly horizontal. Still, each one is aligned with a tangent vector to the string at its point. The tangent unit vector at any point is approximately (1,dz/dx) = (1,z'). Because z' << 1, the magnitude of this vector is about 1. So we can then say that the total force on our little piece of string is this:

$$d\mathbf{F} = \mathbf{T}(x+dx) - \mathbf{T}(x) = T\,(1,z'(x+dx)) - T\,(1,z'(x)) = (0, T\,z''(x)\,dx)$$

Here we have assumed the magnitude of the tension vector T is the same at x+dx as it is at x, certainly true in the limit. This says that the *vertical* force acting on the piece of string is this:

$$dF_z = T\,z''\,dx$$

and we see how the curvature z" has gotten into things. You can also see just from the picture that if the curvature is zero, there will be no force. If it cups down as shown, the total force is down (negative) and so on.

Now we go the next step to assume that the string has mass density ρ per unit length, so the dx piece of string has mass ρdx, and we set $F = ma = (\rho dx)\ddot{z} = T\,z''\,dx$ and we find that

$$\rho\ddot{z} = T\,z'' \quad \Rightarrow \quad \rho\ddot{z} - T\,z'' = 0$$

which is a standard hyperbolic second order ODE, the wave equation in 1D. If the string is at rest, then we have z" = 0, and the only solution to this equation is that the string is in a straight line, which makes pretty good sense. The exact straight line would be determined by the "boundary conditions" at the two ends of the string. For example, it at the left end z = 0 and at the right end z = .25, then the straight line tips up to the right. If the height is the same at both ends, the string is horizontal. In any event, if there is a maximum height of the string, it must occur at one end or the other.

2. Next, suppose the string is replaced by a thin ribbon of width dy. Suppose this ribbon has ρ' mass per unit AREA, and suppose the tension on it is T' which is now not just force, but force PER UNIT WIDTH of the piece of ribbon that is being tugged on. Then we can make this connection to our string problem:

$\rho = \rho' dy \qquad T = T' dy$

Then we could write our equation above as (so the only change is to add the primes!)

$$\rho' \ddot{z} = T' z_{xx}$$

Now, if we think of our ribbon as embedded in a 2D rubber sheet, we also get a contribution to the vertical force from the y direction, so we can just add that into the above equation:

$$\rho' \ddot{z} = T' z_{xx} + T' z_{yy} = T' (z_{xx} + z_{yy}) = T' \nabla^2 z \qquad \nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 \qquad //2D$$

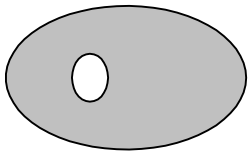
and this is the 2D wave equation. Again, it is hyperbolic because it has only second derivatives AND, when everything is put on one side of the equation, one of the signs is different from all the others. This is the 2D wave equation. Now, if we require the sheet to be at rest, we get

$$(z_{xx} + z_{yy}) = 0 \qquad \text{or} \qquad \nabla^2 z = 0 \qquad \text{// Laplace equation in 2D}$$

and we have then shown why it is that a rubber sheet z(x,y) obeys the Laplace equation. The solution of this equation depends on the "boundary conditions" as with the string, but now the boundary is a 2D boundary surrounding the surface. If z is a constant on ALL of whatever boundary you specify, then z is constant on the entire rubber sheet, which seems pretty reasonable, since there is nothing to pull it up or down anywhere.

Also, as with the string, the max or min value of z(x,y) must occur on the boundary somewhere. Here is one explanation of why that is true. Consider the equation $(z_{xx} + z_{yy}) = 0$ out in the middle of the sheet somewhere. If the x curvature is positive, the y curvature must be negative, so every point is really a saddle point. If you had a maximum there, both curvatures would have to have the same sign, which is not possible from the equation. The case that both curvatures are 0 leads to the flat sheet of constant height.

3. Topologically, the 2D boundary can look like this:



where the rubber sheet is the gray area. You could then imagine the inner part of the boundary to be at a different height from the outer. If the outer part is all at z=0 and the inner all at z=h, and the inner part is taken to be very small, you have a rubber sheet held up by a stick in the middle. The entire surface is then described by Laplace, assuming h is not too large. This is certainly easy to visualize, and you can imagine that at any point on the sheet, things are "saddle like".

Waves on drum heads are described by the Laplace equation. Because a drum boundary is round, the Laplace equation is usually converted to polar coordinates, and solutions of the ODE in those coordinates involve Bessel functions.

4. If we lived in 4D, we could add another dimension to things. Instead of having z(x,y), let z be the third variable and v be the fourth, so the rubber surface in 4D is then v(x,y,z). We would redefine ρ' and T' similarly to the way we did it before, for example, ρ" = ρ'dz and the same for T". Then we add the third z-force on our piece of 3D membrane to get the new wave equation in 4D space

$$\rho'' \ddot{v} = T''(v_{xx} + v_{yy} + v_{zz}) = T'' \nabla^2 v \qquad \nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$$

where now we have the 3D Laplacian operator ∇^2 appearing.

A function of 3D space like v(x,y,z) is called a scalar field. If we insist that things be static in time, then we get this equation for the field,

$\nabla^2 v = 0$ // Laplace equation

5. In the study of electrostatics, the electric "potential" is exactly such a field v(x,y,z), and its units are volts. At each point in 3D space there is a "voltage" that can in theory be measured, and it satisfies the Laplace equation. As in the 1D and 2D cases, the field is completely determined by its value on a boundary, but now the boundary is a 3D one that surrounds a 3D volume of interest. The field must take its maximum value on the boundary. If the field is constant on the boundary, it is constant everywhere in the interior volume.

It you take a hollowed out metal object (like a sphere) and put another metal object inside it, then the two pieces of metal combined form a boundary for the volume between them, somewhat like our 2D picture above for the 2D rubber sheet with a stick in the middle. If you wire a 9 volt battery with one terminal to each of these metal objects, you apply non-zero voltage boundary conditions. The electric potential in the inside volume will be described by the Laplace equation. This is an example of a capacitor.

If there is nothing inside the outer metal object, then, since the entire boundary is then at the same voltage (potential), the potential *inside* the object is constant. Since the electric field is the spatial derivative of the potential, $\mathbf{E} = -\nabla v$ meaning, for example, $E_x = -\partial v/\partial x$, this shows that the electric field inside a metal box of any shape is zero. This is called a Faraday cage and shields whatever is inside from electric fields caused by something happening on the outside (such as lightning bolts).