

## Tables of Integral Transforms

CALIFORNIA INSTITUTE OF TECHNOLOGY  
BATEMAN MANUSCRIPT PROJECT

A. ERDÉLYI, *Editor*

W. MAGNUS, F. OBERHETTINGER, F. G. TRICOMI, *Research Associates*

Higher Transcendental Functions, 3 volumes.  
Tables of Integral Transforms, 2 volumes.



517.75  
BA  
V.2  
cop. 2

TABLES OF INTEGRAL TRANSFORMS

Volume II

A.M.

Based, in part, on notes left by

Harry Bateman

*Late Professor of Mathematics, Theoretical Physics, and Aeronautics at  
the California Institute of Technology*

and compiled by the

Staff of the Bateman Manuscript Project

Prepared at the California Institute of Technology under Contract  
No. N6onr-244 Task Order XIV with the Office of Naval Research

Project Designation Number: NR 043-045

NEW YORK TORONTO LONDON  
McGRAW-HILL BOOK COMPANY, INC.  
1954

TABLES OF INTEGRAL TRANSFORMS, vol.II

COPYRIGHT, 1954, BY THE  
MCGRAW-HILL BOOK COMPANY, INC.

---

PRINTED IN THE UNITED STATES OF AMERICA

All rights reserved except those granted to the United States Government. Otherwise, this book, or parts thereof, may not be reproduced in any form without permission of the publishers.

Library of Congress Catalog Card Number: 54-6214

19550

9 10 11 12 13 14 15 16-MAMM-7 6 5 4

This work is dedicated to the  
memory of

HARRY BATEMAN

as a tribute to the imagination which  
led him to undertake a project of this  
magnitude, and the scholarly dedication  
which inspired him to carry it so far  
toward completion.



## STAFF OF THE BATEMAN MANUSCRIPT PROJECT

### *Director*

Arthur Erdélyi

### *Research Associates*

Wilhelm Magnus (1948-50)

Fritz Oberhettinger (1948-51)

Francesco G. Tricomi (1948-51)

### *Research Assistants*

David Bertin (1951-52)

W.B. Fulks (1949-50)

A.R. Harvey (1948-49)

D.L. Thomsen, Jr. (1950-51)

Maria A. Weber (1949-51)

E.L. Whitney (1948-49)

### *Vari-typist*

Rosemarie Stampfel



## PREFACE

The aims, the history, and the organization of these *Tables of Integral Transforms* were described in the Introduction to vol. I. A little more than one half of the present second, and last, volume consists of tables of further integral transforms, the remaining part of this volume contains integrals of higher transcendental functions.

Under the generic name *Bessel transforms* we list not only the familiar Hankel transforms but also other transformations whose kernels are Bessel functions in the widest sense of the word. In addition to these we list fractional integrals, and also Stieltjes and Hilbert transforms. As far as we know, no extensive tables exist for any of the transformations included in this volume, in fact, for some of them there are comparatively few known transform pairs. A list of all transforms included in this work is given on p. xiff.

The second part of the volume contains miscellaneous integrals involving higher transcendental functions. Some of these integrals cannot be written as transforms, others were not included in the transform tables and are given here. Generally speaking, an integral which can be written as a transform is more likely to be found in the transform tables than among integrals of higher transcendental functions. The latter are arranged according to their integrands. The "hierarchy" of functions given on p. xii of vol. I has been followed and, as in vol. I, composite functions are classified according to the "highest" function occurring in them. A list of definitions of higher transcendental functions is given in the Appendix.

Acknowledgments and thanks are due to the same persons and organizations as in connection with vol. I. Acknowledgments are also due to Mr. John B. Johnston who read the proofs and rendered other valuable technical assistance.

Corrections of errors, additions, and suggestions for improvement will be received gratefully by the Editor.

A. ERDÉLYI





## STANDARD FORMS

Fourier cosine transform ( $\mathfrak{F}_c$ , Chapter I)

$$\int_0^{\infty} f(x) \cos(xy) dx$$

Fourier sine transform ( $\mathfrak{F}_s$ , Chapter II)

$$\int_0^{\infty} f(x) \sin(xy) dx$$

Exponential Fourier transform ( $\mathfrak{F}_e$ , Chapter III)

$$\int_{-\infty}^{\infty} f(x) e^{-ixy} dx$$

Laplace transform ( $\mathfrak{L}$ , Chapter IV)

$$\int_0^{\infty} f(t) e^{-pt} dt$$

Inverse Laplace transform (Chapter V)

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} g(p) e^{pt} dp$$

Mellin transform ( $\mathfrak{M}$ , Chapter VI)

$$\int_0^{\infty} f(x) x^{s-1} dx$$

Inverse Mellin transform (Chapter VII)

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} g(s) x^{-s} ds$$

Hankel transform ( $\mathfrak{H}_\nu$ , Chapter VIII)

$$\int_0^{\infty} f(x) J_\nu(xy) (xy)^{1/2} dx$$

Y - transform ( $\mathfrak{Y}_\nu$ , Chapter IX)

$$\int_0^{\infty} f(x) Y_\nu(xy) (xy)^{1/2} dx$$

$K$ -transform ( $\mathfrak{K}_\nu$ , Chapter X)

$$\int_0^\infty f(x) K_\nu(xy) (xy)^{\frac{1}{2}} dx$$

$\mathbf{H}$ -transform (Chapter XI)

$$\int_0^\infty f(x) \mathbf{H}_\nu(xy) (xy)^{\frac{1}{2}} dx$$

Kontorovich-Lebedev transform (Chapter XII)

$$\int_0^\infty f(x) K_{ix}(y) dx$$

Riemann-Liouville fractional integral ( $\mathfrak{I}_\mu$ , Chapter XIII)

$$\frac{1}{\Gamma(\mu)} \int_0^y f(x) (y-x)^{\mu-1} dx$$

Weyl fractional integral ( $\mathfrak{W}_\mu$ , Chapter XIII)

$$\frac{1}{\Gamma(\mu)} \int_y^\infty f(x) (x-y)^{\mu-1} dx$$

Stieltjes transform ( $\mathfrak{S}$ , Chapter XIV)

$$\int_0^\infty \frac{f(x)}{x+y} dx$$

Generalized Stieltjes transform ( $\mathfrak{S}_\rho$ , Chapter XIV)

$$\int_0^\infty \frac{f(x)}{(x+y)^\rho} dx$$

Hilbert transform (Chapter XV)

$$\frac{1}{\pi} \int_{-\infty}^\infty \frac{f(x)}{x-y} dx$$

## CONTENTS

|                          |    |
|--------------------------|----|
| PREFACE . . . . .        | ix |
| STANDARD FORMS . . . . . | xi |

## BESSEL TRANSFORMS

### CHAPTER VIII

### HANKEL TRANSFORMS

|                                                                                    |    |
|------------------------------------------------------------------------------------|----|
| 8.1. General formulas . . . . .                                                    | 5  |
| 8.2. Hankel transforms of order zero; Elementary functions . .                     | 7  |
| 8.3. Hankel transforms of order zero; Higher transcendental<br>functions . . . . . | 13 |
| 8.4. Hankel transforms of order unity . . . . .                                    | 18 |

### HANKEL TRANSFORMS OF ORDER $\nu$

|                                                                                  |    |
|----------------------------------------------------------------------------------|----|
| 8.5. Algebraic functions and powers with arbitrary index . . .                   | 21 |
| 8.6. Exponential and logarithmic functions . . . . .                             | 28 |
| 8.7. Trigonometric and inverse trigonometric functions . . .                     | 32 |
| 8.8. Hyperbolic and inverse hyperbolic functions . . . . .                       | 41 |
| 8.9. Orthogonal polynomials . . . . .                                            | 42 |
| 8.10. Legendre functions . . . . .                                               | 44 |
| 8.11. Bessel functions of argument $kx$ . . . . .                                | 47 |
| 8.12. Bessel functions of other arguments . . . . .                              | 56 |
| 8.13. Modified Bessel functions of argument $kx$ . . . . .                       | 63 |
| 8.14. Modified Bessel functions of other arguments . . . . .                     | 67 |
| 8.15. Functions related to Bessel functions . . . . .                            | 72 |
| 8.16. Parabolic cylinder functions . . . . .                                     | 76 |
| 8.17. Gauss' hypergeometric function . . . . .                                   | 80 |
| 8.18. Confluent hypergeometric functions . . . . .                               | 82 |
| 8.19. Generalized hypergeometric series and miscellaneous<br>functions . . . . . | 87 |

## CHAPTER IX

## Y - TRANSFORMS

|      |                                                            |     |
|------|------------------------------------------------------------|-----|
| 9.1. | General formulas . . . . .                                 | 95  |
| 9.2. | Algebraic functions and powers with an arbitrary index . . | 96  |
| 9.3. | Other elementary functions . . . . .                       | 105 |
| 9.4. | Higher transcendental functions . . . . .                  | 108 |

## CHAPTER X

## K - TRANSFORMS

|       |                                           |     |
|-------|-------------------------------------------|-----|
| 10.1. | General formulas . . . . .                | 121 |
| 10.2. | Elementary functions . . . . .            | 127 |
| 10.3. | Higher transcendental functions . . . . . | 134 |

## CHAPTER XI

## H - TRANSFORMS

|       |                                           |     |
|-------|-------------------------------------------|-----|
| 11.1. | General formulas . . . . .                | 157 |
| 11.2. | Elementary functions . . . . .            | 158 |
| 11.3. | Higher transcendental functions . . . . . | 162 |

## CHAPTER XII

## KONTOROVICH-LEBEDEV TRANSFORMS

|       |                    |     |
|-------|--------------------|-----|
| 12.1. | Formulas . . . . . | 175 |
|-------|--------------------|-----|

## MISCELLANEOUS TRANSFORMS

## CHAPTER XIII

## FRACTIONAL INTEGRALS

|       |                                                  |     |
|-------|--------------------------------------------------|-----|
| 13.1. | Riemann-Liouville fractional integrals . . . . . | 185 |
| 13.2. | Weyl fractional integrals . . . . .              | 201 |

## CHAPTER XIV

## STIELTJES TRANSFORMS

|       |                                            |     |
|-------|--------------------------------------------|-----|
| 14.1. | General formulas . . . . .                 | 215 |
| 14.2. | Elementary functions . . . . .             | 216 |
| 14.3. | Higher transcendental functions . . . . .  | 224 |
| 14.4. | Generalized Stieltjes transforms . . . . . | 233 |

## CHAPTER XV

## HILBERT TRANSFORMS

|       |                                           |     |
|-------|-------------------------------------------|-----|
| 15.1. | General formulas . . . . .                | 243 |
| 15.2. | Elementary functions . . . . .            | 243 |
| 15.3. | Higher transcendental functions . . . . . | 253 |

## INTEGRALS OF HIGHER TRANSCENDENTAL FUNCTIONS

## CHAPTER XVI

## ORTHOGONAL POLYNOMIALS

|       |                                  |     |
|-------|----------------------------------|-----|
| 16.1. | Tchebichef polynomials . . . . . | 271 |
| 16.2. | Legendre polynomials. . . . .    | 276 |
| 16.3. | Gegenbauer polynomials . . . . . | 280 |
| 16.4. | Jacobi polynomials. . . . .      | 284 |
| 16.5. | Hermite polynomials . . . . .    | 288 |
| 16.6. | Laguerre polynomials. . . . .    | 292 |

## CHAPTER XVII

GAMMA FUNCTION, INCOMPLETE GAMMA FUNCTIONS AND  
RELATED FUNCTIONS

|       |                                                        |     |
|-------|--------------------------------------------------------|-----|
| 17.1. | The gamma function . . . . .                           | 297 |
| 17.2. | The $\Psi$ -function . . . . .                         | 305 |
| 17.3. | Incomplete gamma functions and related functions . . . | 306 |

## CHAPTER XVIII

## LEGENDRE FUNCTIONS

|       |                                                                           |     |
|-------|---------------------------------------------------------------------------|-----|
| 18.1. | Legendre functions of variable $\alpha x + \beta$ : finite intervals. . . | 313 |
| 18.2. | Legendre functions of variable $\alpha x + \beta$ : infinite intervals .  | 320 |
| 18.3. | Legendre functions of other variables . . . . .                           | 326 |

## CHAPTER XIX

## BESSEL FUNCTIONS

|       |                                                                               |     |
|-------|-------------------------------------------------------------------------------|-----|
| 19.1. | Bessel functions of argument $x$ . Finite intervals . . . . .                 | 333 |
| 19.2. | Bessel functions of argument $x$ . Infinite intervals . . . . .               | 339 |
| 19.3. | Bessel functions of arguments $\alpha x + \beta$ , $x^2$ , $x^{-1}$ . . . . . | 349 |
| 19.4. | Bessel functions of other arguments . . . . .                                 | 358 |
| 19.5. | Modified Bessel functions of argument $x$ . . . . .                           | 364 |
| 19.6. | Modified Bessel functions of other arguments . . . . .                        | 372 |
| 19.7. | Bessel functions and modified Bessel functions of<br>variable order . . . . . | 379 |
| 19.8. | Functions related to Bessel functions . . . . .                               | 383 |

## CHAPTER XX

## HYPERGEOMETRIC FUNCTIONS

|       |                                              |     |
|-------|----------------------------------------------|-----|
| 20.1. | Parabolic cylinder functions . . . . .       | 395 |
| 20.2. | Gauss' hypergeometric series . . . . .       | 398 |
| 20.3. | Confluent hypergeometric functions . . . . . | 401 |
| 20.4. | MacRobert's $E$ -function . . . . .          | 414 |
| 20.5. | Meijer's $G$ -function . . . . .             | 417 |

## APPENDIX

|                                                                           |     |
|---------------------------------------------------------------------------|-----|
| Notations and definitions of higher transcendental<br>functions . . . . . | 423 |
| INDEX OF NOTATIONS . . . . .                                              | 449 |

## BESSEL TRANSFORMS

or

integral transforms whose kernels are Bessel functions or functions  
related to Bessel functions.





## CHAPTER VIII

### HANKEL TRANSFORMS

We call

$$g(y; \nu) = \mathfrak{H}_\nu \{f(x); y\} = \int_0^\infty f(x) J_\nu(xy) (xy)^{\frac{1}{2}} dx$$

the *Hankel transform of order  $\nu$*  of  $f(x)$  and take  $y$  to be a positive real variable. For the sake of brevity we often write  $g(y)$  instead of  $g(y; \nu)$ . This form of the Hankel transform has the advantage of reducing to the Fourier sine or cosine transform when  $\nu = \pm \frac{1}{2}$ . Many authors regard

$$\int_0^\infty f(x) J_\nu(xy) x dx$$

or

$$\int_0^\infty f(x) J_\nu [2(xy)^{\frac{1}{2}}] dx$$

as the Hankel transform of order  $\nu$  of  $f(x)$ . The Hankel transform is self-reciprocal [see 8.1(1)] and no table of inverse transforms is required.

Hankel's inversion theorem is proved in detail, and many Hankel transforms are evaluated in Watson's (1922) book on Bessel functions. The theory and application of Hankel transforms is described in several books on Fourier integrals, among which we mention Sneddon (1951) and Titchmarsh (1937).

From the transform pairs given in this chapter further transform pairs may be derived by means of the methods indicated in the introduction to volume I of this work, and also by means of the general formulas given in sec. 8.1. Tricomi (1935) discovered the relation

$$\mathfrak{L} \{t^{\frac{1}{2}\nu-\frac{1}{4}} g[(2t)^{\frac{1}{2}}; \nu]; s\} = s^{-\nu-1} \mathfrak{L} \{t^{\frac{1}{2}\nu-\frac{1}{4}} f[(2t)^{\frac{1}{2}}]; s^{-1}\}$$

between Hankel transforms and Laplace transforms, and this relation may be used to evaluate Hankel transforms by means of the tables of Laplace transforms and inverse Laplace transforms given in chapters IV and V of volume I.

## REFERENCES

- Sneddon, I.N., 1951: *Fourier transforms*, McGraw-Hill, New York.
- Titchmarsh, E.C., 1937: *Introduction to the theory of Fourier integrals*, Oxford.
- Tricomi, Francesco, 1935: *Rend. dei Lincei* (6) 22, 564-571.
- Watson, G. N., 1922: *A treatise on the theory of Bessel functions*, Cambridge.

# HANKEL TRANSFORMS

## 8.1. General formulas

|     |                                                                                                |                                                                                                                                                |
|-----|------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------|
|     | $f(x)$                                                                                         | $\int_0^\infty f(x) J_\nu(xy) (xy)^{\frac{1}{2}} dx$<br>$= g(y; \nu)$<br>$y > 0$                                                               |
| (1) | $\int_0^\infty g(y) J_\nu(xy) (xy)^{\frac{1}{2}} dy$<br>$\operatorname{Re} \nu > -\frac{1}{2}$ | $g(y)$                                                                                                                                         |
| (2) | $f(ax)$<br>$a > 0$                                                                             | $a^{-1} g(a^{-1} y; \nu)$                                                                                                                      |
| (3) | $x^m f(x)$<br>$m = 0, 1, 2, \dots$                                                             | $y^{\frac{1}{2}-\nu} \left( \frac{d}{y dy} \right)^m [y^{\nu-\frac{1}{2}+m} g(y; \nu+m)]$                                                      |
| (4) | $x^m f(x)$<br>$m = 0, 1, 2, \dots$                                                             | $(-1)^m y^{\frac{1}{2}+\nu} \left( \frac{d}{y dy} \right)^m$<br>$\times [y^{-\frac{1}{2}+m-\nu} g(y; \nu-m)]$                                  |
| (5) | $2\nu x^{-1} f(x)$                                                                             | $y g(y; \nu-1) + y g(y; \nu+1)$                                                                                                                |
| (6) | $x^{-1} f(x)$                                                                                  | $y^{\frac{1}{2}-\nu} \int_0^y \eta^{\nu-\frac{1}{2}} g(\eta; \nu-1) d\eta$                                                                     |
| (7) | $x^{-1} f(x)$                                                                                  | $y^{\frac{1}{2}+\nu} \int_y^\infty \eta^{-\nu-\frac{1}{2}} g(\eta; \nu+1) d\eta$                                                               |
| (8) | $x^{-\mu} f(x)$<br>$\operatorname{Re} \nu + 1 > \operatorname{Re} \mu > 0$                     | $2^{1-\mu} [\Gamma(\mu)]^{-1} y^{\frac{1}{2}-\nu}$<br>$\times \int_0^y \eta^{\nu-\mu+\frac{1}{2}} (y^2-\eta^2)^{\mu-1} g(\eta; \nu-\mu) d\eta$ |

## General formulas (cont'd)

|      |                                                                                                                                                                                                                                                                                                      |                                                                                                                                                                       |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------|
|      | $f(x)$                                                                                                                                                                                                                                                                                               | $\int_0^\infty f(x) J_\nu(xy) (xy)^{\frac{1}{2}} dx$<br>$= g(y; \nu)$ <span style="float: right;"><math>y &gt; 0</math></span>                                        |
| (9)  | $x^{-\mu} f(x)$<br>$\operatorname{Re} \nu - 3/2 > \operatorname{Re} \mu > 0$                                                                                                                                                                                                                         | $2^{1-\mu} [\Gamma(\mu)]^{-1} y^{\frac{1}{2}+\nu}$<br>$\times \int_y^\infty \eta^{\frac{1}{2}-\mu-\nu} (\eta^2 - y^2)^{\mu-1} g(\eta; \nu+\mu) d\eta$                 |
| (10) | $2\nu f'(x)$                                                                                                                                                                                                                                                                                         | $(\nu - \frac{1}{2}) y g(y; \nu+1)$<br>$- (\nu + \frac{1}{2}) y g(y; \nu-1)$                                                                                          |
| (11) | $x^{\frac{1}{2}-\nu} \left( \frac{d}{x dx} \right)^m [x^{\nu+m-\frac{1}{2}} f(x)]$<br>$m = 0, 1, 2, \dots$                                                                                                                                                                                           | $y^m g(y; \nu+m)$                                                                                                                                                     |
| (12) | $x^{\frac{1}{2}+\nu} \left( \frac{d}{x dx} \right)^m [x^{\nu-m-\frac{1}{2}} f(x)]$<br>$m = 0, 1, 2, \dots$                                                                                                                                                                                           | $(-y)^m g(y; \nu-m)$                                                                                                                                                  |
| (13) | $x^{\frac{1}{2}-\nu} \int_0^x \xi^{\nu-\mu+\frac{1}{2}} (x^2 - \xi^2)^{\mu-1}$<br>$\times f(\xi) d\xi$<br>$\operatorname{Re} \nu + \frac{1}{2} > \operatorname{Re} \mu > 0$                                                                                                                          | $2^{\mu-1} \Gamma(\mu) y^{-\mu} g(y; \nu-\mu)$                                                                                                                        |
| (14) | $x^{\frac{1}{2}+\nu} \int_x^\infty \xi^{\frac{1}{2}-\mu-\nu} (\xi^2 - x^2)^{\mu-1}$<br>$\times f(\xi) d\xi$<br>$\operatorname{Re} \nu + 1 > \operatorname{Re} \mu > 0$                                                                                                                               | $2^{\mu-1} \Gamma(\mu) y^{-\mu} g(y; \nu+\mu)$                                                                                                                        |
| (15) | $2^\lambda \Gamma(\lambda) x^{\frac{1}{2}-\nu}$<br>$\times \int_0^x \xi^{\frac{1}{2}-\lambda-\mu+\nu} (x^2 - \xi^2)^{\mu-1}$<br>$\times f(\xi) d\xi$<br>$\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \mu > 0$<br>$\operatorname{Re} \nu > \operatorname{Re}(\lambda + \mu) - \frac{1}{2}$ | $2^\mu \Gamma(\mu) y^{\frac{1}{2}-\nu}$<br>$\times \int_0^y \eta^{\frac{1}{2}-\lambda-\mu+\nu} (y^2 - \eta^2)^{\lambda-1}$<br>$\times g(\eta; \nu-\lambda-\mu) d\eta$ |

## General formulas (cont'd)

|      |                                                                                                                                                                                                                                                                |                                                                                                                                                                                |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
|      | $f(x)$                                                                                                                                                                                                                                                         | $\int_0^\infty f(x) J_\nu(xy) (xy)^{\frac{1}{2}} dx$<br>$= g(y; \nu)$ $y > 0$                                                                                                  |
| (16) | $2^\lambda \Gamma(\lambda) x^{\frac{1}{2}+\nu}$<br>$\times \int_x^\infty \xi^{\frac{1}{2}-\lambda-\mu-\nu} (\xi^2 - x^2)^{\mu-1}$<br>$\times f(\xi) d\xi$ $\text{Re } \lambda > 0$<br>$\text{Re } \mu > 0, \text{ Re } \nu >  \text{Re } (\lambda - \mu)  - 1$ | $2^\mu \Gamma(\mu) y^{\frac{1}{2}+\nu}$<br>$\times \int_y^\infty \eta^{\frac{1}{2}-\lambda-\mu-\nu} (\eta^2 - y^2)^{\lambda-1}$<br>$\times g(\eta; \nu + \lambda + \mu) d\eta$ |

## 8.2. Hankel transforms of order zero; Elementary functions

|     |                                                                                |                                                                                                                                      |
|-----|--------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------|
|     | $f(x)$                                                                         | $\int_0^\infty f(x) J_0(xy) (xy)^{\frac{1}{2}} dx$ $y > 0$                                                                           |
| (1) | $x^{-\frac{1}{2}}$                                                             | $y^{-\frac{1}{2}}$                                                                                                                   |
| (2) | $x^{-\frac{1}{2}}$ $0 < x < 1$<br>0 $1 < x < \infty$                           | $y^{\frac{1}{2}} J_0(y) + \frac{1}{2} \pi y^{\frac{1}{2}} [J_1(y) \mathbf{H}_0(y)$<br>$- J_0(y) \mathbf{H}_1(y)]$                    |
| (3) | 0 $0 < x < 1$<br>$x^{-\frac{1}{2}}$ $1 < x < \infty$                           | $y^{-\frac{1}{2}} - y^{\frac{1}{2}} J_0(y) + \frac{1}{2} \pi y^{\frac{1}{2}} [J_0(y) \mathbf{H}_1(y)$<br>$- J_1(y) \mathbf{H}_0(y)]$ |
| (4) | $x^{\frac{1}{2}} (a^2 + x^2)^{-\frac{1}{2}}$ $\text{Re } a > 0$                | $y^{-\frac{1}{2}} e^{-ay}$                                                                                                           |
| (5) | $x^{\frac{1}{2}} (a^2 - x^2)^{-\frac{1}{2}}$ $0 < x < a$<br>0 $a < x < \infty$ | $y^{-\frac{1}{2}} \sin(ay)$                                                                                                          |
| (6) | 0 $0 < x < a$<br>$x^{\frac{1}{2}} (x^2 - a^2)^{-\frac{1}{2}}$ $a < x < \infty$ | $y^{-\frac{1}{2}} \cos(ay)$                                                                                                          |
| (7) | $x^{\frac{1}{2}} (x^2 + a^2)^{-3/2}$ $\text{Re } a > 0$                        | $a^{-1} y^{\frac{1}{2}} e^{-ay}$                                                                                                     |
| (8) | $x^{\frac{1}{2}} (x^4 + a^4)^{-1}$ $ \arg a  < \frac{1}{4} \pi$                | $-a^{-2} y^{\frac{1}{2}} \text{kei}_0(ay)$                                                                                           |

## Elementary functions; Order zero (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                                   | $\int_0^\infty f(x) J_0(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                              |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (9)  | $x^{5/2} (x^4 + a^4)^{-1} \quad  \arg a  < \frac{1}{4} \pi$                                                                                                                                                                                              | $y^{\frac{1}{2}} \ker_0(\alpha y)$                                                                                                                          |
| (10) | $x^{5/2} (x^4 - a^4)^{-1} \quad a > 0$                                                                                                                                                                                                                   | $\frac{1}{2} y^{\frac{1}{2}} [K_0(\alpha y) - \frac{1}{2} \pi Y_0(\alpha y)]$<br>The integral is a Cauchy Principal Value                                   |
| (11) | $x^{-\frac{1}{2}} \frac{(x^2 + a^2)^{\frac{1}{2}} - x}{(x^2 + a^2)^{\frac{1}{2}} + x} \quad \operatorname{Re} a > 0$                                                                                                                                     | $y^{-1/2} + 2a^{-2} y^{-5/2} (\alpha y e^{-\alpha y} + e^{-\alpha y} - 1)$                                                                                  |
| (12) | $x^{\frac{1}{2}} (x^4 + 2a^2 x^2 + b^4)^{-\frac{1}{2}} \quad a > b > 0$                                                                                                                                                                                  | $y^{\frac{1}{2}} I_0[2^{-\frac{1}{2}} (a^2 - b^2)^{\frac{1}{2}} y] \times K_0[2^{-\frac{1}{2}} (a^2 + b^2)^{\frac{1}{2}} y]$                                |
| (13) | $x^{\frac{1}{2}} (x^4 + 2a^2 x^2 + b^4)^{-\frac{1}{2}} \quad b > a > 0$                                                                                                                                                                                  | $y^{\frac{1}{2}} J_0[2^{-\frac{1}{2}} (b^2 - a^2)^{\frac{1}{2}} y] \times K_0[2^{-\frac{1}{2}} (a^2 + b^2)^{\frac{1}{2}} y]$                                |
| (14) | $x^{1/2} (b^2 - x^2)(x^4 \pm 2a^2 x^2 + b^4)^{-3/2} \quad 0 < a < b$                                                                                                                                                                                     | $2^{-1/2} (b^2 \mp a^2)^{-1/2} y^{3/2} \times J_1[(\frac{1}{2} b^2 \mp \frac{1}{2} a^2)^{1/2} y] \times K_0[(\frac{1}{2} b^2 \pm \frac{1}{2} a^2)^{1/2} y]$ |
| (15) | $x^{-\frac{1}{2}} (a^2 - x^2)^{-\frac{1}{2}} \{ [x + (x^2 - a^2)^{\frac{1}{2}}]^{2n} + [x - (x^2 - a^2)^{\frac{1}{2}}]^{2n} \} \quad 0 < x < a$<br>$0 \quad a < x < \infty$                                                                              | $(-1)^n \pi a^{2n+\frac{1}{2}} y^{\frac{1}{2}} [J_n(\frac{1}{2} \alpha y)]^2$                                                                               |
| (16) | $x^{-\frac{1}{2}} (a^2 + x^2)^{-\frac{1}{2}} [(a^2 + x^2)^{\frac{1}{2}} + x]^{2\mu} \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \mu < \frac{3}{4}$                                                                                            | $y^{\frac{1}{2}} a^{2\mu} K_\mu(\frac{1}{2} \alpha y) I_{-\mu}(\frac{1}{2} \alpha y)$                                                                       |
| (17) | $x^{\frac{1}{2}} (x^2 + a^2)^{-\frac{1}{2}} (x^2 + \beta^2)^{-\frac{1}{2}} \times [(x^2 + a^2)^{\frac{1}{2}} + (x^2 + \beta^2)^{\frac{1}{2}}] \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \beta > 0$<br>$\operatorname{Re} \mu < \frac{3}{4}$ | $y^{\frac{1}{2}} (a^2 - \beta^2)^\mu K_\mu[\frac{1}{2} y (a + \beta)] \times I_{-\mu}[\frac{1}{2} y (a - \beta)]$                                           |

## Elementary functions; Order zero (cont'd)

|      | $f(x)$                                                                                                                                                                                                            | $\int_0^\infty f(x) J_0(xy) (xy)^{1/2} dx \quad y > 0$                                                                                                                                         |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (18) | $x^{-1/2} e^{-ax} \quad \operatorname{Re} a > 0$                                                                                                                                                                  | $y^{1/2} (y^2 + a^2)^{-1/2}$                                                                                                                                                                   |
| (19) | $x^{-3/2} (1 - e^{-ax}) \quad \operatorname{Re} a > 0$                                                                                                                                                            | $y^{1/2} \sinh^{-1}(a/y)$                                                                                                                                                                      |
| (20) | $x^{n-1/2} e^{-ax} \quad \operatorname{Re} a > 0$                                                                                                                                                                 | $\frac{n! y^{1/2}}{(a^2 + y^2)^{1/2} n + 1/2} P_n \left[ \frac{a}{(a^2 + y^2)^{1/2}} \right]$                                                                                                  |
| (21) | $x^{2\mu-3/2} e^{-x^2/2} \quad \operatorname{Re} \mu > 0$                                                                                                                                                         | $2^{\mu-1} \Gamma(\mu) y^{1/2} {}_1F_1(\mu; 1; -1/2 y^2)$                                                                                                                                      |
| (22) | $x^{-1} \exp(-2\beta x^{1/2}) \quad \operatorname{Re} \beta > 0$                                                                                                                                                  | $\pi^{-1} \beta y^{-1/2} K_{1/2}(\frac{1}{2} e^{1/2} i \pi \beta^2 y^{-1})$<br>$\times K_{1/2}(\frac{1}{2} e^{-1/2} i \pi \beta^2 y^{-1})$                                                     |
| (23) | $x^{1/2} \exp[-a(x^2 + \beta^2)^{1/2}]$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \beta > 0$                                                                                                           | $ay^{1/2} (y^2 + a^2)^{-3/2} \exp[-\beta(y^2 + a^2)^{1/2}]$<br>$\times [1 + \beta(y^2 + a^2)^{1/2}]$                                                                                           |
| (24) | $x^{1/2} (x^2 + \beta^2)^{-1/2}$<br>$\times \exp[-a(x^2 + \beta^2)^{1/2}]$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \beta > 0$                                                                        | $y^{1/2} (y^2 + a^2)^{-1/2} \exp[-\beta(y^2 + a^2)^{1/2}]$                                                                                                                                     |
| (25) | $x^{1/2} (x^2 + \beta^2)^{-1/2}$<br>$\times \exp[\pm ia(x^2 + \beta^2)^{1/2}]$<br>$a > 0, \quad \operatorname{Re} \beta > 0$                                                                                      | $\pm iy^{1/2} (a^2 - y^2)^{-1/2}$<br>$\times \exp[\pm i\beta(a^2 - y^2)^{1/2}] \quad 0 < y < a$<br>$y^{1/2} (y^2 - a^2)^{-1/2}$<br>$\times \exp[-\beta(y^2 - a^2)^{1/2}] \quad a < y < \infty$ |
| (26) | $\pm ix^{1/2} (b^2 - x^2)^{-1/2}$<br>$\times \exp[\pm ia(b^2 - x^2)^{1/2}]$<br>$0 < y < b$<br>$x^{1/2} (x^2 - b^2)^{-1/2}$<br>$\times \exp[-a(x^2 - b^2)^{1/2}]$<br>$b < y < \infty$<br>$\operatorname{Re} a > 0$ | $y^{1/2} (y^2 + a^2)^{-1/2} \exp[\pm ib(y^2 + a^2)^{1/2}]$                                                                                                                                     |

## Elementary functions; Order zero (cont'd)

|      | $f(x)$                                                                                                                                               | $\int_0^\infty f(x) J_0(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                       |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------|
| (27) | $x^{-\frac{1}{2}} \log x$                                                                                                                            | $-y^{-\frac{1}{2}} \log(2y\gamma)$                                                                                                                  |
| (28) | $x^{-\frac{1}{2}} (x^2 + a^2)^{-\frac{1}{2}} \times \log[x + (x^2 + a^2)^{\frac{1}{2}}] \quad a > 0$                                                 | $y^{\frac{1}{2}} [\frac{1}{2} K_0^2(\frac{1}{2}a\gamma) + \log a I_0(\frac{1}{2}a\gamma) K_0(\frac{1}{2}a\gamma)]$                                  |
| (29) | $x^{-\frac{1}{2}} (x^2 + a^2)^{-\frac{1}{2}} \times \log \frac{(x^2 + a^2)^{\frac{1}{2}} + x}{(x^2 + a^2)^{\frac{1}{2}} - x} \quad \text{Re } a > 0$ | $y^{\frac{1}{2}} K_0^2(\frac{1}{2}a\gamma)$                                                                                                         |
| (30) | $x^{\frac{1}{2}} \log(1 + a^2 x^{-2}) \quad \text{Re } a > 0$                                                                                        | $2y^{-\frac{1}{2}} [y^{-1} - a K_1(a\gamma)]$                                                                                                       |
| (31) | $x^{\frac{1}{2}} \log[ax^{-1} + (1 + a^2 x^{-2})^{\frac{1}{2}}] \quad \text{Re } a > 0$                                                              | $y^{-3/2} (1 - e^{-a\gamma})$                                                                                                                       |
| (32) | $x^{-\frac{1}{2}} \sin(ax) \quad a > 0$                                                                                                              | $y^{\frac{1}{2}} (a^2 - y^2)^{-\frac{1}{2}} \quad 0 < y < a$<br>$0 \quad a < y < \infty$                                                            |
| (33) | $x^{-3/2} \sin(ax) \quad a > 0$                                                                                                                      | $\frac{1}{2} \pi y^{\frac{1}{2}} \quad 0 < y < a$<br>$y^{\frac{1}{2}} \sin^{-1}(a/\gamma) \quad a < y < \infty$                                     |
| (34) | $x^{-\frac{1}{2}} (1+x)^{-1} \sin(1+x)$                                                                                                              | $\frac{1}{2} \pi y^{\frac{1}{2}} J_0(y) \quad 1 \leq y < \infty$                                                                                    |
| (35) | $x^{-\frac{1}{2}} (\beta^2 + x^2)^{-1} \sin(ax) \quad a > 0, \quad \text{Re } \beta > 0$                                                             | $y^{\frac{1}{2}} \beta^{-1} \sinh(\beta a) K_0(\beta y) \quad a < y < \infty$                                                                       |
| (36) | $x^{\frac{1}{2}} (\beta^2 + x^2)^{-1} \sin(ax) \quad a > 0, \quad \text{Re } \beta > 0$                                                              | $\frac{1}{2} \pi y^{\frac{1}{2}} e^{-a\beta} I_0(y\beta) \quad 0 < y < a$                                                                           |
| (37) | $x^{-3/2} e^{-bx} \sin(ax)$                                                                                                                          | $y^{\frac{1}{2}} \sin^{-1} \left( \frac{2a}{r_1 + r_2} \right)$<br>$r_1^2 = b^2 + (a+y)^2, \quad r_1 > 0$<br>$r_2^2 = b^2 + (a-y)^2, \quad r_2 > 0$ |



## Elementary functions; Order zero (cont'd)

|      | $f(x)$                                                                                                                                   | $\int_0^\infty f(x) J_0(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                         |
|------|------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (38) | $x^{\frac{1}{2}} \sin(\frac{1}{2} a^2 x^2) \quad a > 0$                                                                                  | $a^{-2} y^{\frac{1}{2}} \cos(\frac{1}{2} a^{-2} y^2)$                                                                                                                                  |
| (39) | $x^{-3/2} \sin(\frac{1}{2} a^2 x^2)$                                                                                                     | $\frac{1}{2} y^{\frac{1}{2}} \text{si}(\frac{1}{2} a^{-2} y^2)$                                                                                                                        |
| (40) | $x^{-1} e^{-ax^{\frac{1}{2}}} \sin(ax^{\frac{1}{2}})$<br>$ \arg a  < \frac{1}{4} \pi$                                                    | $\frac{1}{2} y^{-\frac{1}{2}} a I_{\frac{1}{2}}(\frac{1}{4} a^2/y) K_{\frac{1}{2}}(\frac{1}{4} a^2/y)$                                                                                 |
| (41) | $x^{\frac{1}{2}} (\beta^2 + x^2)^{-\frac{1}{2}}$<br>$\times \sin[a(\beta^2 + x^2)^{\frac{1}{2}}]$<br>$a > 0, \quad \text{Re } \beta > 0$ | $y^{\frac{1}{2}} (a^2 - y^2)^{-\frac{1}{2}} \cos[\beta(a^2 - y^2)^{\frac{1}{2}}]$<br>$0 < y < a$<br>$0 \quad a < y < \infty$                                                           |
| (42) | $x^{-\frac{1}{2}} \cos(ax) \quad a > 0$                                                                                                  | $0 \quad 0 < y < a$<br>$y^{\frac{1}{2}} (y^2 - a^2)^{-\frac{1}{2}} \quad a < y < \infty$                                                                                               |
| (43) | $x^{-3/2} [1 - \cos(ax)] \quad a > 0$                                                                                                    | $y^{\frac{1}{2}} \cosh^{-1}(a/y) \quad 0 < y < a$<br>$0 \quad a < y < \infty$                                                                                                          |
| (44) | $x^{-\frac{1}{2}} (x^2 + \beta^2)^{-1} \cos(ax)$<br>$a > 0, \quad \text{Re } \beta > 0$                                                  | $\frac{1}{2} \beta^{-1} \pi y^{\frac{1}{2}} e^{-a\beta} I_0(\beta y)$<br>$0 < y < a$                                                                                                   |
| (45) | $x^{\frac{1}{2}} (\beta^2 + x^2)^{-1} \cos(ax)$<br>$a > 0, \quad \text{Re } \beta > 0$                                                   | $y^{\frac{1}{2}} \cosh(\beta a) K_0(\beta y)$<br>$a < y < \infty$                                                                                                                      |
| (46) | $x^{-\frac{1}{2}} e^{-bx} \cos(ax)$                                                                                                      | $y^{\frac{1}{2}} 2^{-\frac{1}{2}} [(b^2 + y^2 - a^2)^2 + 4a^2 b^2]^{-\frac{1}{2}}$<br>$\times \{[(b^2 + y^2 - a^2)^2 + 4a^2 b^2]^{\frac{1}{2}}$<br>$+ b^2 + y^2 - a^2\}^{\frac{1}{2}}$ |
| (47) | $x^{\frac{1}{2}} \cos(\frac{1}{2} a^2 x^2) \quad a > 0$                                                                                  | $a^{-2} y^{\frac{1}{2}} \sin(\frac{1}{2} a^{-2} y^2)$                                                                                                                                  |
| (48) | $x^{-3/2} [1 - \cos(\frac{1}{2} a^2 x^2)]$                                                                                               | $-\frac{1}{2} y^{\frac{1}{2}} \text{Ci}(\frac{1}{2} a^{-2} y^2)$                                                                                                                       |

## Elementary functions; Order zero (cont'd)

|      | $f(x)$                                                                                                                                          | $\int_0^\infty f(x) J_0(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                  |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (49) | $x^{-1} e^{-\alpha x^{\frac{1}{2}}} \cos(\alpha x^{\frac{1}{2}})$<br>$ \arg \alpha  < \frac{1}{4} \pi$                                          | $\frac{1}{2} \alpha y^{-\frac{1}{2}} I_{-\frac{1}{2}}(\frac{1}{4} \alpha^2 y^{-1})$<br>$\times K_{\frac{1}{2}}(\frac{1}{4} \alpha^2 y^{-1})$                                                                    |
| (50) | $x^{\frac{1}{2}} (\beta^2 + x^2)^{-\frac{1}{2}}$<br>$\times \cos[a(\beta^2 + x^2)^{\frac{1}{2}}]$<br>$a > 0, \quad \operatorname{Re} \beta > 0$ | $-y^{\frac{1}{2}} (a^2 - y^2)^{-\frac{1}{2}} \sin[\beta(a^2 - y^2)^{\frac{1}{2}}]$<br>$0 < y < a$<br>$y^{\frac{1}{2}} (y^2 - a^2)^{-\frac{1}{2}} e^{-\beta(y^2 - a^2)^{\frac{1}{2}}}$<br>$a < y < \infty$       |
| (51) | $x^{-3/2} e^{-\alpha x} \sinh(\alpha x) \quad \operatorname{Re} \alpha > 0$                                                                     | $\frac{1}{2} y^{\frac{1}{2}} \log[2 \alpha y^{-1} + (1 + 4 \alpha^2 y^{-2})^{\frac{1}{2}}]$                                                                                                                     |
| (52) | $x^{-\frac{1}{2}} e^{-\alpha x} \sinh(\beta x)$<br>$\operatorname{Re} \alpha >  \operatorname{Re} \beta $                                       | $(\alpha \beta y)^{\frac{1}{2}} r_1^{-1} r_2^{-1} (r_2 - r_1)^{\frac{1}{2}} (r_2 + r_1)^{-\frac{1}{2}}$<br>$r_1 = [y^2 + (\beta - \alpha)^2]^{\frac{1}{2}}$<br>$r_2 = [y^2 + (\beta + \alpha)^2]^{\frac{1}{2}}$ |
| (53) | $x^{\frac{1}{2}} e^{-\alpha/x} \sinh(\alpha x^{-1}) \quad \operatorname{Re} \alpha > 0$                                                         | $2 \alpha y^{-\frac{1}{2}} J_1(2 \alpha^{\frac{1}{2}} y^{\frac{1}{2}}) K_1(2 \alpha^{\frac{1}{2}} y^{\frac{1}{2}})$                                                                                             |
| (54) | $x^{-\frac{1}{2}} e^{-\alpha x} \cosh(\beta x)$<br>$\operatorname{Re} \alpha >  \operatorname{Re} \beta $                                       | $(\alpha \beta y)^{\frac{1}{2}} r_1^{-1} r_2^{-1} (r_2 + r_1)^{\frac{1}{2}} (r_2 - r_1)^{-\frac{1}{2}}$<br>$r_1 = [y^2 + (\beta - \alpha)^2]^{\frac{1}{2}}$<br>$r_2 = [y^2 + (\beta + \alpha)^2]^{\frac{1}{2}}$ |
| (55) | $x^{\frac{1}{2}} \sinh^{-1}(\alpha x^{-1}) \quad \operatorname{Re} \alpha > 0$                                                                  | $y^{-3/2} (1 - e^{-\alpha y})$                                                                                                                                                                                  |
| (56) | $x^{-\frac{1}{2}} (1 + x^2)^{-\frac{1}{2}}$<br>$\times \sinh(2\mu \sinh^{-1} x)$<br>$ \operatorname{Re} \mu  < \frac{1}{2}$                     | $\pi^{-1} \sin(\pi \mu) y^{\frac{1}{2}} [K_{\mu}(\frac{1}{2} y)]^2$                                                                                                                                             |
| (57) | $x^{-\frac{1}{2}} (1 + x^2)^{-\frac{1}{2}}$<br>$\times \cosh(2\mu \sinh^{-1} x)$<br>$ \operatorname{Re} \mu  < \frac{1}{2}$                     | $\frac{1}{2} y^{\frac{1}{2}} K_{\mu}(\frac{1}{2} y)$<br>$\times [I_{\mu}(\frac{1}{2} y) + I_{-\mu}(\frac{1}{2} y)]$                                                                                             |

## 8.3. Hankel transforms of order zero; Higher transcendental functions

|      | $f(x)$                                                                                                                                    | $\int_0^\infty f(x) J_0(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                    |
|------|-------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------|
| (1)  | $x^{\frac{1}{2}} P_n(1-2x^2) \quad 0 < x < 1$<br>$0 \quad 1 < x < \infty$                                                                 | $y^{-\frac{1}{2}} J_{2n+1}(y)$                                                                                                    |
| (2)  | $x^{5/2} P_n(1-2x^2) \quad 0 < x < 1$<br>$0 \quad 1 < x < \infty$                                                                         | $y^{-\frac{1}{2}} (2n+1)^{-1} \times [(n+1) J'_{2n+2}(y) - n J'_{2n}(y)]$                                                         |
| (3)  | $x^{\frac{1}{2}} e^{-\frac{1}{2}x^2} L_n(x^2)$                                                                                            | $(-1)^n e^{-\frac{1}{2}y^2} y^{\frac{1}{2}} L_n(y^2)$                                                                             |
| (4)  | $x^{\frac{1}{2}} \exp(-\frac{1}{2}ax^2) L_n(\frac{1}{2}\beta x^2)$<br>$\text{Re } a > 0$                                                  | $y^{\frac{1}{2}} \frac{(a-\beta)^n}{a^{n+1}} \exp(-\frac{1}{2}a^{-1}x^2) \times L_n \left[ \frac{\beta y^2}{2a(\beta-a)} \right]$ |
| (5)  | $x^{\frac{1}{2}} e^{-x^2} L_n(x^2)$                                                                                                       | $[n!]^{-1} 2^{-2n-1} y^{2n+\frac{1}{2}} e^{-\frac{1}{4}y^2}$                                                                      |
| (6)  | $x^{-\frac{1}{2}} \text{si}(ax) \quad a > 0$                                                                                              | $-y^{-\frac{1}{2}} \sin^{-1}(y/a) \quad 0 < y < a$<br>$0 \quad a < y < \infty$                                                    |
| (7)  | $x^{\frac{1}{2}} \text{si}(a^2 x^2) \quad a > 0$                                                                                          | $-2y^{-3/2} \sin(\frac{1}{4}x^2/a^2)$                                                                                             |
| (8)  | $x^{\frac{1}{2}} \text{Ci}(a^2 x^2) \quad a > 0$                                                                                          | $2y^{-3/2} [1 - \cos(\frac{1}{4}x^2/a^2)]$                                                                                        |
| (9)  | $x^{-\frac{1}{2}} \text{Ci}(a^2 x^2) \quad a > 0$                                                                                         | $y^{-\frac{1}{2}} [\text{Ci}(\frac{1}{4}x^2/a^2) + \log(\frac{1}{4}y^2x^2/a^2)]$                                                  |
| (10) | $x^{\frac{1}{2}} (1+x^2)^{-\nu-1} \times P_\nu[(1-x^2)(1+x^2)^{-1}]$<br>$\text{Re } \nu > 0$                                              | $[2^\nu \Gamma(\nu+1)]^{-2} y^{2\nu+\frac{1}{2}} K_0(y)$                                                                          |
| (11) | $x^{\frac{1}{2}} \{P_{\lambda-\frac{1}{2}}[(1+a^2x^2)^{\frac{1}{2}}]\}^2$<br>$\text{Re } a > 0, \quad  \text{Re } \lambda  < \frac{1}{4}$ | $2\pi^{-2} \cos(\lambda\pi) a^{-1} y^{-\frac{1}{2}} [K_\lambda(\frac{1}{2}y/a)]^2$                                                |

## Higher functions; Order zero (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                | $\int_0^\infty f(x) J_0(xy) (xy)^{\frac{1}{2}} dx \quad \gamma > 0$                                                                                                                                                                                |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (12) | $x^{\frac{1}{2}} P_{\sigma-\frac{1}{2}} [(1+a^2 x^2)^{\frac{1}{2}}]$<br>$\times Q_{\sigma-\frac{1}{2}} [(1+a^2 x^2)^{\frac{1}{2}}]$<br>$\text{Re } a > 0, \quad \text{Re } \sigma > -\frac{1}{4}$                                     | $a^{-1} \gamma^{-\frac{1}{2}} I_{\sigma} (\frac{1}{2} \gamma / a) K_{\sigma} (\frac{1}{2} \gamma / a)$                                                                                                                                             |
| (13) | $x^{\frac{1}{2}} \{P_{\sigma-\frac{1}{2}}^{\mu} [(1+a^2 x^2)^{\frac{1}{2}}]\}^2$<br>$\text{Re } a > 0$<br>$ \text{Re } \sigma  < \frac{1}{4}, \quad \text{Re } \mu < 1$                                                               | $-i \pi^{-1} \gamma^{-3/2} W_{\mu, \sigma} (\gamma / a)$<br>$\times [W_{\mu, \sigma} (e^{\pi i} \gamma / a) - W_{\mu, \sigma} (e^{-\pi i} \gamma / a)]$                                                                                            |
| (14) | $x^{\frac{1}{2}} P_{\sigma-\frac{1}{2}}^{\mu} [(1+a^2 x^2)^{\frac{1}{2}}]$<br>$\times P_{\sigma-\frac{1}{2}}^{-\mu} [(1+a^2 x^2)^{\frac{1}{2}}]$<br>$\text{Re } a > 0, \quad  \text{Re } \sigma  < \frac{1}{4}$                       | $2 \pi^{-1} \gamma^{-3/2} \cos(\sigma \pi)$<br>$\times W_{\mu, \sigma} (\gamma / a) W_{-\mu, \sigma} (\gamma / a)$                                                                                                                                 |
| (15) | $x^{\frac{1}{2}} P_{\sigma-\frac{1}{2}}^{\mu} [(1+a^2 x^2)^{\frac{1}{2}}]$<br>$\times Q_{\sigma-\frac{1}{2}}^{\mu} [(1+a^2 x^2)^{\frac{1}{2}}]$<br>$\text{Re } a > 0$<br>$\text{Re } \sigma > -\frac{1}{4}, \quad \text{Re } \mu < 1$ | $e^{\mu \pi i} \frac{\Gamma(\frac{1}{2} + \sigma - \mu)}{\Gamma(1 + 2\sigma)} \gamma^{-3/2}$<br>$\times W_{\mu, \sigma} (\gamma / a) M_{-\mu, \sigma} (\gamma / a)$                                                                                |
| (16) | $x^{-3/2} [1 - J_0(ax)] \quad a > 0$                                                                                                                                                                                                  | $0 \quad \gamma > a$<br>$\gamma^{\frac{1}{2}} \log(a/\gamma) \quad \gamma < a$                                                                                                                                                                     |
| (17) | $x^{-\frac{1}{2}} J_0(ax) e^{-\beta x}$<br>$\text{Re } \beta >  \text{Im } a $                                                                                                                                                        | $2 \pi^{-1} \gamma^{\frac{1}{2}} K(2 a^{\frac{1}{2}} \gamma^{\frac{1}{2}} / k) k^{-\frac{1}{2}}$<br>$k = [(a + \gamma)^2 + \beta^2]^{\frac{1}{2}}$                                                                                                 |
| (18) | $x^{-\frac{1}{2}} J_1(ax) \quad a > 0$                                                                                                                                                                                                | $a^{-1} \gamma^{\frac{1}{2}} \quad 0 < \gamma < a$<br>$0 \quad a < \gamma < \infty$                                                                                                                                                                |
| (19) | $x^{-3/2} J_1(ax) \quad a > 0$                                                                                                                                                                                                        | $2 \pi^{-1} \gamma^{\frac{1}{2}} E(\gamma/a) \quad 0 < \gamma < a$<br>$\frac{2 \gamma^{3/2}}{\pi a} \left[ K\left(\frac{a}{\gamma}\right) - \left(1 - \frac{a^2}{\gamma^2}\right) E\left(\frac{a}{\gamma}\right) \right]$<br>$a < \gamma < \infty$ |

**Higher functions; Order zero (cont'd)**

|      | $f(x)$                                                                                            | $\int_0^\infty f(x) J_0(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                              |
|------|---------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (20) | $x^{\frac{1}{2}} J_\nu^2(\frac{1}{2}ax) \quad \text{Re } \nu > -\frac{1}{2}$                      | $2\pi^{-1} y^{-\frac{1}{2}} (a^2 - y^2)^{-\frac{1}{2}} \times \cos[2\nu \sin^{-1}(y/a)]$ $0 < y < a$ $0 \quad a < y < \infty$                                                                                                                                                                                               |
| (21) | $x^{\frac{1}{2}} J_0(ax) Y_0(ax) \quad a > 0$                                                     | $0 \quad 0 < y < 2a$ $-2\pi^{-1} y^{-\frac{1}{2}} (y^2 - 4a^2)^{-\frac{1}{2}} \quad 2a < y < \infty$                                                                                                                                                                                                                        |
| (22) | $x^{-\frac{1}{2}} [H_0(ax) - Y_0(ax)] \quad a > 0$                                                | $4\pi^{-1} (a+y)^{-1} y^{\frac{1}{2}} \times K[ a-y  (a+y)^{-1}]$                                                                                                                                                                                                                                                           |
| (23) | $x^{-\frac{1}{2}} \cosh(\beta x) K_0(ax) \quad \text{Re } a >  \text{Re } \beta $                 | $y^{\frac{1}{2}} (u+v)^{-\frac{1}{2}} K(k)$ $u = \frac{1}{2} \{[(a^2 + \beta^2 + y^2)^2 - 4a^2\beta^2]^{\frac{1}{2}} + a^2 - \beta^2 - y^2\}$ $v = \frac{1}{2} \{[(a^2 + \beta^2 + y^2)^2 - 4a^2\beta^2]^{\frac{1}{2}} - a^2 + \beta^2 + y^2\}$ $k^2 = v(u+v)^{-1}$                                                         |
| (24) | $x^{-\frac{1}{2}} \sinh(\beta x) K_1(ax) \quad \text{Re } a >  \text{Re } \beta $                 | $a^{-1} y^{\frac{1}{2}} [u E(k) - K(k) E(u) + K(k) \text{sn } u \text{dn } u / (\text{cn } u)]$ $\text{cn}^2 u = 2y^2 \{[(a^2 + \beta^2 + y^2)^2 - 4a^2\beta^2]^{\frac{1}{2}} - a^2 + \beta^2 + y^2\}^{-1}$ $k^2 = \frac{1}{2} \{1 - (a^2 - \beta^2 - y^2) \times [(a^2 + \beta^2 + y^2)^2 - 4a^2\beta^2]^{-\frac{1}{2}}\}$ |
| (25) | $x^{\frac{1}{2}} J_0(ax) K_0(\beta x) \quad \text{Re } \beta >  \text{Im } a $                    | $y^{\frac{1}{2}} (\beta^4 + a^4 + y^4 - 2a^2y^2 + 2a^2\beta^2 + 2\beta^2y^2)^{-\frac{1}{2}}$                                                                                                                                                                                                                                |
| (26) | $x^{3/2} J_1(ax) K_0(\beta x) \quad \text{Re } \beta \geq  \text{Im } a , \quad \text{Re } a > 0$ | $2ay^{\frac{1}{2}} (a^2 + \beta^2 - y^2) \times [(a^2 + \beta^2 + y^2)^2 - 4y^2a^2]^{-3/2}$                                                                                                                                                                                                                                 |

**Higher functions; Order zero (cont'd)**

|      | $f(x)$                                                                                                            | $\int_0^\infty f(x) J_0(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                      |
|------|-------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (27) | $x^{\frac{1}{2}} I_0(ax) K_0(\beta x)$<br>$\text{Re } \beta > \text{Re } a$                                       | $y^{\frac{1}{2}} (a^4 + \beta^4 + y^4 - 2a^2\beta^2 + 2a^2y^2 + 2\beta^2y^2)^{-\frac{1}{2}}$                                                                                                                                                                                                        |
| (28) | $x^{3/2} I_0(ax) K_1(\beta x)$<br>$\text{Re } \beta >  \text{Re } a $                                             | $2y^{\frac{1}{2}} \beta (\beta^2 + y^2 - a^2) \times [(y^2 + a^2 + \beta^2)^2 - 4a^2\beta^2]^{-3/2}$                                                                                                                                                                                                |
| (29) | $x^{\frac{1}{2}} I_1(ax) K_1(\beta x)$<br>$\text{Re } \beta > \text{Re } a > 0$                                   | $2^{-1} y^{\frac{1}{2}} a^{-1} \beta^{-1} \{ (a^2 + \beta^2 + y^2) \times [(a^2 + \beta^2 + y^2)^2 - 4a^2\beta^2]^{-\frac{1}{2}} - 1 \}$                                                                                                                                                            |
| (30) | $x^{\frac{1}{2}} K_\mu(ax) I_\mu(\beta x)$<br>$\text{Re } \mu > -\frac{1}{2}, \text{Re } a >  \text{Re } \beta $  | $y^{\frac{1}{2}} r_1^{-1} r_2^{-1} (r_2 - r_1)^\mu (r_2 + r_1)^{-\mu}$<br>$r_1 = [y^2 + (\beta - a)^2]^{\frac{1}{2}}$<br>$r_2 = [y^2 + (\beta + a)^2]^{\frac{1}{2}}$                                                                                                                                |
| (31) | $x^{-\frac{1}{2}} I_\mu(\frac{1}{2}ax) K_\mu(\frac{1}{2}ax)$<br>$\text{Re } a > 0, \text{Re } \mu > -\frac{1}{2}$ | $a^{-1} y^{\frac{1}{2}} P_{\mu-\frac{1}{2}}[(1+y^2/a^2)^{\frac{1}{2}}] \times Q_{\mu-\frac{1}{2}}[(1+y^2/a^2)^{\frac{1}{2}}]$                                                                                                                                                                       |
| (32) | $x^{\frac{1}{2}} K_0^2(ax)$<br>$\text{Re } a > 0$                                                                 | $y^{-\frac{1}{2}} (y^2 + 4a^2)^{-\frac{1}{2}} \log \frac{(y^2 + 4a^2)^{\frac{1}{2}} + y}{(y^2 + 4a^2)^{\frac{1}{2}} - y}$                                                                                                                                                                           |
| (33) | $x^{\frac{1}{2}} K_\mu^2(ax)$<br>$\text{Re } a > 0,  \text{Re } \mu  < 1$                                         | $\pi 2^{-1-2\mu} a^{-2\mu} (\sin \mu\pi)^{-1} y^{-\frac{1}{2}} \times (y^2 + 4a^2)^{-\frac{1}{2}} \{ [(y^2 + 4a^2)^{\frac{1}{2}} + y]^{2\mu} - [(y^2 + 4a^2)^{\frac{1}{2}} - y]^{2\mu} \}$                                                                                                          |
| (34) | $x^{-\frac{1}{2}} J_\mu(a^2x^{-1}) J_{-\mu}(a^2x^{-1})$<br>$a > 0,  \text{Re } \mu  < \frac{1}{4}$                | $-i \csc(2\mu\pi) y^{-\frac{1}{2}} \times [e^{2\pi i \mu} J_{2\mu}(2ay^{\frac{1}{2}} e^{-\frac{1}{2}\pi i}) \times J_{-2\mu}(2ay^{\frac{1}{2}} e^{\frac{1}{2}\pi i}) - e^{-2\pi i \mu} J_{2\mu}(2ay^{\frac{1}{2}} e^{\frac{1}{2}\pi i}) \times J_{-2\mu}(2ay^{\frac{1}{2}} e^{-\frac{1}{2}\pi i})]$ |

## Higher functions; Order zero (cont'd)

|      | $f(x)$                                                                                                                                                                                      | $\int_0^\infty f(x) J_0(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                               |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (35) | $x^{-\frac{1}{2}} [J_\mu^2(a^2 x^{-1}) - J_{-\mu}^2(a^2 x^{-1})]$<br>$a > 0, \quad  \operatorname{Re} \mu  < \frac{1}{4}$                                                                   | $\sec(\mu\pi) y^{-\frac{1}{2}}$<br>$\times [J_{2\mu}(2ay^{\frac{1}{2}} e^{\frac{1}{4}\pi i}) J_{2\mu}(2ay^{\frac{1}{2}} e^{-\frac{1}{4}\pi i})$<br>$- J_{-2\mu}(2ay^{\frac{1}{2}} e^{\frac{1}{4}\pi i}) J_{-2\mu}(2ay^{\frac{1}{2}} e^{-\frac{1}{4}\pi i})]$ |
| (36) | $x^{-\frac{1}{2}} H_\mu^{(1)}(a^2 x^{-1}) H_\mu^{(2)}(a^2 x^{-1})$<br>$ \arg a  < \frac{1}{4}\pi, \quad  \operatorname{Re} \mu  < \frac{1}{4}$                                              | $16\pi^{-2} \cos(\mu\pi) y^{-\frac{1}{2}}$<br>$\times K_{2\mu}(2ay^{\frac{1}{2}} e^{\frac{1}{4}\pi i}) K_{2\mu}(2ay^{\frac{1}{2}} e^{-\frac{1}{4}\pi i})$                                                                                                    |
| (37) | $x^{-\frac{1}{2}} I_\mu(a^2 x^{-1}) K_\mu(a^2 x^{-1})$<br>$ \arg a  < \frac{1}{4}\pi, \quad \operatorname{Re} \mu > -\frac{1}{4}$                                                           | $2y^{-\frac{1}{2}} J_{2\mu}(2ay^{\frac{1}{2}}) K_{2\mu}(2ay^{\frac{1}{2}})$                                                                                                                                                                                  |
| (38) | $x^{-\frac{1}{2}} J_\mu(ax^{\frac{1}{2}}) K_\mu(ax^{\frac{1}{2}})$<br>$ \arg a  < \frac{1}{4}\pi, \quad \operatorname{Re} \mu > -1$                                                         | $\frac{1}{2} y^{-\frac{1}{2}} I_{\frac{1}{2}\mu}(\frac{1}{4} a^2 y^{-1}) K_{\frac{1}{2}\mu}(\frac{1}{4} a^2 y^{-1})$                                                                                                                                         |
| (39) | $x^{-\frac{1}{2}} Y_0(ax^{\frac{1}{2}}) K_0(ax^{\frac{1}{2}})$<br>$ \arg a  < \frac{1}{4}\pi$                                                                                               | $-\frac{1}{2} \pi^{-1} y^{-\frac{1}{2}} [K_0(\frac{1}{4} a^2 / y)]^2$                                                                                                                                                                                        |
| (40) | $x^{-\frac{1}{2}} Y_\mu(ax^{\frac{1}{2}}) K_\mu(ax^{\frac{1}{2}})$<br>$ \arg a  < \frac{1}{4}\pi, \quad  \operatorname{Re} \mu  < 1$                                                        | $-\frac{1}{2} y^{-\frac{1}{2}} \sec(\frac{1}{2}\mu\pi) K_{\frac{1}{2}\mu}(\frac{1}{4} a^2 y^{-1})$<br>$\times [\pi^{-1} K_{\frac{1}{2}\mu}(\frac{1}{4} a^2 y^{-1})$<br>$+ \sin(\frac{1}{2}\mu\pi) I_{\frac{1}{2}\mu}(\frac{1}{4} a^2 y^{-1})]$               |
| (41) | $x^{-\frac{1}{2}} K_\mu(ae^{\frac{1}{4}\pi i} x^{\frac{1}{2}})$<br>$\times K_\mu(ae^{-\frac{1}{4}\pi i} x^{\frac{1}{2}})$<br>$ \arg a  < \frac{1}{4}\pi, \quad  \operatorname{Re} \mu  < 1$ | $2^{-4} \pi^2 [\cos(\frac{1}{2}\mu\pi)]^{-1}$<br>$\times H_{\frac{1}{2}\mu}^{(1)}(\frac{1}{4} a^2 / y) H_{\frac{1}{2}\mu}^{(2)}(\frac{1}{4} a^2 / y)$                                                                                                        |
| (42) | $x^{-\frac{1}{2}} D_n(ax) D_{n+1}(ax)$<br>$ \arg a  < \frac{1}{4}\pi$                                                                                                                       | $(-1)^n y^{-\frac{1}{2}} D_n(y/a) D_{n+1}(y/a)$                                                                                                                                                                                                              |
| (43) | $x^{-1} D_\nu(a^{\frac{1}{2}} x^{\frac{1}{2}}) D_{-\nu-1}(a^{\frac{1}{2}} x^{\frac{1}{2}})$<br>$\operatorname{Re} a > 0$                                                                    | $2^{-3/2} \pi a^{-1/2} y^{1/2}$<br>$\times P_{-\frac{1}{2}}^{\nu/2+1/4} [(1+4y^2/a^2)^{1/2}]$<br>$\times P_{-\frac{1}{2}}^{-\nu/2-1/4} [(1+4y^2/a^2)^{1/2}]$                                                                                                 |

**Higher functions; Order zero (cont'd)**

|      | $f(x)$                                                                                                                                              | $\int_0^\infty f(x) J_0(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                    |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (44) | $x^{-3/2} W_{\kappa, \mu}(ax) M_{-\kappa, \mu}(ax)$<br>$\text{Re } a > 0$<br>$\text{Re } \mu > -\frac{1}{2}, \quad \text{Re } \kappa < \frac{3}{4}$ | $e^{-i\kappa\pi} \frac{\Gamma(1+2\mu)}{\Gamma(\frac{1}{2}+\mu+\kappa)} y^{\frac{1}{2}}$<br>$\times P_{\mu-\frac{1}{2}}^{\kappa} [(1+y^2/a^2)^{\frac{1}{2}}]$<br>$\times Q_{\mu-\frac{1}{2}}^{\kappa} [(1+y^2/a^2)^{\frac{1}{2}}]$ |
| (45) | $x^{-3/2} W_{k, \mu}(ax) W_{-k, \mu}(ax)$<br>$\text{Re } a > 0, \quad -\frac{1}{2} < \text{Re } \mu < \frac{1}{2}$                                  | $\frac{1}{2}\pi \cos(\mu\pi) y^{\frac{1}{2}} P_{\mu-\frac{1}{2}}^k [(1+y^2/a^2)^{\frac{1}{2}}]$<br>$\times P_{\mu-\frac{1}{2}}^{-k} [(1+y^2/a^2)^{\frac{1}{2}}]$                                                                  |
| (46) | $x^{\frac{1}{2}} {}_1F_1(\lambda; 1; -x^2) \quad \text{Re } \lambda > 0$                                                                            | $[2^{2\lambda-1} \Gamma(\lambda)]^{-1} y^{2\lambda-3/2} \exp(-\frac{1}{4}y^2)$                                                                                                                                                    |

**8.4. Hankel transforms of order unity**

|     | $f(x)$                                                                                    | $\int_0^\infty f(x) J_1(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$ |
|-----|-------------------------------------------------------------------------------------------|----------------------------------------------------------------|
| (1) | $x^{-\frac{1}{2}} \quad 0 < x < a$<br>$0 \quad a < x < \infty$                            | $y^{-\frac{1}{2}} [1 - J_0(ay)]$                               |
| (2) | $0 \quad 0 < x < a$<br>$x^{-\frac{1}{2}} \quad a < x < \infty$                            | $y^{-\frac{1}{2}} J_0(ay)$                                     |
| (3) | $x^{-\frac{1}{2}} (x^2 + a^2)^{-\frac{1}{2}} \quad \text{Re } a > 0$                      | $a^{-1} y^{-\frac{1}{2}} (1 - e^{-ay})$                        |
| (4) | $x^{-\frac{1}{2}} (a^2 - x^2)^{-\frac{1}{2}} \quad 0 < x < a$<br>$0 \quad a < x < \infty$ | $a^{-1} y^{-\frac{1}{2}} [1 - \cos(ay)]$                       |
| (5) | $0 \quad 0 < x < a$<br>$x^{-\frac{1}{2}} (x^2 - a^2)^{-\frac{1}{2}} \quad a < x < \infty$ | $a^{-1} y^{-\frac{1}{2}} \sin(ay)$                             |



## Order unity (cont'd)

|      | $f(x)$                                                                                                                                      | $\int_0^\infty f(x) J_1(xy) (xy)^{1/2} dx \quad y > 0$                                  |
|------|---------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------|
| (6)  | $x^{-1/2} e^{-ax} \quad \operatorname{Re} a > 0$                                                                                            | $y^{-1/2} [1 - a(a^2 + y^2)^{-1/2}]$                                                    |
| (7)  | $x^{-1/2} e^{-1/4 ax^2} \quad \operatorname{Re} a > 0$                                                                                      | $y^{-1/2} (1 - e^{-y^2/a})$                                                             |
| (8)  | $x^{3/2} e^{-ax^2/4} \quad \operatorname{Re} a > 0$                                                                                         | $4a^{-2} y^{3/2} e^{-y^2/a}$                                                            |
| (9)  | $x^{-1/2} \exp[-a(x^2 + \beta^2)^{1/2}]$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \beta > 0$                                    | $y^{-1/2} [e^{-\beta a} - a(a^2 + y^2)^{-1/2}$<br>$\times e^{-\beta(a^2 + y^2)^{1/2}}]$ |
| (10) | $x^{-1/2} (\beta^2 + x^2)^{-1/2}$<br>$\times \exp[-a(\beta^2 + x^2)^{1/2}]$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \beta > 0$ | $y^{-1/2} \beta^{-1} [e^{-\beta a} - e^{-\beta(a^2 + y^2)^{1/2}}]$                      |
| (11) | $x^{-1/2} \log x$                                                                                                                           | $-y^{-1/2} \log(1/2 y y)$                                                               |
| (12) | $x^{-1/2} \log(a^2 + x^2)$                                                                                                                  | $2y^{-1/2} [K_0(ay) + \log a]$                                                          |
| (13) | $x^{-1/2} \log(1 + x^4)^{1/2}$                                                                                                              | $2y^{-1/2} \ker_0 y$                                                                    |
| (14) | $x^{-1/2} \sin(ax) \quad a > 0$                                                                                                             | $0 \quad 0 < y < a$<br>$ay^{-1/2} (y^2 - a^2)^{-1/2} \quad a < y < \infty$              |
| (15) | $x^{-3/2} e^{-ax} \sin(bx)$                                                                                                                 | $by^{-1/2} (1 - r)$<br>$b^2 = \frac{y^2}{1 - r^2} - \frac{a^2}{r^2}$                    |
| (16) | $x^{-1/2} \sin(1/4 ax^2) \quad a > 0$                                                                                                       | $y^{-1/2} \sin(y^2/a)$                                                                  |
| (17) | $x^{-1/2} \sin^2(1/4 ax^2) \quad a > 0$                                                                                                     | $1/2 y^{-1/2} \cos(1/2 y^2/a)$                                                          |

## Order unity (cont'd)

|      | $f(x)$                                                                                                                    | $\int_0^\infty f(x) J_1(xy) (xy)^{\frac{1}{2}} dx \quad \gamma > 0$                                                                                                                                                            |
|------|---------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (18) | $x^{-\frac{1}{2}} (x^2 + a^2)^{-\frac{1}{2}} \times \sin [b(x^2 + a^2)^{\frac{1}{2}}]$<br>$\text{Re } a > 0, \quad b > 0$ | $a^{-1} \gamma^{-\frac{1}{2}} \{ \sin(ab) - \sin[a(b^2 - \gamma^2)^{\frac{1}{2}}] \}$<br>$0 < \gamma < b$<br>$a^{-1} \gamma^{-\frac{1}{2}} \sin(ab) \quad b < \gamma < \infty$                                                 |
| (19) | $x^{-\frac{1}{2}} \cos(ax) \quad a > 0$                                                                                   | $\gamma^{-\frac{1}{2}} [1 - a(a^2 - \gamma^2)^{-\frac{1}{2}}]$<br>$\gamma^{-\frac{1}{2}} \quad 0 < \gamma < a$<br>$a < \gamma < \infty$                                                                                        |
| (20) | $x^{-\frac{1}{2}} \cos(\frac{1}{4}ax^2) \quad a > 0$                                                                      | $2\gamma^{-\frac{1}{2}} \sin^2(\frac{1}{2}\gamma^2/a)$                                                                                                                                                                         |
| (21) | $x^{-\frac{1}{2}} (x^2 + a^2)^{-\frac{1}{2}} \times \cos [b(x^2 + a^2)^{\frac{1}{2}}]$<br>$\text{Re } a > 0, \quad b > 0$ | $a^{-1} \gamma^{-\frac{1}{2}} \{ -\cos[a(b^2 - \gamma^2)^{\frac{1}{2}}] + \cos(ab) \}$<br>$0 < \gamma < b$<br>$a^{-1} \gamma^{-\frac{1}{2}} \{ \cos(ab) - \exp[-a(\gamma^2 - b^2)^{\frac{1}{2}}] \} \quad b < \gamma < \infty$ |
| (22) | $x^{-\frac{1}{2}} \tan^{-1}(x^2)$                                                                                         | $-2\gamma^{\frac{1}{2}} \text{kei}_0 \gamma$                                                                                                                                                                                   |
| (23) | $x^{3/2} P_n(1 - 2x^2) \quad 0 < x < 1$<br>$0 \quad 1 < x < \infty$                                                       | $(2n+1)^{-1} \gamma^{-\frac{1}{2}} [(n+1) J_{2n+2}(\gamma) - n J_{2n}(\gamma)]$                                                                                                                                                |
| (24) | $x^{-\frac{1}{2}} [D_n(ax)]^2 \quad  \arg a  < \frac{1}{4}\pi$                                                            | $(-1)^{n-1} \gamma^{-\frac{1}{2}} [D_n(\gamma/a)]^2$                                                                                                                                                                           |
| (25) | $x^{-\frac{1}{2}} \text{si}(a^2 x^2) \quad a > 0$                                                                         | $\gamma^{-\frac{1}{2}} [-\text{si}(\frac{1}{4}x^2/a^2) - \frac{1}{2}\pi]$                                                                                                                                                      |
| (26) | $x^{-\frac{1}{2}} J_0(ax) \quad a > 0$                                                                                    | $0 \quad 0 < \gamma < a$<br>$\gamma^{-\frac{1}{2}} \quad a < \gamma < \infty$                                                                                                                                                  |
| (27) | $x^{-3/2} J_0(ax)$                                                                                                        | $2a\pi^{-1} \gamma^{-\frac{1}{2}} [E(\gamma/a) - (1 - \gamma^2/a^2) K(\gamma/a)]$<br>$0 < \gamma < a$<br>$2\pi^{-1} \gamma^{\frac{1}{2}} E(a/\gamma) \quad a < \gamma < \infty$                                                |

## Order unity (cont'd)

|      | $f(x)$                                    | $\int_0^\infty f(x) J_1(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                        |
|------|-------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------|
| (28) | $x^{-5/2} [J_0(ax) - 1] \quad a > 0$      | $-\frac{1}{4} y^{3/2} [1 + 2 \log(a/y)] \quad 0 < y < a$<br>$-\frac{1}{4} y^{1/2} a^2 \quad a < y < \infty$                                          |
| (29) | $x^{-1/2} J_0(ax) J_0(bx) \quad a, b > 0$ | $0 \quad 0 < y <  a - b $<br>$\pi^{-1} y^{-1/2} \cos^{-1}[(a^2 + b^2 - y^2)/(2ab)] \quad  a - b  < y < a + b$<br>$y^{-1/2} \quad a + b < y < \infty$ |
| (30) | $x^{-5/2} J_1(ax) \quad a > 0$            | $\frac{y^{1/2}(y+a)}{\pi} \left[ E\left(\frac{2iy^{1/2}a^{1/2}}{ y-a }\right) - K\left(\frac{2iy^{1/2}a^{1/2}}{ y-a }\right) \right]$                |
| (31) | $x^{-1/2} Y_0(ax) \quad a > 0$            | $-\pi^{-1} y^{-1/2} \log(1 - y^2/a^2) \quad 0 < y < a$                                                                                               |
| (32) | $x^{1/2} \text{kei}_0 x$                  | $-\frac{1}{2} y^{-1/2} \tan^{-1}(y^2)$                                                                                                               |
| (33) | $x^{-1/2} \text{ker}_0 x$                 | $\frac{1}{2} y^{-1/2} \log(1 + y^4)^{1/2}$                                                                                                           |

HANKEL TRANSFORMS OF ORDER  $\nu$ 

## 8.5. Algebraic functions and powers with arbitrary index

|     | $f(x)$                                                                     | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                   |
|-----|----------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $1 \quad 0 < x < 1$<br>$0 \quad 1 < x < \infty$<br>$\text{Re } \nu > -3/2$ | $2^{1/2} y^{-1} \frac{\Gamma(\frac{3}{4} + \frac{1}{2} \nu)}{\Gamma(\frac{1}{4} + \frac{1}{2} \nu)} + (\nu - \frac{1}{2}) J_\nu(y)$<br>$\times S_{-\frac{1}{2}, \nu-1}(y) - J_{\nu-1}(y) S_{\frac{1}{2}, \nu}(y)$ |

## Algebraic functions (cont'd)

|     | $f(x)$                                                                                                     | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad \gamma > 0$                                                                                                                                                                                                                                                                                     |
|-----|------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (2) | $0 \quad 0 < x < 1$<br>$1 \quad 1 < x < \infty$                                                            | $J_{\nu-1}(\gamma) S_{\frac{1}{2}, \nu}(\gamma)$<br>$+ (\frac{1}{2} - \nu) J_\nu(\gamma) S_{-\frac{1}{2}, \nu-1}(\gamma)$                                                                                                                                                                                                                                |
| (3) | $x^{-\frac{1}{2}} \quad \text{Re } \nu > -1$                                                               | $\gamma^{-\frac{1}{2}}$                                                                                                                                                                                                                                                                                                                                  |
| (4) | $x^{\frac{1}{2}-\nu} \quad 0 < x < 1$<br>$0 \quad 1 < x < \infty$                                          | $\frac{2^{1-\nu} \gamma^{\nu-3/2}}{\Gamma(\nu)} - \gamma^{-1/2} J_{\nu-1}(\gamma)$                                                                                                                                                                                                                                                                       |
| (5) | $x^{\nu-\frac{1}{2}} \quad 0 < x < 1$<br>$0 \quad 1 < x < \infty$<br>$\text{Re } \nu > -\frac{1}{2}$       | $2^{\nu-1} \gamma^{\frac{1}{2}-\nu} \pi^{\frac{1}{2}} \Gamma(\nu+\frac{1}{2})$<br>$\times [J_\nu(\gamma) \mathbf{H}_{\nu-1}(\gamma) - \mathbf{H}_\nu(\gamma) J_{\nu-1}(\gamma)]$                                                                                                                                                                         |
| (6) | $x^{\nu+\frac{1}{2}} \quad 0 < x < 1$<br>$0 \quad 1 < x < \infty$<br>$\text{Re } \nu > -1$                 | $\gamma^{-\frac{1}{2}} J_{\nu+1}(\gamma)$                                                                                                                                                                                                                                                                                                                |
| (7) | $x^\mu \quad -\text{Re } \nu - 3/2 < \text{Re } \mu < -1/2$                                                | $2^{\mu+\frac{1}{2}} \gamma^{-\mu-1} \frac{\Gamma(\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{3}{4})}{\Gamma(\frac{1}{2}\nu - \frac{1}{2}\mu + \frac{1}{4})}$                                                                                                                                                                                                |
| (8) | $x^\mu \quad 0 < x < 1$<br>$0 \quad 1 < x < \infty$<br>$\text{Re } (\mu + \nu) > -3/2$                     | $\gamma^{-\mu-1} \left[ (\nu + \mu - \frac{1}{2}) \gamma J_\nu(\gamma) \right.$<br>$\times S_{\mu-\frac{1}{2}, \nu-1}(\gamma) - \gamma J_{\nu-1}(\gamma) S_{\mu+\frac{1}{2}, \nu}(\gamma)$<br>$\left. + 2^{\mu+\frac{1}{2}} \frac{\Gamma(\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{3}{4})}{\Gamma(\frac{1}{2}\nu - \frac{1}{2}\mu + \frac{1}{4})} \right]$ |
| (9) | $x^{\nu-\frac{1}{2}} (x+a)^{-1} \quad  \arg a  < \pi$<br>$-1/2 < \text{Re } \nu < 3/2, \quad \nu \neq 1/2$ | $\frac{1}{2} \pi a^\nu \sec(\nu\pi) \gamma^{\frac{1}{2}} [\mathbf{H}_{-\nu}(a\gamma)$<br>$- Y_{-\nu}(a\gamma)]$                                                                                                                                                                                                                                          |

## Algebraic functions (cont'd)

|      | $f(x)$                                                                                                                            | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                                                                                                                                                                           |
|------|-----------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (10) | $x^{\rho-3/2}(x+a)^{-\mu-1} \quad  \arg a  < \pi$<br>$\operatorname{Re}(\rho + \nu) > 0$<br>$\operatorname{Re}(\rho - \mu) < 5/2$ | $\frac{y^{\frac{1}{2}} \pi a^{\rho-\mu-1}}{\sin(\rho + \nu - \mu) \pi \Gamma(\mu+1)}$ $\times \left\{ \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{1}{2} a y)^{\nu+2m} \Gamma(\rho + \nu + 2m)}{m! \Gamma(\nu + m + 1) \Gamma(\rho + \nu - \mu + 2m)} \right.$ $- \sum_{m=0}^{\infty} \frac{(\frac{1}{2} a y)^{\mu+1-\rho+m} \Gamma(\mu + m + 1)}{m! \Gamma[\frac{1}{2}(\mu + \nu - \rho + m + 3)]}$ $\times \left. \frac{\sin \frac{1}{2}(\rho + \nu - \mu - m) \pi}{\Gamma[\frac{1}{2}(\mu - \nu - \rho + m + 3)]} \right\}$ |
| (11) | $x^{-\frac{1}{2}}(x^2 + a^2)^{-\frac{1}{2}}$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1$                       | $y^{\frac{1}{2}} I_{\frac{1}{2}\nu}(\frac{1}{2} a y) K_{\frac{1}{2}\nu}(\frac{1}{2} a y)$                                                                                                                                                                                                                                                                                                                                                                                                                                 |
| (12) | $x^{\nu+\frac{1}{2}}(x^2 + a^2)^{-1}$<br>$\operatorname{Re} a > 0, \quad -1 < \operatorname{Re} \nu < 3/2$                        | $a^\nu y^{\frac{1}{2}} K_\nu(a y)$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        |
| (13) | $x^{\nu-\frac{1}{2}}(x^2 + a^2)^{-1}$<br>$\operatorname{Re} a > 0, \quad -1/2 < \operatorname{Re} \nu < 5/2$                      | $\frac{1}{2} \pi a^{\nu-1} \sec(\nu \pi) y^{\frac{1}{2}} [I_\nu(a y) - L_{-\nu}(a y)]$                                                                                                                                                                                                                                                                                                                                                                                                                                    |
| (14) | $x^{-\nu-\frac{1}{2}}(x^2 + a^2)^{-1}$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$                   | $\frac{1}{2} \pi a^{-\nu-1} y^{\frac{1}{2}} [I_\nu(a y) - L_\nu(a y)]$                                                                                                                                                                                                                                                                                                                                                                                                                                                    |
| (15) | $x^{\nu+\frac{1}{2}}(x^2 + a^2)^{-\frac{1}{2}}$<br>$\operatorname{Re} a > 0, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}$      | $2^{\frac{1}{2}} \pi^{-\frac{1}{2}} a^{\nu+\frac{1}{2}} K_{\nu+\frac{1}{2}}(a y)$                                                                                                                                                                                                                                                                                                                                                                                                                                         |
| (16) | $x^{\frac{1}{2}-\nu}(x^2 + a^2)^{-\frac{1}{2}}$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$          | $\pi^{\frac{1}{2}} 2^{-\frac{1}{2}} a^{\frac{1}{2}-\nu} [I_{\nu-\frac{1}{2}}(a y) - L_{\nu-\frac{1}{2}}(a y)]$                                                                                                                                                                                                                                                                                                                                                                                                            |
| (17) | $x^{-\nu-\frac{1}{2}}(x^2 + a^2)^{-\nu-\frac{1}{2}}$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$     | $2^\nu a^{-2\nu} y^{\frac{1}{2}+\nu} \frac{\Gamma(\nu+1)}{\Gamma(2\nu+1)}$ $\times I_\nu(\frac{1}{2} a y) K_\nu(\frac{1}{2} a y)$                                                                                                                                                                                                                                                                                                                                                                                         |

## Algebraic functions (cont'd)

|      | $f(x)$                                                                                                                                                  | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad \gamma > 0$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (18) | $x^{\nu+\frac{1}{2}}(x^2+a^2)^{-\nu-\frac{1}{2}}$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$                              | $\frac{\pi^{\frac{1}{2}} \gamma^{\nu-\frac{1}{2}}}{2^\nu e^{a\gamma} \Gamma(\nu+\frac{1}{2})}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |
| (19) | $x^{\nu+1/2}(x^2+a^2)^{-\nu-3/2}$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1$                                                        | $\frac{\gamma^{\nu+\frac{1}{2}} \pi^{\frac{1}{2}}}{2^{\nu+1} a e^{a\gamma} \Gamma(\nu+3/2)}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     |
| (20) | $x^{\nu+\frac{1}{2}}(x^2+a^2)^{-\mu-1}$<br>$\operatorname{Re} a > 0$<br>$-1 < \operatorname{Re} \nu < 2\operatorname{Re} \mu + 3/2$                     | $\frac{a^{\nu-\mu} \gamma^{\mu+\frac{1}{2}} K_{\nu-\mu}(a\gamma)}{2^\mu \Gamma(\mu+1)}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |
| (21) | $x^{\lambda-3/2}(x^2+a^2)^{-\mu-1}$<br>$\operatorname{Re} a > 0$<br>$-\operatorname{Re} \nu < \operatorname{Re} \lambda < 2\operatorname{Re} \mu + 7/2$ | $\frac{\gamma^{\nu+1/2} \Gamma(\frac{1}{2}\lambda+\frac{1}{2}\nu) \Gamma(\mu-\frac{1}{2}\lambda-\frac{1}{2}\nu+1)}{2^{\nu+1} a^{2\mu-\lambda-\nu+2} \Gamma(\mu+1) \Gamma(\nu+1)}$<br>$\times {}_1F_2(\frac{1}{2}\lambda+\frac{1}{2}\nu; \frac{1}{2}\lambda+\frac{1}{2}\nu-\mu, \nu+1; \frac{1}{4}\gamma^2 a^2)$<br>$+\frac{\gamma^{2\mu-\lambda+5/2} \Gamma(\frac{1}{2}\lambda+\frac{1}{2}\nu-\mu-1)}{2^{2\mu-\lambda+3} \Gamma(\frac{1}{2}\nu-\frac{1}{2}\lambda+\mu+2)}$<br>$\times {}_1F_2(\mu+1; \mu+2+\frac{1}{2}\nu-\frac{1}{2}\lambda, \mu+2-\frac{1}{2}\lambda-\frac{1}{2}\nu; \frac{1}{4}\gamma^2 a^2)$ |
| (22) | $x^{-\frac{1}{2}}(a^2-x^2)^{-\frac{1}{2}}$<br>0<br>$0 < x < a$<br>$a < x < \infty$<br>$\operatorname{Re} \nu > -1$                                      | $\frac{1}{2} \pi \gamma^{\frac{1}{2}} [J_{\frac{1}{2}\nu}(\frac{1}{2}a\gamma)]^2$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                |
| (23) | 0<br>$x^{-\frac{1}{2}}(x^2-a^2)^{-\frac{1}{2}}$<br>$0 < x < a$<br>$a < x < \infty$                                                                      | $-\frac{1}{2} \pi \gamma^{\frac{1}{2}} J_{\frac{1}{2}\nu}(\frac{1}{2}a\gamma) Y_{\frac{1}{2}\nu}(\frac{1}{2}a\gamma)$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                            |
| (24) | $x^{\frac{1}{2}-\nu}(a^2-x^2)^{-\frac{1}{2}}$<br>0<br>$0 < x < a$<br>$a < x < \infty$                                                                   | $2^{-\frac{1}{2}} \pi^{\frac{1}{2}} a^{\frac{1}{2}-\nu} \mathbf{H}_{\nu-\frac{1}{2}}(a\gamma)$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |
| (25) | $x^{\nu-\frac{1}{2}}(a^2-x^2)^{\nu-\frac{1}{2}}$<br>0<br>$0 < x < a$<br>$a < x < \infty$<br>$\operatorname{Re} \nu > -\frac{1}{2}$                      | $2^{\nu-1} \pi^{\frac{1}{2}} \Gamma(\nu+\frac{1}{2}) a^{2\nu} \gamma^{\frac{1}{2}-\nu}$<br>$\times [J_\nu(\frac{1}{2}a\gamma)]^2$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                |

## Algebraic functions (cont'd)

|      | $f(x)$                                                                                                                                                          | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                             |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------|
| (26) | $0 \quad 0 < x < a$<br>$x^{-\nu-\frac{1}{2}}(x^2-a^2)^{-\nu-\frac{1}{2}} \quad a < x < \infty$<br>$ \operatorname{Re} \nu  < \frac{1}{2}$                       | $-2^{-\nu-1} a^{-2\nu} \Gamma(\frac{1}{2}-\nu) y^{\nu+\frac{1}{2}} \pi^{\frac{1}{2}}$<br>$\times J_\nu(\frac{1}{2}ay) Y_\nu(\frac{1}{2}ay)$ |
| (27) | $x^{\nu+\frac{1}{2}}(a^2-x^2)^{-\nu-\frac{1}{2}} \quad 0 < x < a$<br>$0 \quad a < x < \infty$<br>$ \operatorname{Re} \nu  < \frac{1}{2}$                        | $\pi^{-\frac{1}{2}} 2^{-\nu} \Gamma(\frac{1}{2}-\nu) y^{\nu-\frac{1}{2}} \sin(ay)$                                                          |
| (28) | $0 \quad 0 < x < a$<br>$x^{-\nu+\frac{1}{2}}(x^2-a^2)^{\nu-\frac{1}{2}} \quad a < x < \infty$<br>$ \operatorname{Re} \nu  < \frac{1}{2}$                        | $\pi^{-\frac{1}{2}} 2^{-\nu} \Gamma(\frac{1}{2}+\nu) y^{-\nu-\frac{1}{2}} \cos(ay)$                                                         |
| (29) | $x^{\nu+1/2}(a^2-x^2)^{-\nu-3/2} \quad 0 < x < a$<br>$0 \quad a < x < \infty$<br>$-1 < \operatorname{Re} \nu < -\frac{1}{2}$                                    | $2^{-1-\nu} \pi^{-\frac{1}{2}} \Gamma(-\frac{1}{2}-\nu) a^{-1} \cos(ay)$<br>$\times y^{\nu+\frac{1}{2}}$                                    |
| (30) | $0 \quad 0 < x < a$<br>$x^{\frac{1}{2}-\nu}(x^2-a^2)^{\nu-3/2} \quad a < x < \infty$<br>$1/2 < \operatorname{Re} \nu < 5/2$                                     | $2^{-\nu-1} \pi^{-\frac{1}{2}} a^{-1} \Gamma(\nu-\frac{1}{2}) y^{\frac{1}{2}-\nu} \sin(ay)$                                                 |
| (31) | $x^{\frac{1}{2}-\nu}(a^2-x^2)^\mu \quad 0 < x < a$<br>$0 \quad a < x < \infty$<br>$\operatorname{Re} \mu > -1$                                                  | $\frac{2^{1-\nu} a^{\mu-\nu+1} s_{\nu+\mu, \mu-\nu+1}(ay)}{y^{\mu+\frac{1}{2}} \Gamma(\nu)}$                                                |
| (32) | $0 \quad 0 < x < a$<br>$x^{\frac{1}{2}-\nu}(x^2-a^2)^\mu \quad a < x < \infty$<br>$\operatorname{Re} \mu > -1, \quad \operatorname{Re}(\nu-2\mu) > \frac{1}{2}$ | $2^\mu \Gamma(\mu+1) a^{1+\mu-\nu} y^{-\mu-\frac{1}{2}} J_{\nu-\mu-1}(ay)$                                                                  |

## Algebraic functions (cont'd)

|      | $f(x)$                                                                                                                                                                                      | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad \gamma > 0$                                                                                                                                                                                      |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (33) | $x^{\nu+\frac{1}{2}}(a^2-x^2)^\mu \quad 0 < x < a$<br>$0 \quad a < x < \infty$<br>$\operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu > -1$                                            | $2^\mu \Gamma(\mu+1) \gamma^{-\mu-\frac{1}{2}} a^{\nu+\mu+1}$<br>$\times J_{\nu+\mu+1}(a\gamma)$                                                                                                                                                          |
| (34) | $x^{\mu-\frac{1}{2}}(a^2-x^2)^\lambda \quad 0 < x < a$<br>$0 \quad a < x < \infty$<br>$\operatorname{Re} \lambda > -1, \quad \operatorname{Re}(\mu+\nu) > -1$                               | $\frac{a^{2\lambda+\mu+\nu+1} \gamma^{\nu+\frac{1}{2}} B(\lambda+1, \frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2})}{2^{\nu+1} \Gamma(\nu+1)}$<br>$\times {}_1F_2\left(\frac{1+\mu+\nu}{2}; \nu+1, \frac{3+\mu+\nu}{2}+\lambda; -\frac{a^2\gamma^2}{4}\right)$ |
| (35) | $x^{-\nu-\frac{1}{2}}(a^2+2x)^{-\frac{1}{2}}$<br>$\times [(a^2+2x)^{\frac{1}{2}}-a]^{2\nu}$<br>$\operatorname{Re} \nu > -\frac{1}{2}$                                                       | $2^\nu \Gamma(\nu+\frac{1}{2}) \pi^{-\frac{1}{2}} D_{-\nu-\frac{1}{2}}(ae^{\frac{1}{2}\pi i} \gamma^{\frac{1}{2}})$<br>$\times D_{-\nu-\frac{1}{2}}(ae^{-\frac{1}{2}\pi i} \gamma^{\frac{1}{2}})$                                                         |
| (36) | $x^{-\frac{1}{2}}(x^2+a^2)^{-\frac{1}{2}}$<br>$\times [(x^2+a^2)^{\frac{1}{2}}+x]^{\nu-1}$<br>$\operatorname{Re} a > 0, \quad -1 < \operatorname{Re} \nu < 5/2$                             | $2\pi^{-1/2} a^{\nu-3/2} \sinh(\frac{1}{2}a\gamma) K_{\nu-\frac{1}{2}}(\frac{1}{2}a\gamma)$                                                                                                                                                               |
| (37) | $x^{-\frac{1}{2}}(x^2+a^2)^{-\frac{1}{2}}$<br>$\times [(x^2+a^2)^{\frac{1}{2}}+x]^{1-\nu}$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$                         | $\pi^{\frac{1}{2}} a^{\frac{1}{2}-\nu} e^{-\frac{1}{2}a\gamma} I_{\nu-\frac{1}{2}}(\frac{1}{2}a\gamma)$                                                                                                                                                   |
| (38) | $x^{-\frac{1}{2}}(x^2+a^2)^{-\frac{1}{2}}$<br>$\times [(x^2+a^2)^{\frac{1}{2}} \pm x]^\mu \quad \operatorname{Re} a > 0$<br>$\operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu < 3/2$ | $\gamma^{\frac{1}{2}} a^\mu I_{\frac{1}{2}(\nu \mp \mu)}(\frac{1}{2}a\gamma) K_{\frac{1}{2}(\nu \pm \mu)}(\frac{1}{2}a\gamma)$                                                                                                                            |
| (39) | $x^{-\nu+\frac{1}{2}}(x^2+a^2)^{-\frac{1}{2}}$<br>$\times [(x^2+a^2)^{\frac{1}{2}}-a]^\nu$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1$                                   | $\gamma^{-\frac{1}{2}} e^{-a\gamma}$                                                                                                                                                                                                                      |
| (40) | $x^{-\mu-\frac{1}{2}}(x^2+a^2)^{-\frac{1}{2}}$<br>$\times [(x^2+a^2)^{\frac{1}{2}}+a]^\mu$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re}(\nu-\mu) > -1$                              | $\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu)}{a\gamma^{\frac{1}{2}} \Gamma(\nu+1)} W_{\frac{1}{2}\mu, \frac{1}{2}\nu}(a\gamma) M_{-\frac{1}{2}\mu, \frac{1}{2}\nu}(a\gamma)$                                                                  |



## Algebraic functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                        | $\int_0^\infty f(x) J_\nu(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                     |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (41) | $0 \quad 0 < x < a$ $x^{-\frac{1}{2}}(x^2 - a^2)^{-\frac{1}{2}} \{ [x + (x^2 - a^2)^{\frac{1}{2}}]^{\nu+1} + [x - (x^2 - a^2)^{\frac{1}{2}}]^{\nu+1} \}$ $a < x < \infty$ $\operatorname{Re} \nu < \frac{1}{2}$                               | $-\pi^{1/2} a^{\nu-3/2} [\sin(\frac{1}{2}ay) J_{\nu+\frac{1}{2}}(\frac{1}{2}ay) + \cos(\frac{1}{2}ay) Y_{\nu+\frac{1}{2}}(\frac{1}{2}ay)]$                                                                                                                           |
| (42) | $0 \quad 0 < x < a$ $x^{-\frac{1}{2}}(x^2 - a^2)^{-\frac{1}{2}} \{ [x + (x^2 - a^2)^{\frac{1}{2}}]^{\nu-1} + [x - (x^2 - a^2)^{\frac{1}{2}}]^{\nu-1} \}$ $a < x < \infty$ $\operatorname{Re} \nu < 5/2$                                       | $\pi^{1/2} a^{\nu-3/2} [\cos(\frac{1}{2}ay) J_{\nu-\frac{1}{2}}(\frac{1}{2}ay) - \sin(\frac{1}{2}ay) Y_{\nu-\frac{1}{2}}(\frac{1}{2}ay)]$                                                                                                                            |
| (43) | $x^{-\frac{1}{2}}(a^2 - x^2)^{-\frac{1}{2}} \{ [x + i(a^2 - x^2)^{\frac{1}{2}}]^\mu + [x - i(a^2 - x^2)^{\frac{1}{2}}]^\mu \} \quad 0 < x < a$ $0 \quad a < x < \infty$ $\operatorname{Re}(\mu + \nu) > -1$                                   | $\pi a^\mu y^{\frac{1}{2}} J_{\frac{1}{2}(\nu+\mu)}(\frac{1}{2}ay) J_{\frac{1}{2}(\nu-\mu)}(\frac{1}{2}ay)$                                                                                                                                                          |
| (44) | $0 \quad 0 < x < a$ $x^{-\frac{1}{2}}(x^2 - a^2)^{-\frac{1}{2}} \{ [x + (x^2 - a^2)^{\frac{1}{2}}]^\mu + [x - (x^2 - a^2)^{\frac{1}{2}}]^\mu \} \quad a < x < \infty$ $\operatorname{Re} \mu < 3/2$                                           | $-\frac{1}{2} \pi y^{\frac{1}{2}} a^\mu [J_{\frac{1}{2}(\mu+\nu)}(\frac{1}{2}ay) Y_{\frac{1}{2}(\nu-\mu)}(\frac{1}{2}ay) + J_{\frac{1}{2}(\nu-\mu)}(\frac{1}{2}ay) Y_{\frac{1}{2}(\nu+\mu)}(\frac{1}{2}ay)]$                                                         |
| (45) | $x^{-2\mu-\frac{1}{2}}(a^2 - x^2)^{-\frac{1}{2}} \times \{ [a + (a^2 - x^2)^{\frac{1}{2}}]^{2\mu} + [a - (a^2 - x^2)^{\frac{1}{2}}]^{2\mu} \} \quad 0 < x < a$ $0 \quad a < x < \infty$ $\operatorname{Re}(2\mu) < \operatorname{Re} \nu + 1$ | $\frac{a^\nu B(\frac{1}{2} + \frac{1}{2}\nu + \mu, \frac{1}{2} + \frac{1}{2}\nu - \mu) y^{\nu+\frac{1}{2}}}{\Gamma(1 + \nu)}$ $\times {}_1F_1(\frac{1}{2} + \frac{1}{2}\nu - \mu; \nu + 1; -iay)$ $\times {}_1F_1(\frac{1}{2} + \frac{1}{2}\nu - \mu; \nu + 1; iay)$ |
| (46) | $x^{\mu-\frac{1}{2}}(1 - 2ax + a^2)^{-\frac{1}{2}} \quad 0 < x < 1$ $0 \quad 1 < x < \infty$ $\operatorname{Re}(\nu + \mu + \frac{1}{2}) > 0$                                                                                                 | <p>see Bose, S. K., 1946: <i>Bull. Calcutta Math. Soc.</i>, 38, 177-180.</p>                                                                                                                                                                                         |

## Algebraic functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                   | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                              |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (47) | $x^{\nu+5/2} (x^4 + 4a^4)^{-\nu-1/2}$<br>$ \arg a  < \pi/4, \quad \operatorname{Re} \nu > 1/6$                                                                                                           | $\frac{\pi^{\frac{1}{2}} y^{\frac{1}{2}+\nu} J_{\nu-1}(ay) K_{\nu-1}(ay)}{2^{3\nu-1} a^{2\nu-2} \Gamma(\nu+\frac{1}{2})}$                                                    |
| (48) | $x^{\nu+\frac{1}{2}} (x^4 + 4a^4)^{-\nu-\frac{1}{2}}$<br>$ \arg a  < \frac{1}{4}\pi, \quad \operatorname{Re} \nu > -\frac{1}{2}$                                                                         | $\frac{y^{\nu+\frac{1}{2}} \pi^{\frac{1}{2}} J_\nu(ay) K_\nu(ay)}{a^{2\nu} 2^{3\nu} \Gamma(\nu+\frac{1}{2})}$                                                                |
| (49) | $x^{\nu+\frac{1}{2}} (x^4 \pm 2a^2 x^2 + b^4)^{-\frac{1}{2}}$<br>$\times [b^2 + x^2 + (x^4 \pm 2a^2 x^2 + b^4)^{\frac{1}{2}}]^{-2\nu}$<br>$0 < a < b$<br>$\operatorname{Re} \nu > -\frac{1}{2}$          | $(b^2 \mp a^2)^{-\nu} 2^\nu y^{\frac{1}{2}} K_0[(\frac{1}{2}b^2 \pm \frac{1}{2}a^2)^{\frac{1}{2}} y]$<br>$\times J_\nu[(\frac{1}{2}b^2 \mp \frac{1}{2}a^2)^{\frac{1}{2}} y]$ |
| (50) | 0<br>$x^{\frac{1}{2}-\nu} (x^2 - a^2)^{-\frac{1}{2}}$<br>$\times \{[a + (a^2 - x^2)^{\frac{1}{2}}]^\nu$<br>$+ [a - (a^2 - x^2)^{\frac{1}{2}}]^\nu\}$<br>$a < x < \infty$<br>$\operatorname{Re} \nu > -1$ | $2y^{-\frac{1}{2}} \cos(ay - \frac{1}{2}\nu\pi)$                                                                                                                             |

## 8.6. Exponential and logarithmic functions

|     |                                                                                           |                                                                                                                                          |
|-----|-------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $x^{-\frac{1}{2}} e^{-ax}$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1$ | $y^{\frac{1}{2}-\nu} (a^2 + y^2)^{-\frac{1}{2}} [(a^2 + y^2)^{\frac{1}{2}} - a]^\nu$                                                     |
| (2) | $x^{-3/2} e^{-ax}$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > 0$          | $\nu^{-1} y^{\frac{1}{2}-\nu} [(a^2 + y^2)^{\frac{1}{2}} - a]^\nu$                                                                       |
| (3) | $x^{m+\frac{1}{2}} e^{-ax} \quad \operatorname{Re} \nu > -m-2$                            | $(-1)^{m+1} y^{\frac{1}{2}-\nu} \frac{d^{m+1}}{da^{m+1}} \{(a^2 + y^2)^{-\frac{1}{2}}$<br>$\times [(a^2 + y^2)^{\frac{1}{2}} - a]^\nu\}$ |

## Exponential and logarithmic functions (cont'd)

|      | $f(x)$                                                                                                 | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                                            |
|------|--------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (4)  | $x^{\nu+\frac{1}{2}} e^{-ax}$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1$           | $\pi^{-1/2} 2^{\nu+1} \Gamma(\nu+3/2) a y^{\nu+1/2}$<br>$\times (a^2+y^2)^{-\nu-3/2}$                                                                                                                                                                                                                                                                                                      |
| (5)  | $x^{\nu-\frac{1}{2}} e^{-ax}$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$ | $2^\nu \pi^{-\frac{1}{2}} \Gamma(\nu+\frac{1}{2}) y^{\nu+\frac{1}{2}} (a^2+y^2)^{-\nu-\frac{1}{2}}$                                                                                                                                                                                                                                                                                        |
| (6)  | $x^{\mu-3/2} e^{-ax}$<br>$a > 0, \quad \operatorname{Re}(\mu+\nu) > 0$                                 | $y^{\frac{1}{2}} (a^2+y^2)^{-\frac{1}{2}} \mu \Gamma(\mu+\nu)$<br>$\times P_{\mu-1}^{-\nu} [a(a^2+y^2)^{-\frac{1}{2}}]$                                                                                                                                                                                                                                                                    |
| (7)  | $x^{\mu-3/2} e^{-ax}$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re}(\mu+\nu) > 0$               | $\frac{y^{\nu+\frac{1}{2}} \Gamma(\mu+\nu)}{2^\nu a^{\mu+\nu} \Gamma(\nu+1)}$<br>$\times {}_2F_1\left(\frac{\mu+\nu}{2}, \frac{\mu+\nu+1}{2}; \nu+1; -\frac{y^2}{a^2}\right)$<br>$= \frac{y^{\nu+\frac{1}{2}} \Gamma(\mu+\nu)}{2^\nu (a^2+y^2)^{\frac{1}{2}(\mu+\nu)} \Gamma(\nu+1)}$<br>$\times {}_2F_1\left[\frac{\mu+\nu}{2}, \frac{1-\mu+\nu}{2}; \nu+1; \frac{y^2}{(a^2+y^2)}\right]$ |
| (8)  | $x^{-\frac{1}{2}} e^{-ax^2}$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1$            | $\frac{\pi^{\frac{1}{2}} y^{\frac{1}{2}}}{2 a^{\frac{1}{2}}} \exp\left(-\frac{y^2}{8a}\right) I_{\frac{1}{2}\nu}\left(\frac{y^2}{8a}\right)$                                                                                                                                                                                                                                               |
| (9)  | $x^{\frac{1}{2}} e^{-ax^2}$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -2$             | $\frac{\pi^{1/2} y^{3/2}}{8 a^{3/2}} \exp\left(-\frac{y^2}{8a}\right)$<br>$\times \left[ I_{\frac{1}{2}\nu-\frac{1}{2}}\left(\frac{y^2}{8a}\right) - I_{\frac{1}{2}\nu+\frac{1}{2}}\left(\frac{y^2}{8a}\right) \right]$                                                                                                                                                                    |
| (10) | $x^{\nu+\frac{1}{2}} e^{-ax^2}$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1$         | $\frac{y^{\nu+\frac{1}{2}}}{(2a)^{\nu+1}} \exp\left(-\frac{y^2}{4a}\right)$                                                                                                                                                                                                                                                                                                                |

## Exponential and logarithmic functions (cont'd)

|      | $f(x)$                                                                                                             | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                                                                                                                              |
|------|--------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (11) | $x^{\nu-3/2} e^{-\alpha x^2}$<br>$\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \nu > 0$                   | $2^{\nu-1} y^{\frac{1}{2}-\nu} \gamma\left(\nu, \frac{y^2}{4\alpha}\right)$                                                                                                                                                                                                                                                                                                                                                                                                  |
| (12) | $x^{\nu+\frac{1}{2}} e^{\pm i\alpha x^2}$<br>$\alpha > 0, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}$          | $\frac{y^{\nu+\frac{1}{2}}}{(2\alpha)^{\nu+1}} \exp\left[\pm i\left(\frac{\nu+1}{2}\pi - \frac{y^2}{4\alpha}\right)\right]$                                                                                                                                                                                                                                                                                                                                                  |
| (13) | $x^{2n+\nu+\frac{1}{2}} e^{-\frac{1}{4}x^2}$<br>$\operatorname{Re} \nu > -1 - 2n$                                  | $2^{2n+\nu+1} n! y^{\nu+\frac{1}{2}} e^{-y^2} L_n^\nu(y^2)$                                                                                                                                                                                                                                                                                                                                                                                                                  |
| (14) | $x^{\mu-\frac{1}{2}} e^{-\alpha x^2}$<br>$\operatorname{Re} \alpha > 0, \quad \operatorname{Re}(\mu + \nu) > -1$   | $\frac{y^{\nu+\frac{1}{2}} \Gamma(\frac{1}{2}\nu + \frac{1}{2}\mu + \frac{1}{2})}{2^{\nu+1} \alpha^{\frac{1}{2}(\mu+\nu+1)} \Gamma(\nu+1)}$<br>$\times {}_1F_1\left(\frac{\nu+\mu+1}{2}; \nu+1; -\frac{y^2}{4\alpha}\right)$<br>$= \frac{\Gamma(\frac{1}{2}\nu + \frac{1}{2}\mu + \frac{1}{2})}{y^{\frac{1}{2}} \alpha^{\frac{1}{2}\mu} \Gamma(\nu+1)} \exp\left(-\frac{y^2}{8\alpha}\right)$<br>$\times M_{\frac{1}{2}\mu, \frac{1}{2}\nu}\left(\frac{y^2}{4\alpha}\right)$ |
| (15) | $x^{-3/2} e^{-\alpha/x}$<br>$\operatorname{Re} \alpha > 0$                                                         | $2y^{\frac{1}{2}} J_\nu[(2\alpha y)^{\frac{1}{2}}] K_\nu[(2\alpha y)^{\frac{1}{2}}]$                                                                                                                                                                                                                                                                                                                                                                                         |
| (16) | $x^{-3/2} e^{-\alpha/x - \beta x}$<br>$\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > 0$            | $2y^{\frac{1}{2}} J_\nu\{(2\alpha)^{\frac{1}{2}}[(\beta^2 + y^2)^{\frac{1}{2}} - \beta]^{\frac{1}{2}}\}$<br>$\times K_\nu\{(2\alpha)^{\frac{1}{2}}[(\beta^2 + y^2)^{\frac{1}{2}} + \beta]^{\frac{1}{2}}\}$                                                                                                                                                                                                                                                                   |
| (17) | $x^{-1} e^{-\alpha x^{\frac{1}{2}}}$<br>$\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$ | $\pi^{-\frac{1}{2}} 2^{\frac{1}{2}} \Gamma(\nu + \frac{1}{2}) D_{-\nu-\frac{1}{2}}(2^{-\frac{1}{2}} \alpha e^{\frac{1}{2}\pi i} y^{-\frac{1}{2}})$<br>$\times D_{-\nu-\frac{1}{2}}(2^{-\frac{1}{2}} \alpha e^{-\frac{1}{2}\pi i} y^{-\frac{1}{2}})$                                                                                                                                                                                                                          |

## Exponential and logarithmic functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                             | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                                                                                |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (18) | $x^{\nu+\frac{1}{2}} e^{a(1-x^2)} \quad 0 < x < 1$<br>$0 \quad 1 < x < \infty$<br>$\operatorname{Re} \nu > -\frac{1}{2}$                                                                                                                           | $(2ia)^{-\nu-1} y^{\frac{1}{2}+\nu} [U_{\nu+1}(2ia, y)$<br>$-i U_{\nu+2}(2ia, y)]$                                                                                                                                                                                                                                                                                                                                             |
| (19) | $x^{\nu+\frac{1}{2}} \exp[-a(x^2+\beta^2)^{\frac{1}{2}}]$<br>$\operatorname{Re} a > 0, \operatorname{Re} \beta > 0, \operatorname{Re} \nu > -1$                                                                                                    | $(\frac{1}{2}\pi)^{-\frac{1}{2}} a \beta^{\nu+3/2} y^{\nu+1/2} (y^2+a^2)^{-\nu-3/4}$<br>$\times K_{\nu+3/2}[\beta(y^2+a^2)^{1/2}]$                                                                                                                                                                                                                                                                                             |
| (20) | $x^{-\frac{1}{2}} (\beta^2+x^2)^{-\frac{1}{2}}$<br>$\times \exp[-a(\beta^2+x^2)^{\frac{1}{2}}]$<br>$\operatorname{Re} a > 0, \operatorname{Re} \beta > 0$<br>$\operatorname{Re} \nu > -1$                                                          | $y^{\frac{1}{2}} I_{\frac{1}{2}\nu} \{ \frac{1}{2} \beta [(a^2+y^2)^{\frac{1}{2}}-a] \}$<br>$\times K_{\frac{1}{2}\nu} \{ \frac{1}{2} \beta [(a^2+y^2)^{\frac{1}{2}}+a] \}$                                                                                                                                                                                                                                                    |
| (21) | $x^{\nu+\frac{1}{2}} (\beta^2+x^2)^{-\frac{1}{2}}$<br>$\times \exp[ia(\beta^2+x^2)^{\frac{1}{2}}] \quad a > 0$<br>$\operatorname{Re} \beta > 0, -1 < \operatorname{Re} \nu < \frac{1}{2}$                                                          | $i 2^{-\frac{1}{2}} \pi^{\frac{1}{2}} \beta^{\frac{1}{2}+\nu} (a^2-y^2)^{-\frac{1}{2}-\frac{1}{2}\nu} y^{\frac{1}{2}+\nu}$<br>$\times H_{-\nu-\frac{1}{2}}^{(1)} [\beta(a^2-y^2)^{\frac{1}{2}}]$<br>$0 < y < a$<br>$2^{\frac{1}{2}} \pi^{-\frac{1}{2}} \beta^{\frac{1}{2}+\nu} y^{\frac{1}{2}+\nu} (y^2-a^2)^{-\frac{1}{2}-\frac{1}{2}\nu}$<br>$\times K_{\nu+\frac{1}{2}} [\beta(y^2-a^2)^{\frac{1}{2}}]$<br>$a < y < \infty$ |
| (22) | $x^{\nu+\frac{1}{2}} (\beta^2+x^2)^{-\frac{1}{2}}$<br>$\times \exp[-a(\beta^2+x^2)^{\frac{1}{2}}]$<br>$\operatorname{Re} a > 0, \operatorname{Re} \beta > 0, \operatorname{Re} \nu > -1$                                                           | $(\frac{1}{2}\pi)^{-\frac{1}{2}} \beta^{\nu+\frac{1}{2}} y^{\nu+\frac{1}{2}} (a^2+y^2)^{-\frac{1}{2}\nu-\frac{1}{2}}$<br>$\times K_{\nu+\frac{1}{2}} [\beta(a^2+y^2)^{\frac{1}{2}}]$                                                                                                                                                                                                                                           |
| (23) | $x^{-\nu+\frac{1}{2}} (x^2+\beta^2)^{-\frac{1}{2}}$<br>$\times [(x^2+\beta^2)^{\frac{1}{2}}-\beta]^\nu$<br>$\times \exp[-a(x^2+\beta^2)^{\frac{1}{2}}]$<br>$\operatorname{Re} a > 0, \operatorname{Re} \beta > 0, \operatorname{Re} \nu > -1$      | $y^{\nu+\frac{1}{2}} [a+(y^2+a^2)^{\frac{1}{2}}]^{-\nu} (y^2+a^2)^{-\frac{1}{2}}$<br>$\times \exp[-\beta(y^2+a^2)^{\frac{1}{2}}]$                                                                                                                                                                                                                                                                                              |
| (24) | $x^{\sigma-\frac{1}{2}} (x^2+\beta^2)^{-\frac{1}{2}} [(x^2+\beta^2)^{\frac{1}{2}}+\beta]^\sigma$<br>$\times \exp[-a(x^2+\beta^2)^{\frac{1}{2}}]$<br>$\operatorname{Re} a > 0, \operatorname{Re} \beta > 0$<br>$\operatorname{Re}(\nu+\sigma) > -1$ | $\frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}\sigma+\frac{1}{2})}{\beta \Gamma(\nu+1)} y^{\frac{1}{2}} M_{\frac{1}{2}\sigma, \frac{1}{2}\nu} \{ \beta[(y^2+a^2)^{\frac{1}{2}}-a] \}$<br>$\times W_{-\frac{1}{2}\sigma, \frac{1}{2}\nu} \{ \beta[(y^2+a^2)^{\frac{1}{2}}+a] \}$                                                                                                                                                      |

**Exponential and logarithmic functions (cont'd)**

|                                                                                      |                                                                              |                                                                                                                                                                                                                                                                                                                  |
|--------------------------------------------------------------------------------------|------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
|                                                                                      | $f(x)$                                                                       | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                  |
| For other Hankel transforms containing exponential functions see Laplace transforms. |                                                                              |                                                                                                                                                                                                                                                                                                                  |
| (25)                                                                                 | $x^\mu \log x$<br>$-\operatorname{Re} \nu - 3/2 < \operatorname{Re} \mu < 0$ | $\frac{2^{\mu-\frac{1}{2}} \Gamma(\frac{1}{2} \mu + \frac{1}{2} \nu + \frac{3}{4})}{\Gamma(\frac{1}{2} \nu - \frac{1}{2} \mu + \frac{1}{4}) y^{\mu+1}}$<br>$\times [\psi(\frac{1}{2} \mu + \frac{1}{2} \nu + \frac{3}{4})$<br>$+ \psi(\frac{1}{2} \nu - \frac{1}{2} \mu + \frac{1}{4}) - \log(\frac{1}{4} y^2)]$ |

**8.7. Trigonometric and inverse trigonometric functions**

|     |                                                                                     |                                                                                                                                                                                                                                            |
|-----|-------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $x^{-\frac{1}{2}} \sin(ax)$<br>$a > 0, \operatorname{Re} \nu > -2$                  | $\cos(\frac{1}{2} \pi \nu) y^{\nu+\frac{1}{2}} (a^2 - y^2)^{-\frac{1}{2}}$<br>$\times [a + (a^2 - y^2)^{\frac{1}{2}}]^{-\nu} \quad 0 < y < a$<br>$y^{\frac{1}{2}} (y^2 - a^2)^{-\frac{1}{2}} \sin[\nu \sin^{-1}(a/y)]$<br>$a < y < \infty$ |
| (2) | $x^{-3/2} \sin(ax)$<br>$a > 0, \operatorname{Re} \nu > -1$                          | $\nu^{-1} \sin(\frac{1}{2} \pi \nu) y^{\nu+\frac{1}{2}}$<br>$\times [a + (a^2 - y^2)^{\frac{1}{2}}]^{-\nu} \quad 0 < y \leq a$<br>$\nu^{-1} y^{\frac{1}{2}} \sin[\nu \sin^{-1}(a/y)]$<br>$a < y < \infty$                                  |
| (3) | $x^{\nu+\frac{1}{2}} \sin(ax)$<br>$a > 0, -3/2 < \operatorname{Re} \nu < -1/2$      | $-2^{1+\nu} \pi^{-1/2} \sin(\nu \pi) \Gamma(\nu+3/2) a$<br>$\times y^{\nu+1/2} (a^2 - y^2)^{-\nu-3/2} \quad 0 < y < a$<br>$-2^{1+\nu} \pi^{-1/2} \Gamma(\nu+3/2)$<br>$\times a y^{\nu+1/2} (y^2 - a^2)^{-\nu-3/2} \quad a < y < \infty$    |
| (4) | $x^{\nu-\frac{1}{2}} \sin(ax)$<br>$a > 0, -1 < \operatorname{Re} \nu < \frac{1}{2}$ | $[\Gamma(\frac{1}{2} - \nu)]^{-1} \pi^{\frac{1}{2}} 2^\nu y^{\nu+\frac{1}{2}}$<br>$\times (a^2 - y^2)^{-\nu-\frac{1}{2}} \quad 0 < y < a$<br>$0 \quad a < y < \infty$                                                                      |

## Trigonometric functions (cont'd)

|      | $f(x)$                                                                                                                                 | $\int_0^\infty f(x) J_\nu(xy)(xy)^{1/2} dx \quad \gamma > 0$                                                                                                                                                                                                                                                                                                                                                                                                     |
|------|----------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (5)  | $x^{1/2-\nu} \sin(ax)$<br>$a > 0, \quad \operatorname{Re} \nu > 1/2$                                                                   | $0 \quad 0 < \gamma < a$<br>$2^{1-\nu} \pi^{1/2} a [\Gamma(\nu-1/2)]^{-1} \gamma^{1/2+\nu}$<br>$\times (y^2 - a^2)^{\nu-3/2} \quad a < \gamma < \infty$                                                                                                                                                                                                                                                                                                          |
| (6)  | $x^{-\nu+2n+1/2} \sin(ax)$<br>$a > 0, \quad \operatorname{Re} \nu > 2n + 1/2$                                                          | $0 \quad 0 < \gamma < a$<br>$(-1)^n 2^{\nu-2n-2} \gamma^{2n-\nu+3/2} (2n+1)!$<br>$\times \Gamma(\nu-2n-1) [\Gamma(2\nu-2n-1)]^{-1}$<br>$\times (y^2 - a^2)^{\nu-2n-3/2} C_{2n+1}^{\nu-2n-1}(a\gamma^{-1})$<br>$a < \gamma < \infty$                                                                                                                                                                                                                              |
| (7)  | $x^{\mu-3/2} \sin(ax)$<br>$a > 0, \quad -\operatorname{Re} \nu < \operatorname{Re} \mu < 3/2$                                          | $\frac{\gamma^{\nu+1/2} \Gamma(\nu+\mu) \sin[1/2 \pi(\nu+\mu)]}{2^\nu a^{\nu+\mu} \Gamma(\nu+1)}$<br>$\times {}_2F_1\left(\frac{1+\nu+\mu}{2}, \frac{\nu+\mu}{2}; \nu+1; \frac{\gamma^2}{a^2}\right)$<br>$0 < \gamma < a$<br><br>$\frac{2^\mu a \Gamma(1/2+1/2 \nu+1/2 \mu)}{\gamma^{\mu+1/2} \Gamma(1/2+1/2 \nu-1/2 \mu)}$<br>$\times {}_2F_1\left(\frac{1+\nu+\mu}{2}, \frac{1+\mu-\nu}{2}; \frac{3}{2}; \frac{a^2}{\gamma^2}\right)$<br>$a < \gamma < \infty$ |
| (8)  | $x^{\nu-1/2} (x^2 + \beta^2)^{-1} \sin(ax) \quad a > 0$<br>$\operatorname{Re} \beta > 0, \quad -1 < \operatorname{Re} \nu < 3/2$       | $\beta^{\nu-1} \sinh(a\beta) \gamma^{1/2} K_\nu(\beta\gamma) \quad \gamma \geq a$                                                                                                                                                                                                                                                                                                                                                                                |
| (9)  | $x^{1/2-\nu} (x^2 + \beta^2)^{-1} \sin(ax) \quad a > 0$<br>$\operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -1/2$           | $1/2 \pi \beta^{-\nu} e^{-a\beta} \gamma^{1/2} I_\nu(\beta\gamma) \quad 0 < \gamma \leq a$                                                                                                                                                                                                                                                                                                                                                                       |
| (10) | $x^{-3/2} e^{-xa \cos \phi \cos \psi} \sin(xa \sin \psi)$<br>$\operatorname{Re} \nu > -1, \quad a > 0, \quad 0 < \phi, \psi < 1/2 \pi$ | $a^{1/2} \nu^{-1} (\sin \phi)^{1/2} (\tan 1/2 \phi)^\nu \sin(\nu \psi)$<br>$\gamma = a \sin \phi$                                                                                                                                                                                                                                                                                                                                                                |

## Trigonometric functions (cont'd)

|      | $f(x)$                                                                                                                                   | $\int_0^\infty f(x) J_\nu(xy)(xy)^{1/2} dx \quad \gamma > 0$                                                                                                                                                                                                                                 |
|------|------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (11) | $x^{\nu+1/2} e^{-ax \cos \phi \cos \psi} \sin(ax \sin \psi)$<br>$a > 0, \quad 0 < \phi, \psi < 1/2\pi$<br>$\operatorname{Re} \nu > -3/2$ | $2^{\nu+1} \pi^{-1/2} \Gamma(\nu+3/2) a^{-\nu-3/2}$<br>$\times (\sin \phi)^{\nu+1/2} (\cos^2 \psi$<br>$+ \sin^2 \psi \cos^2 \phi)^{-\nu-3/2}$<br>$\times \sin[(\nu+3/2)\alpha] \quad \gamma = a \sin \phi$<br>$\tan(1/2\alpha) = \tan \psi \cos \phi$                                        |
| (12) | $x^{\nu-1/2} e^{-xa \cos \phi \cos \psi} \sin(xa \sin \psi)$<br>$a > 0, \quad 0 < \phi, \psi < 1/2\pi$<br>$\operatorname{Re} \nu > -1$   | $2^\nu \pi^{-1/2} a^{-\nu-1/2} \Gamma(\nu+1/2) (\sin \phi)^{\nu+1/2}$<br>$\times (\cos^2 \psi + \sin^2 \psi \cos^2 \phi)^{-\nu-1/2}$<br>$\times \sin[(\nu+3/2)\alpha] \quad \gamma = a \sin \phi$<br>$\tan(1/2\alpha) = \tan \psi \cos \phi$                                                 |
| (13) | $x^{-1/2} \sin(ax^2)$<br>$a > 0, \quad \operatorname{Re} \nu > -3$                                                                       | $-\frac{\pi^{1/2} \gamma^{1/2}}{2a^{1/2}} \sin\left(\frac{\gamma^2}{8a} - \frac{\nu+1}{4}\pi\right)$<br>$\times J_{1/2\nu}\left(\frac{\gamma^2}{8a}\right)$                                                                                                                                  |
| (14) | $x^{1/2} \sin(ax^2)$<br>$a > 0, \quad \operatorname{Re} \nu > -4$                                                                        | $\frac{\pi^{1/2} \gamma^{3/2}}{8a^{3/2}} \left[ \cos\left(\frac{\gamma^2}{8a} - \frac{\nu\pi}{4}\right) J_{1/2\nu-1/2}\left(\frac{\gamma^2}{8a}\right) \right.$<br>$\left. - \sin\left(\frac{\gamma^2}{8a} - \frac{\nu\pi}{4}\right) J_{1/2\nu+1/2}\left(\frac{\gamma^2}{8a}\right) \right]$ |
| (15) | $x^{\nu+1/2} \sin(ax^2)$<br>$a > 0, \quad -2 < \operatorname{Re} \nu < 1/2$                                                              | $\frac{\gamma^{\nu+1/2}}{2^{\nu+1} a^{\nu+1}} \cos\left(\frac{\gamma^2}{4a} - \frac{\nu\pi}{2}\right)$                                                                                                                                                                                       |
| (16) | $x^{\nu+1/2} \sin(ax^2) \quad 0 < x < b$<br>$0 \quad b < x < \infty$<br>$\operatorname{Re} \nu > -2$                                     | $(2a)^{-\nu-1} \gamma^{\nu+1/2} [\sin(ab^2) U_{\nu+1}(2ab^2; by)$<br>$- \cos(ab^2) U_{\nu+2}(2ab^2; by)]$                                                                                                                                                                                    |
| (17) | $x^{-1} e^{-ax^{1/2}} \sin(ax^{1/2})$<br>$a > 0, \quad \operatorname{Re} \nu > -1$                                                       | $i 2^{-1/2} \pi^{-1/2} \Gamma(\nu+1/2) D_{-\nu-1/2}(a\gamma^{-1/2})$<br>$\times [D_{-\nu-1/2}(i a \gamma^{-1/2}) - D_{-\nu-1/2}(-i a \gamma^{-1/2})]$                                                                                                                                        |



## Trigonometric functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                                                     | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                           |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (18) | $x^{\nu+\frac{1}{2}} \sin[a(x^2+\beta^2)^{\frac{1}{2}}] \quad a > 0$<br>$\operatorname{Re} \beta > 0, \quad -1 < \operatorname{Re} \nu < -\frac{1}{2}$                                                                                                                     | $(\frac{1}{2}\pi)^{1/2} a\beta^{\nu+3/2} y^{\nu+1/2} (a^2-y^2)^{-\nu/2-3/4}$<br>$\times \{ \sin(\nu\pi) J_{\nu+3/2} [\beta(a^2-y^2)^{1/2}]$<br>$+ \cos(\nu\pi) Y_{\nu+3/2} [\beta(a^2-y^2)^{1/2}] \}$<br>$0 < y < a$<br>$-(\frac{1}{2}\pi)^{-1/2} a\beta^{\nu+3/2} y^{\nu+1/2} (y^2-a^2)^{\nu/2-3/4}$<br>$\times K_{\nu+3/2} [\beta(y^2-a^2)^{1/2}] \quad a < y < \infty$ |
| (19) | $x^{-\frac{1}{2}} (x^2+\beta^2)^{-\frac{1}{2}}$<br>$\times \sin[a(x^2+\beta^2)^{\frac{1}{2}}] \quad a > 0$<br>$\operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -1$                                                                                              | $\frac{1}{2}\pi y^{\frac{1}{2}} J_{\frac{1}{2}\nu} \{ \frac{1}{2}\beta [a-(a^2-y^2)^{\frac{1}{2}}] \}$<br>$\times J_{-\frac{1}{2}\nu} \{ \frac{1}{2}\beta [a+(a^2-y^2)^{\frac{1}{2}}] \}$<br>$0 < y < a$                                                                                                                                                                  |
| (20) | $x^{\nu+\frac{1}{2}} (\beta^2+x^2)^{-\frac{1}{2}}$<br>$\times \sin[a(\beta^2+x^2)^{\frac{1}{2}}] \quad a > 0$<br>$\operatorname{Re} \beta > 0, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}$                                                                             | $2^{-\frac{1}{2}} \pi^{\frac{1}{2}} \beta^{\frac{1}{2}+\nu} y^{\frac{1}{2}+\nu} (a^2-y^2)^{-\frac{1}{4}-\frac{1}{2}\nu}$<br>$\times J_{-\nu-\frac{1}{2}} [\beta(a^2-y^2)^{\frac{1}{2}}] \quad 0 < y < a$<br>$0 \quad a < y < \infty$                                                                                                                                      |
| (21) | $x^{\nu+\frac{1}{2}} (b^2+x^2)^{-2}$<br>$\times \sin[a(x^2+b^2)^{\frac{1}{2}}] \quad a > 0$<br>$b > 0, \quad -1 < \operatorname{Re} \nu < 7/2$                                                                                                                             | $ay^{\frac{1}{2}} b^\nu K_\nu(yb) \quad y > a$                                                                                                                                                                                                                                                                                                                            |
| (22) | $x^{-\frac{1}{2}} (a^2-x^2)^{-\frac{1}{2}}$<br>$\times \sin[b(a^2-x^2)^{\frac{1}{2}}] \quad 0 < x < a$<br>$-x^{-\frac{1}{2}} (x^2-a^2)^{-\frac{1}{2}}$<br>$\times \exp[-b(x^2-a^2)^{\frac{1}{2}}] \quad a < x < \infty$<br>$b > 0, \quad \operatorname{Re} \nu > -1$       | $\frac{1}{2}\pi y^{\frac{1}{2}} J_{\frac{1}{2}\nu} \{ \frac{1}{2}a [(b^2+y^2)^{\frac{1}{2}}-b] \}$<br>$\times Y_{\frac{1}{2}\nu} \{ \frac{1}{2}a [(b^2+y^2)^{\frac{1}{2}}+b] \}$                                                                                                                                                                                          |
| (23) | $x^{\nu+\frac{1}{2}} (a^2-x^2)^{-\frac{1}{2}}$<br>$\times \sin[b(a^2-x^2)^{\frac{1}{2}}] \quad 0 < x < a$<br>$-x^{\nu+\frac{1}{2}} (x^2-a^2)^{-\frac{1}{2}}$<br>$\times \exp[-b(x^2-a^2)^{\frac{1}{2}}] \quad a < x < \infty$<br>$b > 0, \quad \operatorname{Re} \nu > -1$ | $2^{-\frac{1}{2}} \pi^{\frac{1}{2}} a^{\nu+\frac{1}{2}} (b^2+y^2)^{-\frac{1}{2}\nu-\frac{1}{4}} y^{\nu+\frac{1}{2}}$<br>$\times Y_{\nu+\frac{1}{2}} [a(b^2+y^2)^{\frac{1}{2}}]$                                                                                                                                                                                           |

## Trigonometric functions (cont'd)

|                                                                                        | $f(x)$                                                                                                                                                                   | $\int_0^\infty f(x) J_\nu(xy)(xy)^{1/2} dx \quad y > 0$                                                                                                                                                                                     |
|----------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (24)                                                                                   | $0 \quad 0 < x < a$<br>$x^{1/2-\nu} \sin[b(x^2-a^2)^{1/2}]$<br>$a < x < \infty$<br>$b > 0, \quad \text{Re } \nu > 1/2$                                                   | $0 \quad 0 < y < b$<br>$2^{-1/2} \pi^{1/2} a^{3/2-\nu} b y^{1/2-\nu}$<br>$\times (y^2-b^2)^{\nu/2-3/4}$<br>$\times J_{\nu-3/2}[a(y^2-b^2)^{1/2}]$<br>$b < y < \infty$                                                                       |
| (25)                                                                                   | $x^{1/2-\nu}(x^2+\beta^2)^{-1/2}[(x^2+\beta^2)^{1/2}-\beta]^\nu$<br>$\times \sin[a(x^2+\beta^2)^{1/2}] \quad a > 0$<br>$\text{Re } \beta > 0, \quad \text{Re } \nu > -1$ | $y^{\nu+1/2}[a+(a^2-y^2)^{1/2}]^{-\nu}(a^2-y^2)^{-1/2}$<br>$\times \cos[\beta(a^2-y^2)^{1/2}+1/2\pi\nu]$<br>$0 < y < a$<br>$y^{-1/2}(y^2-a^2)^{-1/2} \exp[-\beta(y^2-a^2)^{1/2}]$<br>$\times \sin[\nu \sin^{-1}(a/y)] \quad a < y < \infty$ |
| (26)                                                                                   | $0 \quad 0 < x < c$<br>$x^{1/2-\nu}(x^2+b^2)^{-1}$<br>$\times \sin[a(x^2-c^2)^{1/2}] \quad c < x < \infty$<br>$\text{Re } \nu > -3/2$                                    | $1/2 \pi y^{1/2} b^{-\nu} e^{-a(c^2+b^2)^{1/2}} I_\nu(by)$<br>$0 < y < a$                                                                                                                                                                   |
| For other Hankel transforms containing sines see the table of Fourier sine transforms. |                                                                                                                                                                          |                                                                                                                                                                                                                                             |
| (27)                                                                                   | $x^{-3/2} \cos(ax) \quad a > 0$<br>$\text{Re } \nu > 0$                                                                                                                  | $\nu^{-1} \cos(1/2 \nu \pi) y^{\nu+1/2} [a+(a^2-y^2)^{1/2}]^{-\nu}$<br>$0 < y \leq a$<br>$\nu^{-1} y^{1/2} \cos[\nu \sin^{-1}(a/y)]$<br>$a < y < \infty$                                                                                    |
| (28)                                                                                   | $x^{\nu+1/2} \cos(ax)$<br>$a > 0, \quad -1 < \text{Re } \nu < -1/2$                                                                                                      | $2^{1+\nu} \pi^{1/2} a [\Gamma(-1/2-\nu)]^{-1} y^{\nu+1/2}$<br>$\times (a^2-y^2)^{-\nu-3/2} \quad 0 < y < a$<br>$0 \quad a < y < \infty$                                                                                                    |
| (29)                                                                                   | $x^{\nu-1/2} \cos(ax)$<br>$a > 0, \quad  \text{Re } \nu  < 1/2$                                                                                                          | $-2^\nu \pi^{-1/2} \sin(\nu \pi) \Gamma(1/2+\nu)$<br>$\times y^{\nu+1/2} (a^2-y^2)^{-\nu-1/2} \quad 0 < y < a$<br>$2^\nu \pi^{-1/2} \Gamma(1/2+\nu) y^{\nu+1/2} (y^2-a^2)^{-\nu-1/2}$<br>$a < y < \infty$                                   |

## Trigonometric functions (cont'd)

|      | $f(x)$                                                                                                                                            | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                                                                                                                           |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (30) | $x^{-\nu-\frac{1}{2}} \cos(ax)$<br>$a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$                                                            | $0 \quad 0 < y < a$<br>$\frac{\pi^{\frac{1}{2}} (y^2 - a^2)^{\nu-\frac{1}{2}}}{2^\nu y^{\nu-\frac{1}{2}} \Gamma(\nu+\frac{1}{2})} \quad a < y < \infty$                                                                                                                                                                                                                                                                                                                   |
| (31) | $x^{-\nu+2n-\frac{1}{2}} \cos(ax)$<br>$a > 0, \quad \operatorname{Re} \nu > 2n - \frac{1}{2}$                                                     | $0 \quad 0 < y < a$<br>$(-1)^n y^{-\nu+2n+\frac{1}{2}} 2^{\nu-2n-1} \Gamma(\nu-2n)$<br>$\times [\Gamma(2\nu-2n)]^{-1} (2n)!$<br>$\times (y^2 - a^2)^{\nu-2n-\frac{1}{2}} C_{2n}^{\nu-2n}(ay^{-1})$<br>$a < y < \infty$                                                                                                                                                                                                                                                    |
| (32) | $x^{\mu-3/2} \cos(ax)$<br>$a > 0, \quad -\operatorname{Re} \nu < \operatorname{Re} \mu < 3/2$                                                     | $\frac{y^{\nu+\frac{1}{2}} \Gamma(\nu+\mu) \cos[\frac{1}{2}\pi(\nu+\mu)]}{2^\nu a^{\nu+\mu} \Gamma(\nu+1)}$<br>$\times {}_2F_1\left(\frac{\nu+\mu}{2}, \frac{\nu+\mu+1}{2}; \nu+1; \frac{y^2}{a^2}\right)$<br>$0 < y < a$<br>$\frac{2^{\mu-1} y^{\frac{1}{2}-\mu} \Gamma(\frac{1}{2}\nu+\frac{1}{2}\mu)}{\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\mu)}$<br>$\times {}_2F_1\left(\frac{\nu+\mu}{2}, \frac{\mu-\nu}{2}; \frac{1}{2}; \frac{a^2}{y^2}\right)$<br>$a < y < \infty$ |
| (33) | $x^{\nu+\frac{1}{2}} (x^2 + \beta^2)^{-1} \cos(ax)$<br>$a > 0, \quad \operatorname{Re} \beta > 0, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}$ | $\beta^\nu \cosh(a\beta) y^{\frac{1}{2}} K_\nu(\beta y) \quad y \geq a$                                                                                                                                                                                                                                                                                                                                                                                                   |
| (34) | $x^{-\nu-\frac{1}{2}} (x^2 + \beta^2)^{-1} \cos(ax)$<br>$a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -3/2$            | $\frac{1}{2} \pi \beta^{-\nu-1} e^{-a\beta} y^{\frac{1}{2}} I_\nu(\beta y)$<br>$0 < y \leq a$                                                                                                                                                                                                                                                                                                                                                                             |

## Trigonometric functions (cont'd)

|      | $f(x)$                                                                                                                                                                | $\int_0^\infty f(x) J_\nu(xy)(xy)^{1/2} dx \quad y > 0$                                                                                                                                                                                                                                                          |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (35) | $x^{-3/2} e^{-xa \cos \phi \cos \psi}$<br>$\times \cos(xa \sin \psi) \quad a > 0$<br>$0 < \phi, \psi < \frac{1}{2}\pi, \quad \operatorname{Re} \nu > 0$               | $a^{1/2} \nu^{-1} (\sin \phi)^{1/2} (\tan \frac{1}{2}\phi)^\nu \cos(\nu\psi)$<br>$y = a \sin \phi$                                                                                                                                                                                                               |
| (36) | $x^{\nu+1/2} e^{-ax \cos \phi \cos \psi}$<br>$\times \cos(ax \sin \psi) \quad a > 0$<br>$0 < \phi, \psi < \frac{1}{2}\pi, \quad \operatorname{Re} \nu > -1$           | $2^{\nu+1} \pi^{-1/2} \Gamma(\nu+3/2) a^{-\nu-3/2}$<br>$\times (\sin \phi)^{\nu+1/2} (\cos^2 \psi$<br>$+ \sin^2 \psi \cos^2 \phi)^{-\nu-3/2}$<br>$\times \cos[(\nu+3/2)a] \quad y = a \sin \phi$<br>$\tan(\frac{1}{2}a) = \tan \psi \cos \phi$                                                                   |
| (37) | $x^{\nu-1/2} e^{-xa \cos \phi \cos \psi}$<br>$\times \cos(xa \sin \psi) \quad a > 0$<br>$0 < \phi, \psi < \frac{1}{2}\pi, \quad \operatorname{Re} \nu > -\frac{1}{2}$ | $2^\nu \pi^{-1/2} a^{-\nu-1/2} \Gamma(\nu+1/2) (\sin \phi)^{\nu+1/2}$<br>$\times (\cos^2 \psi + \sin^2 \psi \cos^2 \phi)^{-\nu-1/2}$<br>$\times \cos[(\nu+1/2)a] \quad y = a \sin \phi$<br>$\tan(\frac{1}{2}a) = \tan \psi \cos \phi$                                                                            |
| (38) | $x^{-1/2} \cos(ax^2)$<br>$a > 0, \quad \operatorname{Re} \nu > -1$                                                                                                    | $\frac{\pi^{1/2} y^{1/2}}{2 a^{1/2}} \cos\left(\frac{y^2}{8a} - \frac{\nu+1}{4} \pi\right)$<br>$\times J_{\frac{1}{2}\nu}\left(\frac{y^2}{8a}\right)$                                                                                                                                                            |
| (39) | $x^{1/2} \cos(ax^2)$<br>$a > 0, \quad \operatorname{Re} \nu > -2$                                                                                                     | $\frac{\pi^{1/2} y^{3/2}}{8 a^{3/2}} \left[ \cos\left(\frac{y^2}{8a} - \frac{\nu\pi}{4}\right) \right.$<br>$\times J_{\frac{1}{2}\nu+\frac{1}{2}}\left(\frac{y^2}{8a}\right)$<br>$\left. + \sin\left(\frac{y^2}{8a} - \frac{\nu\pi}{4}\right) J_{\frac{1}{2}\nu-\frac{1}{2}}\left(\frac{y^2}{8a}\right) \right]$ |
| (40) | $x^{\nu+1/2} \cos(ax^2) \quad -1 < \operatorname{Re} \nu < \frac{1}{2}$                                                                                               | $\frac{y^{\nu+1/2}}{2^{\nu+1} a^{\nu+1}} \sin\left(\frac{y^2}{4a} - \frac{\nu\pi}{2}\right)$                                                                                                                                                                                                                     |

## Trigonometric functions (cont'd)

|      | $f(x)$                                                                                                                                                                                         | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                                                                            |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (41) | $x^{\nu+\frac{1}{2}} \cos(ax^2) \quad 0 < x < b$<br>$0 \quad b < x < \infty$<br>$\operatorname{Re} \nu > -1$                                                                                   | $(2a)^{-\nu-\frac{1}{2}} y^{\nu+\frac{1}{2}}$<br>$\times [\sin(ab^2) U_{\nu+\frac{1}{2}}(2ab^2 by)$<br>$+ \cos(ab^2) U_{\nu+\frac{1}{2}}(2ab^2 by)]$                                                                                                                                                                                                                                                                       |
| (42) | $x^{-1} e^{-ax^{\frac{1}{2}}} \cos(ax^{\frac{1}{2}})$<br>$a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$                                                                                   | $2^{-\frac{1}{2}} \pi^{-\frac{1}{2}} \Gamma(\nu+\frac{1}{2}) D_{-\nu-\frac{1}{2}}(ay^{-\frac{1}{2}})$<br>$\times [D_{-\nu-\frac{1}{2}}(ia y^{-\frac{1}{2}}) + D_{-\nu-\frac{1}{2}}(-ia y^{-\frac{1}{2}})]$                                                                                                                                                                                                                 |
| (43) | $x^{\nu+\frac{1}{2}} \cos[a(x^2+\beta^2)^{\frac{1}{2}}] \quad a > 0$<br>$\operatorname{Re} \beta > 0, \quad -1 < \operatorname{Re} \nu < -\frac{1}{2}$                                         | $(\frac{1}{2}\pi)^{1/2} a \beta^{\nu+3/2} y^{\nu+1/2}$<br>$\times (a^2-y^2)^{-\nu/2-3/4}$<br>$\times \{ \cos(\pi\nu) J_{\nu+3/2}[\beta(a^2-y^2)^{1/2}]$<br>$- \sin(\pi\nu) Y_{\nu+3/2}[\beta(a^2-y^2)^{1/2}] \}$<br>$0 < y < a$<br>$0 \quad a < y < \infty$                                                                                                                                                                |
| (44) | $x^{-\frac{1}{2}} (x^2+\beta^2)^{-\frac{1}{2}}$<br>$\times \cos[a(x^2+\beta^2)^{\frac{1}{2}}] \quad a > 0$<br>$\operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -1$                  | $-\frac{1}{2}\pi y^{\frac{1}{2}} J_{\frac{1}{2}\nu} \{ \frac{1}{2}\beta [a-(a^2-y^2)^{\frac{1}{2}}] \}$<br>$\times Y_{-\frac{1}{2}\nu} \{ \frac{1}{2}\beta [a+(a^2-y^2)^{\frac{1}{2}}] \}$<br>$0 < y < a$                                                                                                                                                                                                                  |
| (45) | $x^{\nu+\frac{1}{2}} (x^2+\beta^2)^{-\frac{1}{2}}$<br>$\times \cos[a(x^2+\beta^2)^{\frac{1}{2}}] \quad a > 0$<br>$\operatorname{Re} \beta > 0, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}$ | $-2^{-\frac{1}{2}} \pi^{-\frac{1}{2}} \beta^{\frac{1}{2}+\nu} y^{\frac{1}{2}+\nu} (a^2-y^2)^{-\frac{1}{2}-\frac{1}{2}\nu}$<br>$\times Y_{-\nu-\frac{1}{2}} [\beta(a^2-y^2)^{\frac{1}{2}}] \quad 0 < y < a$<br>$2^{\frac{1}{2}} \pi^{-\frac{1}{2}} \beta^{\frac{1}{2}+\nu} y^{\frac{1}{2}+\nu} (y^2-a^2)^{-\frac{1}{2}-\frac{1}{2}\nu}$<br>$\times K_{\nu+\frac{1}{2}} [\beta(y^2-a^2)^{\frac{1}{2}}] \quad a < y < \infty$ |
| (46) | $x^{1/2}(x^2+b^2)^{-3/2}$<br>$\times \cos[a(x^2+b^2)^{\frac{1}{2}}] \quad a > 0$<br>$b > 0, \quad -1 < \operatorname{Re} \nu < 5/2$                                                            | $y^{\frac{1}{2}} b^\nu K_\nu(by) \quad y > a$                                                                                                                                                                                                                                                                                                                                                                              |
| (47) | $x^{-\frac{1}{2}} (a^2-x^2)^{-\frac{1}{2}} \cos[b(a^2-x^2)^{\frac{1}{2}}]$<br>$0 < x < a$<br>$0 \quad a < x < \infty$<br>$\operatorname{Re} \nu > -1$                                          | $\frac{1}{2}\pi y^{\frac{1}{2}} J_{\frac{1}{2}\nu} \{ \frac{1}{2}a [(b^2+y^2)^{\frac{1}{2}}-b] \}$<br>$\times J_{\frac{1}{2}\nu} \{ \frac{1}{2}a [(b^2+y^2)^{\frac{1}{2}}+b] \}$                                                                                                                                                                                                                                           |

## Trigonometric functions (cont'd)

|                                                                                            | $f(x)$                                                                                                                                                                                                      | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                               |
|--------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (48)                                                                                       | $x^{\nu+\frac{1}{2}}(a^2-x^2)^{-\frac{1}{2}} \times \cos[b(a^2-x^2)^{\frac{1}{2}}] \quad 0 < x < a$<br>$0 \quad a < x < \infty$<br>$\text{Re } \nu > -1$                                                    | $2^{-\frac{1}{2}} \pi^{\frac{1}{2}} a^{\nu+\frac{1}{2}} y^{\nu+\frac{1}{2}} (b^2+y^2)^{-\frac{1}{2}\nu-\frac{1}{4}}$<br>$\times J_{\nu+\frac{1}{2}}[a(b^2+y^2)^{\frac{1}{2}}]$                                                                                                                                |
| (49)                                                                                       | $0 \quad 0 < x < a$<br>$x^{\frac{1}{2}-\nu}(x^2-a^2)^{-\frac{1}{2}} \times \cos[b(x^2-a^2)^{\frac{1}{2}}] \quad a < x < \infty$<br>$b > 0, \quad \text{Re } \nu > -\frac{1}{2}$                             | $0 \quad 0 < y < b$<br>$2^{-\frac{1}{2}} \pi^{\frac{1}{2}} a^{\frac{1}{2}-\nu} y^{\frac{1}{2}-\nu} (y^2-b^2)^{\frac{1}{2}\nu-\frac{1}{4}}$<br>$\times J_{\nu-\frac{1}{2}}[a(y^2-b^2)^{\frac{1}{2}}] \quad b < y < \infty$                                                                                     |
| (50)                                                                                       | $x^{\frac{1}{2}-\nu}(x^2+\beta^2)^{-\frac{1}{2}} \times [(x^2+\beta^2)^{\frac{1}{2}}-\beta]^\nu \times \cos[a(x^2+\beta^2)^{\frac{1}{2}}] \quad a > 0$<br>$\text{Re } \beta > 0, \quad \text{Re } \nu > -1$ | $-y^{\nu+\frac{1}{2}}[a+(a^2-y^2)^{\frac{1}{2}}]^{-\nu}(a^2-y^2)^{-\frac{1}{2}}$<br>$\times \sin[\beta(a^2-y^2)^{\frac{1}{2}}+\frac{1}{2}\pi\nu] \quad 0 < y < a$<br>$y^{-\frac{1}{2}}(y^2-a^2)^{-\frac{1}{2}} \exp[-\beta(y^2-a^2)^{\frac{1}{2}}]$<br>$\times \cos[\nu \sin^{-1}(a/y)] \quad a < y < \infty$ |
| (51)                                                                                       | $0 \quad 0 < x < c$<br>$x^{\frac{1}{2}-\nu}(x^2+b^2)^{-1}(x^2-c^2)^{-\frac{1}{2}} \times \cos[a(x^2-c^2)^{\frac{1}{2}}] \quad c < x < \infty$<br>$\text{Re } \nu > -5/2$                                    | $\frac{1}{2}\pi y^{\frac{1}{2}} b^{-\nu}(c^2+b^2)^{-\frac{1}{2}}$<br>$\times e^{-a(c^2+b^2)^{\frac{1}{2}}} I_\nu(by) \quad 0 < y < a$                                                                                                                                                                         |
| For other Hankel transforms containing cosines see the table of Fourier cosine transforms. |                                                                                                                                                                                                             |                                                                                                                                                                                                                                                                                                               |
| (52)                                                                                       | $x^{-\frac{1}{2}}(x^2-1)^{-\frac{1}{2}} \times \cos[(\nu-1)\cos^{-1}x] \quad 0 < x < 1$<br>$0 \quad 1 < x < \infty$<br>$\text{Re } \nu > 0$                                                                 | $\pi^{\frac{1}{2}} \sin(\frac{1}{2}y) J_{\nu-\frac{1}{2}}(\frac{1}{2}y)$                                                                                                                                                                                                                                      |
| (53)                                                                                       | $x^{-\frac{1}{2}}(1-x^2)^{-\frac{1}{2}} \times \cos[(\nu+1)\cos^{-1}x] \quad 0 < x < 1$<br>$0 \quad 1 < x < \infty$<br>$\text{Re } \nu > -1$                                                                | $\pi^{\frac{1}{2}} \cos(\frac{1}{2}y) J_{\nu+\frac{1}{2}}(\frac{1}{2}y)$                                                                                                                                                                                                                                      |

## Trigonometric functions (cont'd)

|      | $f(x)$                                                                                                                                           | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                 |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------|
| (54) | $x^{-\frac{1}{2}}(1-x^2)^{-\frac{1}{2}} \cos(\mu \cos^{-1} x)$<br>$0 < x < 1$<br>$0 \quad 1 < x < \infty$<br>$\operatorname{Re}(\mu + \nu) > -1$ | $\frac{1}{2} \pi y^{\frac{1}{2}} J_{\frac{1}{2}(\mu+\nu)}(\frac{1}{2}y) J_{\frac{1}{2}(\nu-\mu)}(\frac{1}{2}y)$ |
| (55) | $0 \quad 0 < x < 1$<br>$x^{\frac{1}{2}}(x^2-1)^{-\frac{1}{2}} \cos(\nu \cos^{-1} x^{-1})$<br>$1 < x < \infty$<br>$\operatorname{Re} \nu > -1$    | $y^{-\frac{1}{2}} \cos(y - \frac{1}{2} \nu \pi)$<br>$.$                                                         |

## 8.8. Hyperbolic and inverse hyperbolic functions

|                                                                                     |                                                                                                                                                |                                                                                                                                                                                            |
|-------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1)                                                                                 | $x^{\nu-\frac{1}{2}} e^{-\frac{1}{2} \pi x} \operatorname{csch}(\frac{1}{2} \pi x)$<br>$\operatorname{Re} \nu > -\frac{1}{2}$                  | $\pi^{-\frac{1}{2}} 2^{\nu+1} \Gamma(\nu+\frac{1}{2}) y^{\nu+\frac{1}{2}}$<br>$\times \sum_{n=1}^{\infty} (n^2 \pi^2 + y^2)^{-\nu-\frac{1}{2}}$                                            |
| (2)                                                                                 | $x^{\frac{1}{2}} \frac{x \cosh x + \sinh x}{\sinh(2x) + 2x}$                                                                                   | For this and similar integrals<br>see Boit, M. A., 1935:<br><i>J. Appl. Phys.</i> , 6, 367-375.                                                                                            |
| (3)                                                                                 | $x^{\nu+\frac{1}{2}} \sinh(\alpha x) \operatorname{csch}(\pi x)$<br>$ \operatorname{Re} \alpha  < \pi, \quad \operatorname{Re} \nu > -1$       | $2 \pi^{-1} y^{\frac{1}{2}} \sum_{n=1}^{\infty} (-1)^{n-1} n^{\nu+1} \sin(n \alpha)$<br>$\times K_\nu(ny)$                                                                                 |
| For other similar integrals see Weber, H., 1873 : <i>J. of Math.</i> 75,<br>75-105. |                                                                                                                                                |                                                                                                                                                                                            |
| (4)                                                                                 | $x^{-\frac{1}{2}}(1+x^2)^{-\frac{1}{2}} \sinh(2\mu \sinh^{-1} x)$<br>$\operatorname{Re} \nu > -1, \quad  \operatorname{Re} \mu  < \frac{3}{4}$ | $\frac{1}{2} y^{\frac{1}{2}} [I_{\frac{1}{2}\nu-\mu}(\frac{1}{2}y) K_{\frac{1}{2}\nu+\mu}(\frac{1}{2}y)$<br>$- I_{\frac{1}{2}\nu+\mu}(\frac{1}{2}y) K_{\frac{1}{2}\nu-\mu}(\frac{1}{2}y)]$ |

**Hyperbolic functions (cont'd)**

|     | $f(x)$                                                                                                                                                              | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                         |
|-----|---------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (5) | $x^{-\frac{1}{2}}(1+x^2)^{-\frac{1}{2}} \cosh(2\mu \sinh^{-1} x)$<br>$\operatorname{Re} \nu > -1, \quad  \operatorname{Re} \mu  < \frac{3}{4}$                      | $\frac{1}{2} y^{\frac{1}{2}} [I_{\frac{1}{2}\nu-\mu}(\frac{1}{2}y) K_{\frac{1}{2}\nu+\mu}(\frac{1}{2}y)$<br>$+ I_{\frac{1}{2}\nu+\mu}(\frac{1}{2}y) K_{\frac{1}{2}\nu-\mu}(\frac{1}{2}y)]$              |
| (6) | $0 \quad 0 < x < 1$<br>$x^{-\frac{1}{2}}(x^2-1)^{-\frac{1}{2}}$<br>$\times \cosh[(\nu-1) \cosh^{-1} x]$<br>$1 < x < \infty$<br>$-1/2 < \operatorname{Re} \nu < 5/2$ | $\frac{1}{2} \pi^{\frac{1}{2}} [\cos(\frac{1}{2}y) J_{\nu-\frac{1}{2}}(\frac{1}{2}y)$<br>$- \sin(\frac{1}{2}y) Y_{\nu-\frac{1}{2}}(\frac{1}{2}y)]$                                                      |
| (7) | $0 \quad 0 < x < 1$<br>$x^{-\frac{1}{2}}(x^2-1)^{-\frac{1}{2}}$<br>$\times \cosh[(\nu+1) \cosh^{-1} x]$<br>$1 < x < \infty$<br>$-5/2 < \operatorname{Re} \nu < 1/2$ | $-\frac{1}{2} \pi^{\frac{1}{2}} [\sin(\frac{1}{2}y) J_{\nu+\frac{1}{2}}(\frac{1}{2}y)$<br>$+ \cos(\frac{1}{2}y) Y_{\nu+\frac{1}{2}}(\frac{1}{2}y)]$                                                     |
| (8) | $0 \quad 0 < x < 1$<br>$x^{-\frac{1}{2}}(x^2-1)^{-\frac{1}{2}} \cosh(\mu \cosh^{-1} x)$<br>$1 < x < \infty$<br>$ \operatorname{Re} \mu  < 3/2$                      | $-\frac{1}{4} \pi y^{\frac{1}{2}} [J_{\frac{1}{2}(\mu+\nu)}(\frac{1}{2}y) Y_{\frac{1}{2}(\nu-\mu)}(\frac{1}{2}y)$<br>$+ J_{\frac{1}{2}(\nu-\mu)}(\frac{1}{2}y) Y_{\frac{1}{2}(\mu+\nu)}(\frac{1}{2}y)]$ |

**8.9. Orthogonal polynomials**

|     |                                                                                                                               |                                                                                                             |
|-----|-------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------|
| (1) | $x^{-\frac{1}{2}}(1-x^2)^{-\frac{1}{2}} T_n(x) \quad 0 < x < 1$<br>$0 \quad 1 < x < \infty$<br>$\operatorname{Re} \nu > -n-1$ | $\frac{1}{2} \pi y^{\frac{1}{2}} J_{\frac{1}{2}(\nu+n)}(\frac{1}{2}y) J_{\frac{1}{2}(\nu-n)}(\frac{1}{2}y)$ |
| (2) | $x^{\nu+\frac{1}{2}} e^{-x^2} L_n^\nu(x^2) \quad \operatorname{Re} \nu > -1$                                                  | $2^{-2n-\nu-1} (n!)^{-1} y^{2n+\nu+\frac{1}{2}} \exp(-\frac{1}{4}y^2)$                                      |
| (3) | $x^{\nu+\frac{1}{2}} e^{-\frac{1}{2}x^2} L_n^\nu(x^2) \quad \operatorname{Re} \nu > -1$                                       | $(-1)^n e^{-\frac{1}{2}y^2} y^{\nu+\frac{1}{2}} L_n^\nu(y^2)$                                               |



## Orthogonal polynomials (cont'd)

|     | $f(x)$                                                                                                                                            | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                              |
|-----|---------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (4) | $x^{2n+\nu+\frac{1}{2}} e^{-\frac{1}{2}x^2} L_n^{\nu+n}(\frac{1}{2}x^2)$<br>$\operatorname{Re} \nu > -1$                                          | $y^{2n+\nu+\frac{1}{2}} e^{-\frac{1}{2}y^2} L_n^{\nu+n}(\frac{1}{2}y^2)$                                                                                                                                                                                                                                                                                                     |
| (5) | $x^{\nu+\frac{1}{2}} e^{-\beta x^2} L_n^\nu(ax^2)$<br>$\operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > 0$                              | $2^{-\nu-1} \beta^{-\nu-n-1} (\beta-a)^n y^{\nu+\frac{1}{2}}$<br>$\times \exp\left(-\frac{y^2}{4\beta}\right) L_n^\nu\left[\frac{ay^2}{4\beta(\alpha-\beta)}\right]$                                                                                                                                                                                                         |
| (6) | $x^{\nu+\frac{1}{2}} e^{-\alpha x^2} [L_n^{\frac{1}{2}\nu}(\alpha x^2)]^2$<br>$\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \nu > -1$    | $(2\alpha)^{-\nu-1} y^{\nu+\frac{1}{2}}$<br>$\times \exp\left(-\frac{y^2}{4\alpha}\right) \left[L_n^{\frac{1}{2}\nu}\left(\frac{y^2}{4\alpha}\right)\right]^2$                                                                                                                                                                                                               |
| (7) | $x^{\nu+\frac{1}{2}} e^{-\beta x^2} [L_n^{\frac{1}{2}\nu}(\alpha x^2)]^2$<br>$\operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -1$      | $\frac{y^{\frac{1}{2}+\nu}}{\pi n!} \Gamma(n+1+\frac{1}{2}\nu)(2\beta)^{-\nu-1}$<br>$\times \exp\left(-\frac{y^2}{4\beta}\right)$<br>$\times \sum_{l=0}^n \frac{(-1)^l \Gamma(n-l+\frac{1}{2}) \Gamma(l+\frac{1}{2})}{\Gamma(l+1+\frac{1}{2}\nu)(n-l)!}$<br>$\times \left(\frac{2\alpha-\beta}{\beta}\right)^{2l} L_{2l}^\nu\left[\frac{ay^2}{2\beta(2\alpha-\beta)}\right]$ |
| (8) | $x^{\nu+\frac{1}{2}} e^{-\alpha x^2} L_m^{\nu-\sigma}(ax^2) L_n^\sigma(ax^2)$<br>$\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \nu > -1$ | $(-1)^{m+n} (2\alpha)^{-\nu-1} y^{\nu+\frac{1}{2}} \exp\left(-\frac{y^2}{4\alpha}\right)$<br>$\times L_n^{\sigma-m+n}\left(\frac{y^2}{4\alpha}\right) L_m^{\nu-\sigma+m-n}\left(\frac{y^2}{4\alpha}\right)$                                                                                                                                                                  |
| (9) | $x^{\nu+\frac{1}{2}} e^{-x^2} L_n^\sigma(x^2) L_n^{\nu-\sigma}(x^2)$<br>$\operatorname{Re} \nu > -1$                                              | $2^{-\nu-1} y^{\nu+\frac{1}{2}} e^{-\frac{1}{4}y^2} L_n^\sigma(\frac{1}{4}y^2)$<br>$\times L_n^{\nu-\sigma}(\frac{1}{4}y^2)$                                                                                                                                                                                                                                                 |

## Orthogonal polynomials (cont'd)

|      | $f(x)$                                                                                                                                                                                                                | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                     |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (10) | $0 \quad 0 < x < a$<br>$x^{2n+\frac{1}{2}-\nu}(x^2-a^2)^{\nu-2n-\frac{1}{2}}$<br>$\times C_{2n}^{\nu-2n}(a/x) \quad a < x < \infty$<br>$2n - \frac{1}{2} < \text{Re } \nu < 2n + \frac{1}{2}$                         | $(-1)^n 2^{2n-\nu+1} \Gamma(2\nu-2n)$<br>$\times [(2n)! \Gamma(\nu-2n)]^{-1} y^{-\nu+2n-\frac{1}{2}}$<br>$\times \cos(ay)$                                                                                                          |
| (11) | $0 \quad 0 < x < a$<br>$x^{2n-\nu+3/2}(x^2-a^2)^{\nu-2n-3/2}$<br>$\times C_{2n+1}^{\nu-2n-1}(a/x) \quad a < x < \infty$<br>$2n + 1/2 < \text{Re } \nu < 2n + 3/2$                                                     | $(-1)^n 2^{2n-\nu+2} \Gamma(2\nu-2n-1)$<br>$\times [(2n+1)! \Gamma(\nu-2n-1)]^{-1}$<br>$\times y^{-\nu+2n+\frac{1}{2}} \sin(ay)$                                                                                                    |
| (12) | $x^{\nu+\frac{1}{2}}(1-x^2)^{-\frac{1}{2}} \sin[a(1-x^2)^{\frac{1}{2}}]$<br>$\times C_{2n+1}^{\nu+\frac{1}{2}}[(1-x^2)^{\frac{1}{2}}] \quad 0 < x < 1$<br>$0 \quad 1 < x < \infty$<br>$\text{Re } \nu > -\frac{1}{2}$ | $(-1)^n 2^{-\frac{1}{2}} \pi^{\frac{1}{2}} y^{\nu+\frac{1}{2}} (a^2+y^2)^{-\frac{1}{2}\nu-\frac{1}{4}}$<br>$\times C_{2n+1}^{\nu+\frac{1}{2}}[a(y^2+a^2)^{-\frac{1}{2}}]$<br>$\times J_{\nu+3/2+2n}[(a^2+y^2)^{\frac{1}{2}}]$       |
| (13) | $x^{\nu+\frac{1}{2}}(1-x^2)^{-\frac{1}{2}} \cos[a(1-x^2)^{\frac{1}{2}}]$<br>$\times C_{2n}^{\nu+\frac{1}{2}}[(1-x^2)^{\frac{1}{2}}] \quad 0 < x < 1$<br>$0 \quad 1 < x < \infty$<br>$\text{Re } \nu > -\frac{1}{2}$   | $(-1)^n 2^{-\frac{1}{2}} \pi^{\frac{1}{2}} y^{\nu+\frac{1}{2}} (a^2+y^2)^{-\frac{1}{2}\nu-\frac{1}{4}}$<br>$\times C_{2n}^{\nu+\frac{1}{2}}[a(y^2+a^2)^{-\frac{1}{2}}]$<br>$\times J_{\nu+\frac{1}{2}+2n}[(a^2+y^2)^{\frac{1}{2}}]$ |

## 8.10. Legendre functions

|     |                                                                                                                                                                                                     |                                                                                                                                    |
|-----|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $(x^2+2)^{-\frac{1}{2}\nu-\frac{1}{4}} P_{\mu}^{-\nu-\frac{1}{2}}(x^2+1)$<br>$\text{Re } \nu > -1$<br>$-3/2 - \text{Re } \nu < \text{Re } \mu < \text{Re } \nu + 1/2$                               | $\frac{2^{\frac{1}{2}-\nu} \pi^{-\frac{1}{2}} [K_{\mu+\frac{1}{2}}(2^{-\frac{1}{2}}y)]^2}{\Gamma(\nu+\mu+3/2)\Gamma(\nu-\mu+1/2)}$ |
| (2) | $0 \quad 0 < x < a$<br>$(x^2-a^2)^{\frac{1}{2}\mu-\frac{1}{4}} P_{-\frac{1}{2}+\nu}^{-\frac{1}{2}-\mu}(ax^{-1})$<br>$a < x < \infty$<br>$ \text{Re } \mu  < \frac{1}{2}, \quad \text{Re } \nu > -1$ | $2^{\frac{1}{2}} \pi^{-\frac{1}{2}} y^{-\mu-\frac{1}{2}} \cos[ay + \frac{1}{2}(\nu-\mu)\pi]$                                       |

## Legendre functions (cont'd)

|     | $f(x)$                                                                                                                                                                                                                                                                 | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                             |
|-----|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (3) | $x^{\nu-\frac{1}{2}}(1-x^2)^{\frac{1}{2}\nu+\frac{1}{2}}$ $\times P_\mu^{-\nu-\frac{1}{2}}(2x^{-2}-1) \quad 0 < x < 1$ $0 \quad 1 < x < \infty$ $-3/2 - \operatorname{Re} \nu < \operatorname{Re} \mu < \operatorname{Re} \nu + 1/2$                                   | $\frac{\Gamma(3/2+\mu+\nu)\Gamma(\frac{1}{2}+\nu-\mu)(2y)^{\nu+\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}[\Gamma(3/2+\nu)]^2}$ $\times {}_1F_1(\nu+\mu+3/2; 2\nu+2; iy)$ $\times {}_1F_1(\nu+\mu+3/2; 2\nu+2; -iy)$ |
| (4) | $x^{\frac{1}{2}}(a^2+x^2)^{-\frac{1}{2}\mu}$ $\times P_{\mu-1}^{-\nu}[a(a^2+x^2)^{-\frac{1}{2}}] \quad \operatorname{Re} a > 0$ $\operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu > \frac{1}{2}$                                                                | $\frac{y^{\mu-3/2} e^{-ay}}{\Gamma(\mu+\nu)}$                                                                                                                                                               |
| (5) | $x^{\nu+\frac{1}{2}}(x^2+a^2)^{\frac{1}{2}\nu}$ $\times P_\nu\left[\frac{x^2+2a^2}{2a(x^2+a^2)^{\frac{1}{2}}}\right]$ $\operatorname{Re} a > 0, \quad -1 < \operatorname{Re} \nu < 0$                                                                                  | $\frac{(2a)^{\nu+1}y^{-\nu-\frac{1}{2}}}{\pi\Gamma(-\nu)} [K_{\nu+\frac{1}{2}}(\frac{1}{2}ay)]^2$                                                                                                           |
| (6) | $x^{\frac{1}{2}-\nu}(x^2+a^2)^{-\frac{1}{2}\nu}$ $\times P_{\nu-1}\left[\frac{x^2+2a^2}{2a(x^2+a^2)^{\frac{1}{2}}}\right]$ $\operatorname{Re} a > 0, \quad 0 < \operatorname{Re} \nu < 1$                                                                              | $\frac{(2a)^{1-\nu}}{\Gamma(\nu)} y^{\nu-\frac{1}{2}} I_{\nu-\frac{1}{2}}(\frac{1}{2}ay)$ $\times K_{\nu-\frac{1}{2}}(\frac{1}{2}ay)$                                                                       |
| (7) | $x^{\frac{1}{2}}\{P_\mu^{-\frac{1}{2}\nu}[(1+a^2x^2)^{\frac{1}{2}}]\}^2$ $\operatorname{Re} a > 0$ $-3/4 < \operatorname{Re} \mu < -1/4, \quad \operatorname{Re} \nu > -1$                                                                                             | $\frac{2[K_{\mu+\frac{1}{2}}(\frac{1}{2}a^{-1}y)]^2}{\pi a\Gamma(1+\mu+\frac{1}{2}\nu)\Gamma(\frac{1}{2}\nu-\mu)y^{\frac{1}{2}}}$                                                                           |
| (8) | $x^{\frac{1}{2}}(1+a^2x^2)^{-\frac{1}{2}}$ $\times P_\mu^{-\frac{1}{2}\nu}[(1+a^2x^2)^{\frac{1}{2}}]$ $\times P_{\mu+1}^{-\frac{1}{2}\nu}[(1+a^2x^2)^{\frac{1}{2}}]$ $\operatorname{Re} \nu > -1, \quad \operatorname{Re} a > 0$ $-7/4 < \operatorname{Re} \mu < -1/4$ | $\frac{y^{\frac{1}{2}}K_{\mu+1/2}(\frac{1}{2}a^{-1}y)K_{\mu+3/2}(\frac{1}{2}a^{-1}y)}{\pi a^2\Gamma(2+\frac{1}{2}\nu+\mu)\Gamma(\frac{1}{2}\nu-\mu)}$                                                       |

## Legendre functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                                                              | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                                           |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (9)  | $x^{\frac{1}{2}}(1+a^2x^2)^{-\frac{1}{2}}$<br>$\times P_{\mu}^{-\frac{1}{2}-\frac{1}{2}\nu}[(1+a^2x^2)^{\frac{1}{2}}]$<br>$\times P_{\mu}^{\frac{1}{2}-\frac{1}{2}\nu}[(1+a^2x^2)^{\frac{1}{2}}]$<br>$\text{Re } \nu > -1, \quad \text{Re } a > 0$<br>$-5/4 < \text{Re } \mu < 1/4$ | $\frac{y^{\frac{1}{2}} [K_{\mu+\frac{1}{2}}(\frac{1}{2}a^{-1}y)]^2}{\pi a^2 \Gamma(\nu/2+\mu+3/2) \Gamma(\nu/2-\mu+1/2)}$                                                                                                                                                                                                                                                                 |
| (10) | $Q_{\nu-\frac{1}{2}}[(a^2+x^2)x^{-1}]$<br>$\text{Re } \nu > -\frac{1}{2}$                                                                                                                                                                                                           | $2^{-\frac{1}{2}} \pi y^{-\frac{1}{2}} \exp[-(a^2-\frac{1}{4})^{\frac{1}{2}}y]$<br>$\times J_{\nu}(\frac{1}{2}y)$                                                                                                                                                                                                                                                                         |
| (11) | $x^{\frac{1}{2}-\mu}(1+a^2x^2)^{-\frac{1}{2}\mu-\frac{1}{4}}$<br>$\times Q_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(\pm iax) \quad \text{Re } a > 0$<br>$-\frac{3}{4} - \frac{1}{2}\text{Re } \nu < \text{Re } \mu < 1+\text{Re } \nu$                                                   | $i(2\pi)^{\frac{1}{2}} e^{i\pi(\mu+\frac{1}{2}\nu+\frac{1}{4})} a^{-1} y^{\mu-\frac{1}{2}}$<br>$\times I_{\nu}(\frac{1}{2}a^{-1}y) K_{\mu}(\frac{1}{2}a^{-1}y)$                                                                                                                                                                                                                           |
| (12) | $(x^2+2)^{-\frac{1}{2}\nu-\frac{1}{4}} Q_{\mu}^{\nu+\frac{1}{2}}(x^2+1)$<br>$\text{Re } \nu > -1$<br>$\text{Re}(2\mu+\nu) > -5/2$                                                                                                                                                   | $2^{-\nu-\frac{1}{2}} \pi^{\frac{1}{2}} e^{(\nu+\frac{1}{2})\pi i} y^{\nu+\frac{1}{2}}$<br>$\times K_{\mu+\frac{1}{2}}(2^{-\frac{1}{2}}y) I_{\mu+\frac{1}{2}}(2^{-\frac{1}{2}}y)$                                                                                                                                                                                                         |
| (13) | $x^{-\nu-\frac{1}{2}} Q_{-\frac{1}{2}}^{\nu-\frac{1}{2}}(1+2a^2/x^2)$<br>$\text{Re } a > 0, \quad 0 < \text{Re } \nu < 3/2$                                                                                                                                                         | $-ie^{i\pi\nu} \pi^{\frac{1}{2}} 2^{-\nu} (y/a)^{\nu-\frac{1}{2}}$<br>$\times I_{\nu-\frac{1}{2}}(\frac{1}{2}ay) K_{\nu-\frac{1}{2}}(\frac{1}{2}ay)$                                                                                                                                                                                                                                      |
| (14) | $x^{\nu-\frac{1}{2}}(a^2+x^2)^{\frac{1}{4}+\frac{1}{2}\nu}$<br>$\times Q_{\mu}^{\nu+\frac{1}{2}}(1+2a^2/x^2) \quad \text{Re } a > 0$<br>$\text{Re}(\mu+\nu) > -3/2$<br>$\text{Re}(\nu-\mu) > -1/2$                                                                                  | $-ie^{i\pi\nu} \pi^{-1/2} 2^{\nu} [\Gamma(3/2+\mu+\nu)]^2$<br>$\times \Gamma(1/2+\nu-\mu) a^{\nu-1/2} y^{-\nu-3/2}$<br>$\times W_{-\mu-\frac{1}{2}, \nu+\frac{1}{2}}(ay)$<br>$\times \left[ \frac{\cos(\mu\pi)}{\Gamma(2+2\nu)} M_{\mu+\frac{1}{2}, \nu+\frac{1}{2}}(ay) \right.$<br>$\left. + \frac{\sin(\pi\nu)}{\Gamma(\nu+\mu+3/2)} W_{\mu+\frac{1}{2}, \nu+\frac{1}{2}}(ay) \right]$ |

## Legendre functions (cont'd)

|      | $f(x)$                                                                                                                                                                           | $\int_0^\infty f(x) J_\nu(xy)(xy)^{1/2} dx \quad y > 0$                                                                                                                                |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (15) | $x^{-\nu-1/2} (x^2 + a^2)^{1/2-\nu}$<br>$\times Q_\mu^{1/2-\nu} (1 + 2a^2/x^2) \quad \text{Re } a > 0$<br>$0 < \text{Re } \nu < \text{Re } \mu + 3/2$                            | $\frac{ie^{-i\pi} \pi^{1/2} \Gamma(3/2 + \mu - \nu)}{2^\nu a^{\nu+1/2} \Gamma(2\nu)} y^{-\nu-3/2}$<br>$\times M_{\mu+1/2, \nu-1/2}(ay) W_{-\mu-1/2, \nu-1/2}(ay)$                      |
| (16) | $x^{1/2} P_\mu^{-1/2} \nu [(1 + a^2 x^2)^{1/2}]$<br>$\times Q_\mu^{-1/2} \nu [(1 + a^2 x^2)^{1/2}] \quad \text{Re } a > 0$<br>$\text{Re } \mu > -3/4, \quad \text{Re } \nu > -1$ | $\frac{e^{-1/2 \nu \pi i} \Gamma(1 + \mu + 1/2 \nu)}{a \Gamma(1 + \mu - 1/2 \nu)} y^{1/2} I_{\mu+1/2} \left( \frac{y}{2a} \right)$<br>$\times K_{\mu+1/2} \left( \frac{y}{2a} \right)$ |

8.11. Bessel functions of argument  $kx$ 

|     |                                                                                                 |                                                                                                                                                                                                                                               |
|-----|-------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $x^{-1/2} J_{\nu-1}(ax)$<br>$a > 0, \quad \text{Re } \nu > -1$                                  | $0 \quad 0 < y < a$<br>$a^{\nu-1} y^{-\nu+1/2} \quad a < y < \infty$                                                                                                                                                                          |
| (2) | $x^{-3/2} J_\nu(ax)$<br>$a > 0, \quad \text{Re } \nu > 0$                                       | $1/2 \nu^{-1} a^{-\nu} y^{\nu+1/2} \quad 0 < y \leq a$<br>$1/2 \nu^{-1} a^\nu y^{-\nu+1/2} \quad a \leq y < \infty$                                                                                                                           |
| (3) | $x^{-1/2} J_{\nu+1}(ax)$<br>$a > 0, \quad \text{Re } \nu > -3/2$                                | $a^{-\nu-1} y^{\nu+1/2} \quad 0 < y < a$<br>$0 \quad a < y < \infty$                                                                                                                                                                          |
| (4) | $x^{-2\lambda-1/2} J_\nu(ax) \quad a > 0$<br>$\text{Re } \nu + 1/2 > \text{Re } \lambda > -1/2$ | $\frac{a^\nu y^{\nu+1/2} \Gamma(\nu - \lambda + 1/2)}{2^{2\lambda} (a+y)^{2\nu-2\lambda+1} \Gamma(\nu+1) \Gamma(\lambda+1/2)}$<br>$\times {}_2F_1 \left[ \nu - \lambda + \frac{1}{2}, \nu + \frac{1}{2}; 2\nu+1; \frac{4ay}{(a+y)^2} \right]$ |
| (5) | $x^{-1/2} J_{\nu+2n+1}(ax)$<br>$a > 0, \quad \text{Re } \nu > -1 - n$                           | $y^{\nu+1/2} a^{-\nu-1} P_n^{(\nu, 0)}(1 - 2y^2/a^2) \quad 0 < y < a$<br>$0 \quad a < y < \infty$                                                                                                                                             |

Bessel functions of  $kx$  (cont'd)

|     | $f(x)$                                                                                                   | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 |
|-----|----------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (6) | $x^{-\frac{1}{2}} J_\mu(ax)$<br>$a > 0, \quad \text{Re}(\mu + \nu) > -1$                                 | $y^{\nu+\frac{1}{2}} a^{-\nu-1} \frac{\Gamma(\frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2})}{\Gamma(\nu+1) \Gamma(\frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2})}$<br>$\times {}_2F_1\left(\frac{\mu+\nu+1}{2}, \frac{\nu-\mu+1}{2}; \nu+1; \frac{y^2}{a^2}\right)$<br>$0 < y < a$<br>For $y > a$ interchange $\mu$ and $\nu$ .                                                                                                                                                                                                                                                 |
| (7) | $x^{\nu-\mu+\frac{1}{2}} J_\mu(ax)$<br>$a > 0, \quad -1 < \text{Re } \nu < \text{Re } \mu$               | $\frac{2^{\nu-\mu+1} y^{\nu+\frac{1}{2}}}{\Gamma(\mu-\nu) a^\mu} (a^2 - y^2)^{\mu-\nu-1}$<br>$0 < y < a$<br>0<br>$a < y < \infty$                                                                                                                                                                                                                                                                                                                                                                                                                                               |
| (8) | $x^{\mu-\nu+\frac{1}{2}} J_\mu(ax)$<br>$a > 0, \quad \text{Re } \nu > \text{Re } \mu > -1$               | 0<br>$0 < y < a$<br>$\frac{2^{\mu-\nu+1} a^\mu}{\Gamma(\nu-\mu) y^{\nu-\frac{1}{2}}} (y^2 - a^2)^{\nu-\mu-1}$<br>$a < y < \infty$                                                                                                                                                                                                                                                                                                                                                                                                                                               |
| (9) | $x^{-\lambda-\frac{1}{2}} J_\mu(ax) \quad a > 0$<br>$\text{Re}(\mu + \nu) + 1 > \text{Re } \lambda > -1$ | $\frac{\Gamma[\frac{1}{2}(\mu+\nu-\lambda+1)] y^{\nu+\frac{1}{2}}}{2^\lambda a^{\nu-\lambda+1} \Gamma(\nu+1) \Gamma[\frac{1}{2}(\lambda+\mu-\nu+1)]}$<br>$\times {}_2F_1\left(\frac{\mu+\nu-\lambda+1}{2}, \frac{\nu-\lambda-\mu+1}{2}; \nu+1; \frac{y^2}{a^2}\right)$<br>$0 < y < a$<br>$\frac{\Gamma[\frac{1}{2}(\mu+\nu-\lambda+1)] a^\mu}{2^\lambda y^{\mu-\lambda+\frac{1}{2}} \Gamma(\mu+1) \Gamma[\frac{1}{2}(\lambda+\nu-\mu+1)]}$<br>$\times {}_2F_1\left(\frac{\mu+\nu-\lambda+1}{2}, \frac{\mu-\lambda-\nu+1}{2}; \mu+1; \frac{a^2}{y^2}\right)$<br>$a < y < \infty$ |

Bessel functions of  $kx$  (cont'd)

|      | $f(x)$                                                                                                                                                                                        | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                           |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (10) | $x^{\frac{1}{2}} (x^2 + \beta^2)^{-1} J_\nu(ax) \quad a > 0$<br>$\operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -1$                                                               | $y^{\frac{1}{2}} I_\nu(y\beta) K_\nu(a\beta) \quad 0 < y < a$<br>$y^{\frac{1}{2}} I_\nu(a\beta) K_\nu(y\beta) \quad a < y < \infty$                                       |
| (11) | $x^{\frac{1}{2}-2n} (\beta^2 + x^2)^{-1} J_\nu(ax)$<br>$a > 0, \quad \operatorname{Re} \beta > 0$<br>$\operatorname{Re} \nu > n - 1, \quad n = 0, 1, 2, \dots$                                | $(-1)^n \beta^{-2n} y^{\frac{1}{2}} I_\nu(y\beta) K_\nu(a\beta) \quad 0 < y < a$<br>$(-1)^n \beta^{-2n} y^{\frac{1}{2}} I_\nu(a\beta) K_\nu(y\beta) \quad a < y < \infty$ |
| (12) | $x^{\nu-\mu+\frac{1}{2}} (\beta^2 + x^2)^{-1} J_\mu(ax)$<br>$a > 0, \quad \operatorname{Re} \beta > 0$<br>$1 + \operatorname{Re} \mu > \operatorname{Re} \nu > -1$                            | $\beta^{\nu-\mu} y^{\frac{1}{2}} I_\mu(a\beta) K_\nu(y\beta) \quad a < y < \infty$                                                                                        |
| (13) | $x^{\nu-\mu+2n+\frac{1}{2}} (\beta^2 + x^2)^{-1} J_\mu(ax)$<br>$a > 0, \quad \operatorname{Re} \beta > 0$<br>$\operatorname{Re} \mu - 2n + 1 > \operatorname{Re} \nu > -n - 1$<br>$n$ integer | $(-1)^n \beta^{\nu-\mu+2n} y^{\frac{1}{2}} I_\mu(a\beta) K_\nu(y\beta) \quad a < y < \infty$                                                                              |
| (14) | $x^{\mu-\nu+\frac{1}{2}} (\beta^2 + x^2)^{-1} J_\mu(ax)$<br>$a > 0, \quad \operatorname{Re} \beta > 0$<br>$1 + \operatorname{Re} \nu > \operatorname{Re} \mu > -1$                            | $y^{\frac{1}{2}} \beta^{\mu-\nu} I_\nu(y\beta) K_\mu(a\beta) \quad 0 < y < a$                                                                                             |
| (15) | $x^{\mu-\nu+2n+\frac{1}{2}} (\beta^2 + x^2)^{-1} J_\mu(ax)$<br>$a > 0, \quad \operatorname{Re} \beta > 0$<br>$\operatorname{Re} \nu - 2n + 1 > \operatorname{Re} \mu > -n - 1$<br>$n$ integer | $(-1)^n \beta^{\mu-\nu+2n} y^{\frac{1}{2}} I_\nu(y\beta) K_\mu(a\beta) \quad 0 < y < a$                                                                                   |
| (16) | $x^{\frac{1}{2}} (x^2 + \beta^2)^{-1} J_{\nu-2n}(ax)$<br>$a > 0, \quad \operatorname{Re} \beta > 0$<br>$\operatorname{Re} \nu > n - 1$                                                        | $(-1)^n y^{\frac{1}{2}} I_\nu(y\beta) K_{\nu-2n}(a\beta) \quad 0 < y < a$<br>$(-1)^n y^{\frac{1}{2}} I_{\nu-2n}(a\beta) K_\nu(y\beta) \quad a < y < \infty$               |

Bessel functions of  $kx$  (cont'd)

|      | $f(x)$                                                                                                                                                                | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                                                   |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (17) | $x^{-\frac{1}{2}} e^{-\alpha x} J_\nu(\beta x)$<br>$\operatorname{Re} \alpha > \operatorname{Im} \beta > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$               | $\pi^{-1} \beta^{-\frac{1}{2}} Q_{\nu-\frac{1}{2}} \left( \frac{\alpha^2 + \beta^2 + \gamma^2}{2\beta\gamma} \right)$                                                                                                                                                                                                                                                                             |
| (18) | $x^{\mu-3/2} e^{-\alpha x} J_\nu(\beta x)$<br>$\operatorname{Re} \alpha >  \operatorname{Im} \beta , \quad \operatorname{Re}(\mu+2\nu) > 0$                           | $\frac{\beta^\nu \gamma^{\nu+\frac{1}{2}} \Gamma(\mu+2\nu)}{\pi \alpha^{\mu+2\nu} \Gamma(2\nu+1)}$<br>$\times \int_0^\pi {}_2F_1 \left( \frac{\mu}{2} + \nu, \frac{\mu+1}{2} + \nu; \nu+1; -\frac{u^2}{a^2} \right)$<br>$\times (\sin \phi)^{2\nu} d\phi$<br>$u^2 = \beta^2 + \gamma^2 - 2\beta\gamma \cos \phi$                                                                                  |
| (19) | $x^{-1} e^{-\alpha a \cos \phi \cos \psi} J_\mu(\alpha x \sin \phi)$<br>$a > 0, \quad 0 < \phi, \psi < \frac{1}{2}\pi$<br>$\operatorname{Re}(\mu+\nu) > -\frac{1}{2}$ | $\Gamma(\mu+\nu+\frac{1}{2}) (\sin \psi)^{\frac{1}{2}}$<br>$\times P_{\nu-\frac{1}{2}}^{-\mu}(\cos \phi) P_{\mu-\frac{1}{2}}^{-\nu}(\cos \psi)$<br>$y = a \sin \psi$                                                                                                                                                                                                                              |
| (20) | $x^{-\frac{1}{2}} e^{-\beta x} J_\mu(\alpha x)$<br>$\operatorname{Re} \beta >  \operatorname{Im} \alpha $<br>$\operatorname{Re}(\mu+\nu+1) > 0$                       | $2\pi^{-1} \alpha^\mu \beta \gamma^{\nu+\frac{1}{2}} \int_0^{\frac{1}{2}\pi} (2\beta \sec \theta)^{\mu+\nu}$<br>$\times (\beta^2 \sec^2 \theta + \gamma^2 - \alpha^2 + u)^{-\mu}$<br>$\times (\beta^2 \sec^2 \theta + \alpha^2 - \gamma^2 + u)^{-\nu}$<br>$\times \sec^2 \theta \cos[(\mu-\nu)\theta] u^{-1} d\theta$<br>$u^2 = (\sec^2 \theta b^2 + \alpha^2 + \gamma^2)^2 - 4\alpha^2 \gamma^2$ |
| (21) | $x^{\mu-\nu-\frac{1}{2}} e^{-\alpha x} J_\mu(\beta x)$<br>$\operatorname{Re} \alpha >  \operatorname{Im} \beta , \quad \operatorname{Re} \mu > -\frac{1}{2}$          | $\frac{\beta^\mu \gamma^{\nu+\frac{1}{2}} \Gamma(\mu+\frac{1}{2})}{2^{\nu-\mu} \pi \Gamma(\nu+\frac{1}{2})} \int_0^\pi (\sin \phi)^{2\nu}$<br>$\times [(a+iy \cos \phi)^2 + \beta^2]^{-\mu-\frac{1}{2}} d\phi$                                                                                                                                                                                    |
| (22) | $x^{\lambda-3/2} e^{-\alpha x} J_\mu(\beta x)$<br>$\operatorname{Re} \alpha > \operatorname{Im} \beta > 0$<br>$\operatorname{Re}(\lambda+\mu+\nu) > 0$                | $y^{\frac{1}{2}} \sum_{m=0}^\infty \frac{\Gamma(\lambda+\mu+\nu+2m)}{m! \Gamma(\mu+m+1)} \left( -\frac{\beta^2}{4a^2} \right)^m$<br>$\times {}_2F_1(-m, -\mu-m; \nu+1; \gamma^2 \beta^{-2})$                                                                                                                                                                                                      |



Bessel functions of  $kx$  (cont'd)

|      | $f(x)$                                                                                                                            | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                       |
|------|-----------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (23) | $x^{\frac{1}{2}} e^{-\beta x^2} J_\nu(ax)$<br>$\operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -1$                     | $\frac{y^{\frac{1}{2}}}{2\beta} \exp\left(-\frac{a^2+y^2}{4\beta}\right) I_\nu\left(\frac{ay}{2\beta}\right)$                                                                                                         |
| (24) | $x^{\lambda+\frac{1}{2}} e^{-ax^2} J_\mu(\beta x) \quad \operatorname{Re} a > 0$<br>$\operatorname{Re}(\mu + \nu + \lambda) > -2$ | $y^{\frac{1}{2}} \sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{1}{2}\nu+\frac{1}{2}\mu+\frac{1}{2}\lambda)}{m! \Gamma(m+\mu+1)} \left(-\frac{\beta^2}{4a}\right)^m$<br>$\times {}_2F_1(-m, -\mu-m; \nu+1; y^2 \beta^{-2})$ |
| (25) | $x^\lambda J_\mu(ax) \begin{matrix} \cos(bx) \\ \sin(bx) \end{matrix}$                                                            | see under Fourier transforms.                                                                                                                                                                                         |
| (26) | $x^{\frac{1}{2}} \sin(ax^2) J_\nu(bx)$<br>$a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -2$                                  | $\frac{y^{\frac{1}{2}}}{2a} \cos\left(\frac{y^2+b^2}{4a} - \frac{\nu\pi}{2}\right) J_\nu\left(\frac{by}{2a}\right)$                                                                                                   |
| (27) | $x^{\frac{1}{2}} \cos(ax^2) J_\nu(bx)$<br>$a > 0, \quad b > 0, \quad \operatorname{Re} \nu > -1$                                  | $\frac{y^{\frac{1}{2}}}{2a} \sin\left(\frac{b^2+y^2}{4a} - \frac{\nu\pi}{2}\right) J_\nu\left(\frac{by}{2a}\right)$                                                                                                   |
| (28) | $x^{-\frac{1}{2}} [J_0(\frac{1}{2}ax)]^2$<br>$a > 0, \quad \operatorname{Re} \nu > -1$                                            | $y^{-\frac{1}{2}} \{P_{\frac{1}{2}\nu-\frac{1}{2}}[(1-a^2/y^2)^{\frac{1}{2}}]\}^2$<br>$a < y < \infty$                                                                                                                |
| (29) | $x^{\frac{1}{2}} [J_{\frac{1}{2}\nu}(\frac{1}{2}ax)]^2$<br>$a > 0, \quad \operatorname{Re} \nu > -1$                              | $2\pi^{-1} y^{-\frac{1}{2}} (a^2-y^2)^{-\frac{1}{2}} \quad 0 < y < a$<br>$0 \quad a < y < \infty$                                                                                                                     |
| (30) | $x^{\frac{1}{2}-\nu} [J_\nu(\frac{1}{2}ax)]^2$<br>$a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$                             | $\frac{2^{1-\nu} y^{\nu-\frac{1}{2}} (a^2-y^2)^{\nu-\frac{1}{2}}}{\pi^{\frac{1}{2}} a^{2\nu} \Gamma(\nu+\frac{1}{2})} \quad 0 < y < a$<br>$0 \quad a < y < \infty$                                                    |

Bessel functions of  $kx$  (cont'd)

|      | $f(x)$                                                                                                                                     | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                                                                                                                                                          |
|------|--------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (31) | $x^{\frac{1}{2}-\nu} J_\nu(ax) J_\nu(bx)$<br>$a, b > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$                                        | $\frac{[y^2 - (a-b)^2]^{\nu-\frac{1}{2}} [(a+b)^2 - y^2]^{\nu-\frac{1}{2}}}{y^{\nu-\frac{1}{2}} 2^{3\nu-1} \pi^{\frac{1}{2}} (ab)^\nu \Gamma(\nu + \frac{1}{2})}$ $ a-b  < y < a+b$<br>0<br>$0 < y <  a-b  \quad \text{or} \quad a+b < y < \infty$                                                                                                                                                                                                                                                       |
| (32) | $x^{\frac{1}{2}} J_{\frac{1}{2}(\nu+n)}(\frac{1}{2}ax) J_{\frac{1}{2}(\nu-n)}(\frac{1}{2}ax)$<br>$a > 0, \quad \operatorname{Re} \nu > -1$ | $2\pi^{-1} y^{-\frac{1}{2}} (a^2 - y^2)^{-\frac{1}{2}} T_n(a^{-1}y)$ $0 < y < a$<br>0<br>$a < y < \infty$                                                                                                                                                                                                                                                                                                                                                                                                |
| (33) | $x^{-\frac{1}{2}} J_\mu^2(\frac{1}{2}ax)$<br>$a > 0$<br>$\operatorname{Re} \nu + \operatorname{Re} 2\mu > -1$                              | $(\frac{1}{2}a)^{2\mu} y^{-2\mu-\frac{1}{2}} \frac{\Gamma(\frac{1}{2} + \frac{1}{2}\nu + \mu)}{[\Gamma(\mu+1)]^2 \Gamma(\frac{1}{2} + \frac{1}{2}\nu - \mu)}$ $\times \frac{1}{2} {}_2F_1[\frac{1}{2} - \frac{1}{2}\nu + \mu, \frac{1}{2} + \frac{1}{2}\nu + \mu; \mu+1; \frac{1}{2} - \frac{1}{2}(1 - a^2/y^2)^{\frac{1}{2}}]^2$ $a < y < \infty$                                                                                                                                                       |
| (34) | $x^{\frac{1}{2}-\mu} J_\mu(ax) J_\nu(bx)$<br>$a, b > 0$<br>$\operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu > -\frac{1}{2}$        | $\frac{y^{\mu-\frac{1}{2}} (\sinh u)^{\mu-\frac{1}{2}}}{(\frac{1}{2}\pi^3)^{\frac{1}{2}} a^\mu b^{1-\mu}} e^{(\mu-\frac{1}{2})\pi i}$ $\times \sin[(\nu-\mu)\pi] Q_{\nu-\frac{1}{2}}^{-\mu}(\cosh u)$ $0 < y < a-b$<br>$\frac{b^{\mu-1} y^{\mu-\frac{1}{2}}}{(2\pi)^{\frac{1}{2}} a^\mu} (\sin v)^{\mu-\frac{1}{2}} P_{\nu-\frac{1}{2}}^{-\mu}(\cos v)$ $ a-b  < y < a+b$<br>0<br>$0 < y < b-a \quad \text{or} \quad a+b < y < \infty$<br>$2by \cosh u = a^2 - b^2 - y^2$ $2by \cos v = b^2 + y^2 - a^2$ |

Bessel functions of  $kx$  (cont'd)

|      | $f(x)$                                                                                                                                                       | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                                                                                                                                                |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (35) | $x^{\frac{1}{2}-\nu} J_\mu(ax) J_\mu(bx) \quad a, b > 0$<br>$\operatorname{Re} \mu > -1, \quad \operatorname{Re} \nu > -\frac{1}{2}$                         | $0 \quad 0 < y <  a-b $<br>$\frac{(ab)^{\nu-1}}{(2\pi)^{\frac{1}{2}} y^{\nu-\frac{1}{2}}} (\sin u)^{\nu-\frac{1}{2}} P_{\mu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cos u)$ $ a-b  < y < a+b$<br>$\frac{(ab)^{\nu-1} (\sinh v)^{\nu-\frac{1}{2}}}{(\frac{1}{2}\pi^3)^{\frac{1}{2}} y^{\nu-\frac{1}{2}}} e^{(\nu-\frac{1}{2})\pi i}$ $\times \sin[(\mu-\nu)\pi] Q_{\mu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cosh v)$ $a+b < y < \infty$<br>$2ab \cos u = a^2 + b^2 - y^2$<br>$2ab \cosh v = y^2 - a^2 - b^2$ |
| (36) | $x^{\rho-\mu-\nu+\frac{1}{2}} J_\mu(ax) J_\rho(bx)$<br>$b > a > 0$<br>$\operatorname{Re} \rho > -1, \quad \operatorname{Re}(\rho - \mu - \nu) < \frac{1}{2}$ | $0 \quad 0 < y < b-a$                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |
| (37) | $x^{\rho-\mu-\nu-3/2} J_\mu(ax) J_\rho(bx)$<br>$b > a > 0$<br>$\operatorname{Re} \rho > 0, \quad \operatorname{Re}(\rho - \mu - \nu) < 5/2$                  | $\frac{2^{\rho-\mu-\nu-1} y^{\nu+\frac{1}{2}} a^\mu \Gamma(\rho)}{b^\rho \Gamma(\mu+1) \Gamma(\nu+1)}$ $0 < y < b-a$                                                                                                                                                                                                                                                                                                                                                                           |
| (38) | $x^{-\frac{1}{2}} J_\mu(xa \sin \phi \cos \psi) J_\rho(ax)$<br>$a > 0, \quad 0 < \phi, \psi < \frac{1}{2}\pi$<br>$\operatorname{Re}(\mu + \nu + \rho) > -1$  | $\frac{a^{-\frac{1}{2}} \Gamma[\frac{1}{2}(1+\sigma+\rho)]}{\Gamma(\mu+1) \Gamma(\nu+1) \Gamma[\frac{1}{2}(1-\sigma+\rho)]}$ $\times (\sin \phi \cos \psi)^\mu (\sin \psi \cos \phi)^{\nu+\frac{1}{2}}$ $\times {}_2F_1\left(\frac{1+\sigma-\rho}{2}, \frac{1+\sigma+\rho}{2}; \mu+1; \sin^2 \phi\right)$ $\times {}_2F_1\left(\frac{1+\sigma-\rho}{2}, \frac{1+\sigma+\rho}{2}; \nu+1; \sin^2 \psi\right)$<br>$\sigma = \mu + \nu, \quad y = a \cos \phi \sin \psi$                           |

Bessel functions of  $kx$  (cont'd)

|      | $f(x)$                                                                                                                                                                       | $\int_0^\infty f(x) J_\nu(xy)(xy)^{1/2} dx \quad y > 0$                                                                                                                                                       |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (39) | $x^{1/2} J_\mu(xa \sin \phi \cos \psi) J_{\nu-\mu}(ax)$<br>$a > 0, \quad 0 < \phi, \psi < \frac{1}{2}\pi$<br>$\operatorname{Re} \nu > -1$                                    | $2\pi^{-1} a^{-3/2} \sin(\mu\pi)(\sin \phi)^\mu (\sin \psi)^{\nu+1/2}$<br>$\times (\cos \phi)^{1/2-\nu} (\cos \psi)^{-\mu}$<br>$\times [\cos(\phi+\psi) \cos(\phi-\psi)]^{-1}$<br>$y = a \cos \phi \sin \psi$ |
| (40) | $x^\lambda J_\mu(ax) J_\rho(bx)$                                                                                                                                             | see Bailey, W. N., 1936: <i>Proc. London Math. Soc.</i> (2), 40, 37-48.                                                                                                                                       |
| (41) | $x^{2n-\mu-3/2} (x^2+c^2)^{-1}$<br>$\times J_\nu(ax) J_\mu(bx) \quad a > b > 0$<br>$\operatorname{Re} \nu > \frac{1}{2} - n$<br>$\operatorname{Re} \mu > 2n - 9/2$           | $(-1)^{n+1} c^{2n-\mu-2} y^{1/2} I_\mu(bc)$<br>$\times I_\nu(yc) K_\nu(ac) \quad 0 < y < a - b$                                                                                                               |
| (42) | $x^{\nu-M+1/2} \prod_{i=1}^k J_{\mu_i}(a_i x)$<br>$a_i > 0, \quad \sum_{i=1}^k \mu_i = M$<br>$-1 < \operatorname{Re} \nu < \operatorname{Re} M + \frac{1}{2}k - \frac{1}{2}$ | 0<br>$\sum_{i=1}^k a_i < y < \infty$                                                                                                                                                                          |
| (43) | $x^{\nu-M-3/2} \prod_{i=1}^k J_{\mu_i}(a_i x)$<br>$a_i > 0, \quad M = \sum_{i=1}^k \mu_i$<br>$0 < \operatorname{Re} \nu < \operatorname{Re} M + k/2 + 3/2$                   | $2^{\nu-M-1} y^{1/2-\nu} \Gamma(\nu) \prod_{i=1}^k \frac{a_i^{\mu_i}}{\Gamma(1+\mu_i)}$<br>$\sum_{i=1}^k a_i < y < \infty$                                                                                    |
| (44) | $x^{-\lambda-1/2} Y_\mu(ax)$                                                                                                                                                 | see under Mellin transforms                                                                                                                                                                                   |
| (45) | $x^{1/2} (x^2+\beta^2)^{-1} Y_{\nu-2n-1}(ax)$<br>$a > 0, \quad \operatorname{Re} \beta > 0$<br>$\operatorname{Re} \nu > n - \frac{1}{2}$                                     | $(-1)^n y^{1/2} I_\nu(y\beta) K_{\nu-2n-1}(a\beta)$<br>$0 < y < a$                                                                                                                                            |

**Bessel functions of  $kx$  (cont'd)**

|      | $f(x)$                                                                                                                                                                                                                                                                                 | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                           |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (46) | $x^{\frac{1}{2}}(x^2 + \beta^2)^{-1}$<br>$\times \{ \cos [\frac{1}{2}(\nu - \mu)\pi] J_\mu(ax)$<br>$+ \sin [\frac{1}{2}(\nu - \mu)\pi] Y_\mu(ax) \}$<br>$a > 0, \quad \text{Re } \beta > 0$<br>$\text{Re}(\nu \pm \mu) > -2$                                                           | $y^{\frac{1}{2}} I_\nu(\beta y) K_\mu(a\beta) \quad 0 < y \leq a$                                                                                                         |
| (47) | $x^{\rho + \frac{1}{2}}(\beta^2 + x^2)^{-1}$<br>$\times \{ \cos [\frac{1}{2}(\rho - \mu + \nu)\pi] J_\mu(ax)$<br>$+ \sin [\frac{1}{2}(\rho - \mu + \nu)\pi] Y_\mu(ax) \}$<br>$a > 0, \quad \text{Re } \beta > 0$<br>$\text{Re}(\nu \pm \mu + \rho) > -2, \quad \text{Re } \rho < 1$    | $\beta^\rho y^{\frac{1}{2}} I_\nu(\beta y) K_\mu(a\beta) \quad 0 < y < a$                                                                                                 |
| (48) | $x^{\frac{1}{2}} J_{\frac{1}{2}\nu}(\frac{1}{2}ax) Y_{\frac{1}{2}\nu}(\frac{1}{2}ax)$<br>$a > 0, \quad \text{Re } \nu > -1$                                                                                                                                                            | $0 \quad 0 < y < a$<br>$-2\pi^{-1} y^{-\frac{1}{2}} (y^2 - a^2)^{-\frac{1}{2}} \quad a < y < \infty$                                                                      |
| (49) | $x^{\nu + \frac{1}{2}} J_\nu(\frac{1}{2}ax) Y_\nu(\frac{1}{2}ax)$<br>$a > 0, \quad  \text{Re } \nu  < \frac{1}{2}$                                                                                                                                                                     | $0 \quad 0 < y < a$<br>$-\frac{2^{\nu+1} a^{2\nu} y^{-\nu-\frac{1}{2}} (y^2 - a^2)^{-\nu-\frac{1}{2}}}{\pi^{\frac{1}{2}} \Gamma(\frac{1}{2} - \nu)} \quad a < y < \infty$ |
| (50) | $x^{\rho + \frac{1}{2}}(x^2 + y^2)^{-1} J_\mu(bx)$<br>$\times \{ \cos [\frac{1}{2}(\rho + \mu)\pi] J_\nu(ax)$<br>$+ \sin [\frac{1}{2}(\rho + \mu)\pi] Y_\nu(ax) \}$<br>$a > b > 0, \quad \text{Re } \rho < 3/2$<br>$\text{Re}(\mu + \rho + 2\nu) > -2$<br>$\text{Re}(\mu + \rho) > -2$ | $y^\rho y^{\frac{1}{2}} I_\mu(b y) I_\nu(y y) K_\nu(a y) \quad 0 < y < a - b$                                                                                             |

Bessel functions of  $kx$  (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                                                                                                 | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                  |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------|
| (51) | $x^{\rho+\frac{1}{2}}(x^2+\beta^2)^{-1} \prod_{i=1}^k [J_{\mu_i}(c_i x)]$ $\times \{ \cos [\frac{1}{2}(\rho+M-\mu)\pi] J_\mu(ax)$ $+ \sin [\frac{1}{2}(\rho+M-\mu)\pi] Y_\mu(ax) \}$ $a > 0, \quad c_i > 0$ $\nu + \sum_{i=1}^k \mu_i = M, \quad \text{Re } \rho < (k+3)/2$ $\text{Re}(\rho+M) >  \text{Re } \mu  - 2$ | $\beta^\rho \gamma^{\frac{1}{2}} I_\nu(\beta y) K_\mu(a\beta) \prod_{i=1}^k I_{\mu_i}(\beta c_i)$ $0 < y < a - \sum_{i=1}^k c_i$ |

## 8.12. Bessel functions of other arguments

|     |                                                                                                                                                   |                                                                                                                                                                                |
|-----|---------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $x^{\frac{1}{2}} J_{\frac{1}{2}\nu}(\frac{1}{4}ax^2)$ $a > 0, \quad \text{Re } \nu > -1$                                                          | $2a^{-1} \gamma^{\frac{1}{2}} J_{\frac{1}{2}\nu}(\gamma^2/a)$                                                                                                                  |
| (2) | $x^{\frac{1}{2}} \exp(-\frac{1}{4}ax^2) J_{\frac{1}{2}\nu}(\frac{1}{4}\beta x^2)$ $\text{Re } a >  \text{Im } \beta , \quad \text{Re } \nu > -1$  | $2(a^2+\beta^2)^{-\frac{1}{2}} \gamma^{\frac{1}{2}} \exp\left(-\frac{a\gamma^2}{a^2+\beta^2}\right)$ $\times J_{\frac{1}{2}\nu}\left(\frac{\beta\gamma^2}{a^2+\beta^2}\right)$ |
| (3) | $x^{(\frac{1}{2}-\nu)/3} \exp(\frac{1}{4}ax^2 i)$ $\times J_{(\nu-\frac{1}{2})/3}(\frac{1}{4}ax^2)$ $\text{Im } a > 0, \quad \text{Re } \nu > -1$ | $a^{(\nu-2)/3} \gamma^{(\frac{1}{2}-\nu)/3} \exp\left(\frac{\nu+1}{6}\pi i - \frac{\gamma^2}{4a}i\right)$ $\times J_{(\nu-\frac{1}{2})/3}\left(\frac{\gamma^2}{4a}\right)$     |
| (4) | $x^{(\frac{1}{2}-\nu)/3} \sin(\frac{1}{4}ax^2)$ $\times J_{(\nu-\frac{1}{2})/3}(\frac{1}{4}ax^2)$ $a > 0, \quad \text{Re } \nu > -5/2$            | $a^{(\nu-2)/3} \gamma^{(\frac{1}{2}-\nu)/3} \sin\left(\frac{\nu+1}{6}\pi - \frac{\gamma^2}{4a}\right)$ $\times J_{(\nu-\frac{1}{2})/3}\left(\frac{\gamma^2}{4a}\right)$        |

## Bessel functions of other arguments (cont'd)

|      | $f(x)$                                                                                                                                            | $\int_0^\infty f(x) J_\nu(xy)(xy)^{1/2} dx \quad \gamma > 0$                                                                                                                                                                                                                                               |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (5)  | $x^{(\frac{1}{2}-\nu)/3} \cos(\frac{1}{4}ax^2)$<br>$\times J_{(\nu-\frac{1}{2})/3}(\frac{1}{4}ax^2)$<br>$a > 0, \quad \operatorname{Re} \nu > -1$ | $a^{(\nu-2)/3} \gamma^{(\frac{1}{2}-\nu)/3} \cos\left(\frac{\nu+1}{6}\pi - \frac{\gamma^2}{4a}\right)$<br>$\times J_{(\nu-\frac{1}{2})/3}\left(\frac{\gamma^2}{4a}\right)$                                                                                                                                 |
| (6)  | $x^{1/2} [J_{\frac{1}{4}\nu}(\frac{1}{4}ax^2)]^2$<br>$a > 0, \quad \operatorname{Re} \nu > -1$                                                    | $-\frac{\gamma^{1/2}}{a} J_{\frac{1}{4}\nu}\left(\frac{\gamma^2}{4a}\right) Y_{\frac{1}{4}\nu}\left(\frac{\gamma^2}{4a}\right)$                                                                                                                                                                            |
| (7)  | $x^{1/2} J_{\frac{1}{4}\nu}(\frac{1}{4}ax^2) J_{-\frac{1}{4}\nu}(\frac{1}{4}ax^2)$<br>$a > 0, \quad \operatorname{Re} \nu > -2$                   | $\frac{\gamma^{1/2}}{a} J_{\frac{1}{4}\nu}\left(\frac{\gamma^2}{4a}\right) \left[ J_{\frac{1}{4}\nu}\left(\frac{\gamma^2}{4a}\right) \sin\left(\frac{\pi\nu}{4}\right) \right.$<br>$\left. - Y_{\frac{1}{4}\nu}\left(\frac{\gamma^2}{4a}\right) \cos\left(\frac{\pi\nu}{4}\right) \right]$                 |
| (8)  | $x^{1/2} J_{\frac{1}{4}\nu-\mu}(ax^2) J_{\frac{1}{4}\nu+\mu}(ax^2)$<br>$a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$                        | $\frac{2}{\pi\gamma^{3/2}} [e^{\frac{1}{4}\nu\pi i} W_{\mu, \frac{1}{4}\nu}(u) W_{-\mu, \frac{1}{4}\nu}(u)$<br>$+ e^{-\frac{1}{4}\nu\pi i} W_{\mu, \frac{1}{4}\nu}(v) W_{-\mu, \frac{1}{4}\nu}(v)]$<br>$u = \frac{\gamma^2}{8a} e^{\frac{1}{2}\pi i}, \quad v = \frac{\gamma^2}{8a} e^{-\frac{1}{2}\pi i}$ |
| (9)  | $x^{-1/2} J_\nu(ax^{-1})$<br>$a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$                                                                  | $\gamma^{-1/2} J_{2\nu}(2a^{1/2}\gamma^{1/2})$                                                                                                                                                                                                                                                             |
| (10) | $x^{-5/2} J_\nu(ax^{-1})$<br>$a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$                                                                  | $a^{-1} \gamma^{1/2} J_{2\nu}(2a^{1/2}\gamma^{1/2})$                                                                                                                                                                                                                                                       |
| (11) | $x^{-3/2} J_{\nu-1}(ax^{-1})$<br>$a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$                                                              | $a^{-1/2} J_{2\nu-1}(2a^{1/2}\gamma^{1/2})$                                                                                                                                                                                                                                                                |

## Bessel functions of other arguments (cont'd)

|      | $f(x)$                                                                                                                                                                                                                            | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad \gamma > 0$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (12) | $x^{-2\nu} J_{\frac{1}{2}-\nu}(ax^{-1})$<br>$a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < 3$                                                                                                                               | $-\frac{1}{2} i \csc(2\nu\pi)(\gamma/a)^{\nu-\frac{1}{2}}$<br>$\times [e^{2\nu\pi i} J_{1-2\nu}(u) J_{2\nu-1}(v)$<br>$- e^{-2\nu\pi i} J_{2\nu-1}(u) J_{1-2\nu}(v)]$<br>$u = (\frac{1}{2}a\gamma)^{\frac{1}{2}} e^{\frac{1}{4}\pi i}$<br>$v = (\frac{1}{2}a\gamma)^{\frac{1}{2}} e^{-\frac{1}{4}\pi i}$                                                                                                                                                                                                                                                                                        |
| (13) | $x^{\rho-3/2} J_\mu(x^{-1})$<br>$-3/2 - \operatorname{Re} \nu < \operatorname{Re} \rho < \operatorname{Re} \mu + 3/2$                                                                                                             | $\frac{1}{2}\pi \csc[\frac{1}{2}(\mu-\nu-\rho)\pi] \gamma^{\nu+\frac{1}{2}}$<br>$\times \left[ A {}_0F_3\left(1+\nu, 1+\frac{\rho-\mu+\nu}{2}, 1+\frac{\mu+\nu+\rho}{2}; \frac{\gamma^2}{16}\right)$<br>$- \gamma^\mu B {}_0F_3\left(1+\mu, 1+\frac{\mu+\nu-\rho}{2}, 1+\frac{\mu-\nu-\rho}{2}; \frac{\gamma^2}{16}\right) \right]$<br>$A^{-1} = 2^{\nu+\rho} \Gamma(1+\nu) \Gamma[1+\frac{1}{2}(\rho-\mu+\nu)]$<br>$\times \Gamma[1+\frac{1}{2}(\rho+\mu+\nu)]$<br>$B^{-1} = 2^{2\mu-\rho} \Gamma(1+\mu) \Gamma[1+\frac{1}{2}(\mu+\nu-\rho)]$<br>$\times \Gamma[1+\frac{1}{2}(\mu-\nu-\rho)]$ |
| (14) | $x^{\frac{1}{2}} (\beta^2+x^2)^{-\frac{1}{2}} \exp\left(-\frac{a^2\beta}{\beta^2+x^2}\right)$<br>$\times J_\nu\left(\frac{a^2x}{\beta^2+x^2}\right)$<br>$\operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$ | $\gamma^{-\frac{1}{2}} e^{-\beta\gamma} J_{2\nu}(2a\gamma^{\frac{1}{2}})$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      |
| (15) | $J_{2\nu-1}(ax^{\frac{1}{2}})$<br>$\operatorname{Re} \nu > -\frac{1}{2}$                                                                                                                                                          | $\frac{1}{2}a\gamma^{-3/2} J_{\nu-1}(\frac{1}{4}a^2\gamma^{-1})$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               |
| (16) | $x^{-\frac{1}{2}} J_{2\nu}(ax^{\frac{1}{2}})$<br>$\operatorname{Re} \nu > -\frac{1}{2}$                                                                                                                                           | $\gamma^{-\frac{1}{2}} J_\nu(\frac{1}{4}a^2\gamma^{-1})$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |
| (17) | $x^{-\frac{1}{2}} e^{-\beta x} J_{2\nu}(2ax^{\frac{1}{2}})$<br>$\operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$                                                                                          | $\gamma^{\frac{1}{2}} (\gamma^2+\beta^2)^{-\frac{1}{2}} \exp\left(-\frac{a^2\beta}{\beta^2+\gamma^2}\right)$<br>$\times J_\nu\left(\frac{a^2\gamma}{\beta^2+\gamma^2}\right)$                                                                                                                                                                                                                                                                                                                                                                                                                  |



## Bessel functions of other arguments (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                                                        | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                           |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (18) | $x^{\nu+\frac{1}{2}}(x^2+\beta^2)^{-\frac{1}{2}\mu}$<br>$\times J_\mu[a(x^2+\beta^2)^{\frac{1}{2}}] \quad a > 0$<br>$\operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > \operatorname{Re} \nu > -1$                                                                   | $a^{-\mu} y^{\nu+\frac{1}{2}} \beta^{-\mu+\nu+1} (a^2-y^2)^{\frac{1}{2}\mu-\frac{1}{2}\nu-\frac{1}{2}}$<br>$\times J_{\mu-\nu-1}[\beta(a^2-y^2)^{\frac{1}{2}}] \quad 0 < y < a$<br>$0 \quad a < y < \infty$                                                               |
| (19) | $x^{\nu+\frac{1}{2}}(x^2+\beta^2)^{-\frac{1}{2}\mu-1}$<br>$\times J_{\mu-1}[a(x^2+\beta^2)^{\frac{1}{2}}] \quad a > 0$<br>$\operatorname{Re} \beta > 0, \operatorname{Re}(\mu+2) > \operatorname{Re} \nu > -1$                                                                | $(\frac{1}{2}a)^{\mu-1} \beta^\nu [\Gamma(\mu)]^{-1} y^{\frac{1}{2}} K_\nu(\beta y)$<br>$a < y < \infty$                                                                                                                                                                  |
| (20) | $x^{\nu-3/2}(x^2+\beta^2)^{-\mu/2}$<br>$\times J_\mu[a(x^2+\beta^2)^{1/2}] \quad a > 0$<br>$\operatorname{Re} \beta > 0, \operatorname{Re}(\mu+2) > \operatorname{Re} \nu > 0$                                                                                                | $\beta^{-\mu} 2^{\nu-1} \Gamma(\nu) y^{\frac{1}{2}-\nu} J_\mu(a\beta)$<br>$a < y < \infty$                                                                                                                                                                                |
| (21) | $x^{\nu+\frac{1}{2}}(x^2+a^2)^{-1}(x^2+\beta^2)^{-\frac{1}{2}\mu}$<br>$\times J_\mu[c(x^2+\beta^2)^{\frac{1}{2}}] \quad a > 0$<br>$\operatorname{Re} a > 0, \operatorname{Re} \beta > 0, \quad c > 0$<br>$-1 < \operatorname{Re} \nu < 2 + \operatorname{Re} \mu$             | $a^\nu y^{\frac{1}{2}} (\beta^2-a^2)^{-\frac{1}{2}\mu} J_\mu[c(\beta^2-a^2)^{\frac{1}{2}}]$<br>$\times K_\nu(ay) \quad c \leq y < \infty$                                                                                                                                 |
| (22) | $x^{\nu+2n-3/2}(x^2+a^2)^{-1}(x^2+\beta^2)^{-\mu/2}$<br>$\times J_\mu[c(x^2+\beta^2)^{1/2}] \quad a > 0$<br>$\operatorname{Re} a > 0, \operatorname{Re} \beta > 0, \quad c > 0$<br>$-n < \operatorname{Re} \nu < 4-2n + \operatorname{Re} \mu$                                | $(-1)^{n+1} y^{\frac{1}{2}} a^{\nu+2n-2} (\beta^2-a^2)^{-\frac{1}{2}\mu}$<br>$\times J_\mu[c(\beta^2-a^2)^{\frac{1}{2}}] K_\nu(ay) \quad c < y < \infty$                                                                                                                  |
| (23) | $x^{\nu+\frac{1}{2}}(x^2+\beta^2)^{-\frac{1}{2}\nu-\frac{1}{2}}$<br>$\times C_{2n+1}^{\nu+\frac{1}{2}}[\beta(x^2+\beta^2)^{-\frac{1}{2}}]$<br>$\times J_{\nu+3/2+2n}[a(x^2+\beta^2)^{\frac{1}{2}}]$<br>$a > 0, \operatorname{Re} \beta > 0, \operatorname{Re} \nu > -1$       | $(-1)^n 2^{\frac{1}{2}} \pi^{-\frac{1}{2}} a^{\frac{1}{2}-\nu} y^{\nu+\frac{1}{2}} (a^2-y^2)^{-\frac{1}{2}}$<br>$\times \sin[\beta(a^2-y^2)^{\frac{1}{2}}]$<br>$\times C_{2n+1}^{\nu+\frac{1}{2}}[(1-y^2/a^2)^{\frac{1}{2}}] \quad 0 < y < a$<br>$0 \quad a < y < \infty$ |
| (24) | $x^{\nu+\frac{1}{2}}(x^2+\beta^2)^{-\frac{1}{2}\nu-\frac{1}{2}}$<br>$\times C_{2n}^{\nu+\frac{1}{2}}[\beta(x^2+\beta^2)^{-\frac{1}{2}}]$<br>$\times J_{\nu+\frac{1}{2}+2n}[a(x^2+\beta^2)^{\frac{1}{2}}]$<br>$a > 0, \operatorname{Re} \beta > 0, \operatorname{Re} \nu > -1$ | $(-1)^n 2^{\frac{1}{2}} \pi^{-\frac{1}{2}} a^{\frac{1}{2}-\nu} y^{\nu+\frac{1}{2}} (a^2-y^2)^{-\frac{1}{2}}$<br>$\times \cos[\beta(a^2-y^2)^{\frac{1}{2}}]$<br>$\times C_{2n}^{\nu+\frac{1}{2}}[(1-y^2/a^2)^{\frac{1}{2}}] \quad 0 < y < a$<br>$0 \quad a < y < \infty$   |

## Bessel functions of other arguments (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                                 | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                       |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (25) | $x^{\nu-3/2} (x^2 + \beta^2)^{-n\mu/2}$<br>$\times \prod_{i=1}^n J_{\mu_i} [a_i (x^2 + \beta^2)^{\frac{1}{2}}]$<br>$a_i > 0, \quad \operatorname{Re} \beta > 0$<br>$\operatorname{Re} (n\mu + \frac{1}{2}n + \frac{1}{2}) > \operatorname{Re} \nu > 0$ | $2^{\nu-1} \beta^{-n\mu} \Gamma(\nu) y^{\frac{1}{2}-\nu} \prod_{i=1}^n J_{\mu_i}(a_i \beta)$<br>$\sum_{i=1}^n a_i < y < \infty$                                                                       |
| (26) | $x^{\nu+\frac{1}{2}} \prod_{i=1}^n z_i^{-\mu_i} J_{\mu_i}(a_i z_i)$<br>$a_i > 0, \quad \operatorname{Re} \beta_i > 0$<br>$z_i = (x^2 + \beta_i^2)^{\frac{1}{2}}$<br>$\frac{1}{2}n + \sum_{i=1}^n \mu_i - \frac{1}{2} > \operatorname{Re} \nu > -1$     | $0$<br>$\sum_{i=1}^n a_i < y < \infty$                                                                                                                                                                |
| (27) | $x^{\nu-3/2} \prod_{i=1}^n z_i^{-\mu_i} J_{\mu_i}(a_i z_i)$<br>$a_i > 0, \quad \operatorname{Re} \beta_i > 0$<br>$z_i = (x^2 + \beta_i^2)^{\frac{1}{2}}$<br>$\frac{1}{2}n + \sum_{i=1}^n \mu_i + 3/2 > \operatorname{Re} \nu > 0$                      | $2^{\nu-1} \Gamma(\nu) y^{\frac{1}{2}-\nu}$<br>$\times \prod_{i=1}^n [\beta_i^{-\mu_i} J_{\mu_i}(a_i \beta_i)]$<br>$\sum_{i=1}^n a_i < y < \infty$                                                    |
| (28) | $x^{\nu+\frac{1}{2}} (1-x^2)^{\frac{1}{2}\mu} J_{\mu} [a(1-x^2)^{\frac{1}{2}}]$<br>$0 < x < 1$<br>$0$<br>$1 < x < \infty$<br>$\operatorname{Re} \mu > -1, \quad \operatorname{Re} \nu > -1$                                                            | $a^{\mu} y^{\nu+\frac{1}{2}} (a^2 + y^2)^{-\frac{1}{2}(\mu+\nu+1)}$<br>$\times J_{\mu+\nu+1} [(a^2 + y^2)^{\frac{1}{2}}]$                                                                             |
| (29) | $0$<br>$x^{\frac{1}{2}-\nu} (x^2 - c^2)^{\frac{1}{2}\mu} J_{\mu} [a(x^2 - c^2)^{\frac{1}{2}}]$<br>$c < x < \infty$<br>$a > 0, \quad \operatorname{Re} \nu > \operatorname{Re} \mu > -1$                                                                | $0$<br>$0 < y < a$<br>$a^{\mu} c^{1+\mu-\nu} y^{-\nu+\frac{1}{2}} (y^2 - a^2)^{\frac{1}{2}\nu-\frac{1}{2}\mu-\frac{1}{2}}$<br>$\times J_{\nu-\mu-1} [c(y^2 - a^2)^{\frac{1}{2}}]$<br>$a < y < \infty$ |

## Bessel functions of other arguments (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                       | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                           |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (30) | $0 \qquad 0 < x < c$ $x^{\frac{1}{2}-\nu}(x^2+\beta^2)^{-1}(x^2-c^2)^{\frac{1}{2}\mu}$ $\times J_\mu[a(x^2-c^2)^{\frac{1}{2}}] \quad c < x < \infty$ $a > 0, \quad \text{Re } \beta > 0$ $-1 < \text{Re } \mu < 2 + \text{Re } \nu$          | $\beta^{-\nu}(c^2+\beta^2)^{\frac{1}{2}\mu} y^{\frac{1}{2}} K_\mu[a(c^2+\beta^2)^{\frac{1}{2}}]$ $\times I_\nu(y\beta) \qquad 0 < y < a$                                                                                                                  |
| (31) | $0 \qquad 0 < x < c$ $x^{\frac{1}{2}-\nu}(x^2+\beta^2)^{-1}(x^2-c^2)^{\frac{1}{2}\mu+n-1}$ $\times J_\mu[a(x^2-c^2)^{\frac{1}{2}}] \quad c < x < \infty$ $a > 0, \quad \text{Re } \beta > 0$ $-n < \text{Re } \mu < 4 - 2n + \text{Re } \nu$ | $(-1)^{n+1} \beta^{-\nu}(\beta^2+c^2)^{\frac{1}{2}\mu+n-1} y^{\frac{1}{2}}$ $\times K_\mu[a(\beta^2+c^2)^{\frac{1}{2}}] I_\nu(by)$ $0 < y < a$                                                                                                            |
| (32) | $x^{\nu+2n+\frac{1}{2}}(1-x^2)^{\frac{1}{2}\lambda+m}$ $\times J_\lambda[a(1-x^2)^{\frac{1}{2}}] \quad 0 < x < 1$ $0 \qquad 1 < x < \infty$ $a > 0, \quad \text{Re } \lambda > -1, \quad \text{Re } \nu > -1$                                | $a^{-\lambda} y^{-\nu+\frac{1}{2}} \left( \frac{d}{ada} \right)^m \left( \frac{d}{ydy} \right)^n$ $\times \{ a^{2\lambda+2m} y^{2\nu+2n}$ $\times (a^2+y^2)^{-\frac{1}{2}(\lambda+\nu+m+n+1)}$ $\times J_{\lambda+\nu+m+n+1}[(a^2+y^2)^{\frac{1}{2}}] \}$ |
| (33) | $x^\rho(1-x^2)^\mu J_\lambda[a(1-x^2)^{\frac{1}{2}}]$ $0 < x < 1$ $0 \qquad 1 < x < \infty$                                                                                                                                                  | <p>see Bailey, W. N., 1938: <i>Quart. J. Math. Oxford Series</i> 9, 141-147.</p>                                                                                                                                                                          |
| (34) | $x^{\frac{1}{2}} J_{\frac{1}{2}\nu} \{ \frac{1}{2}a[(x^2+\beta^2)^{\frac{1}{2}}-\beta] \}$ $\times J_{\frac{1}{2}\nu} \{ \frac{1}{2}a[(x^2+\beta^2)^{\frac{1}{2}}+\beta] \}$ $a > 0, \quad \text{Re } \nu > -1$                              | $2\pi^{-1} y^{-\frac{1}{2}} (a^2-y^2)^{-\frac{1}{2}}$ $\times \cos [\beta(a^2-y^2)^{\frac{1}{2}}] \qquad 0 < y < a$ $0 \qquad a < y < \infty$                                                                                                             |
| (35) | $x^{\frac{1}{2}} Y_{\frac{1}{2}\nu} (\frac{1}{4}ax^2)$ $a > 0, \quad \text{Re } \nu > -1$                                                                                                                                                    | $-2a^{-1} y^{\frac{1}{2}} \mathbf{H}_{\frac{1}{2}\nu}(y^2/a)$                                                                                                                                                                                             |
| (36) | $x^{\frac{1}{2}} J_{\frac{1}{2}\nu} (\frac{1}{4}ax^2) Y_{\frac{1}{2}\nu} (\frac{1}{4}ax^2)$ $a > 0, \quad \text{Re } \nu > -1$                                                                                                               | $-2a^{-1} y^{\frac{1}{2}} \left[ J_{\frac{1}{2}\nu} \left( \frac{y^2}{4a} \right) \right]^2$                                                                                                                                                              |

## Bessel functions of other arguments (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                                                                                       | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                   |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (37) | $x^{-\frac{1}{2}} Y_\nu(ax^{-1})$<br>$a > 0, \quad  \operatorname{Re} \nu  < \frac{1}{2}$                                                                                                                                                                                                                    | $-2\pi^{-1} y^{-\frac{1}{2}} [K_{2\nu}(2a^{\frac{1}{2}} y^{\frac{1}{2}})$<br>$- \frac{1}{2} \pi Y_{2\mu}(2a^{\frac{1}{2}} y^{\frac{1}{2}})]$                                                                                                                                                                      |
| (38) | $x^{-5/2} Y_\nu(ax^{-1})$<br>$a > 0, \quad  \operatorname{Re} \nu  < \frac{1}{2}$                                                                                                                                                                                                                            | $2y^{\frac{1}{2}} a^{-1} \pi^{-1} [K_{2\nu}(2a^{\frac{1}{2}} y^{\frac{1}{2}})$<br>$+ \frac{1}{2} \pi Y_{2\nu}(2a^{\frac{1}{2}} y^{\frac{1}{2}})]$                                                                                                                                                                 |
| (39) | $x^{-\frac{1}{2}} Y_{2\nu}(2ax^{\frac{1}{2}})$<br>$a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$                                                                                                                                                                                                        | $2 \sec(\nu\pi) y^{-\frac{1}{2}} [\frac{1}{2} \cos(\nu\pi) Y_\nu(a^2/y)$<br>$- Y_{-\nu}(a^2/y) + H_{-\nu}(a^2/y)]$                                                                                                                                                                                                |
| (40) | $0 \quad 0 < x < c$<br>$x^{\frac{1}{2}-\nu} (x^2 + \beta^2)^{-1} (x^2 - c^2)^{\frac{1}{2}\mu+n-\frac{1}{2}}$<br>$\times Y_\mu[a(x^2 - c^2)^{\frac{1}{2}}] \quad c < x < \infty$<br>$a > 0, \quad \operatorname{Re} \beta > 0$<br>$-\frac{1}{2} - n < \operatorname{Re} \mu < 3 - 2n + \operatorname{Re} \nu$ | $(-1)^{n+1} \beta^{-\nu} y^{\frac{1}{2}} (\beta^2 + c^2)^{\frac{1}{2}\mu+n-\frac{1}{2}}$<br>$\times K_\mu[a(\beta^2 + c^2)^{\frac{1}{2}}] I_\nu(\beta y)$<br>$0 < y < a$                                                                                                                                          |
| (41) | $x^{\frac{1}{2}} J_{\frac{1}{2}\nu} \{ \frac{1}{2} a [(x^2 + \beta^2)^{\frac{1}{2}} - \beta] \}$<br>$\times Y_{\frac{1}{2}\nu} \{ \frac{1}{2} a [(x^2 + \beta^2)^{\frac{1}{2}} + \beta] \}$<br>$a > 0, \quad \operatorname{Re} \nu > -1$                                                                     | $2\pi^{-1} y^{-\frac{1}{2}} (a^2 - y^2)^{-\frac{1}{2}}$<br>$\times \sin[\beta(a^2 - y^2)^{\frac{1}{2}}] \quad 0 < y < a$<br>$-2\pi^{-1} y^{-\frac{1}{2}} (y^2 - a^2)^{-\frac{1}{2}}$<br>$\times \exp[-\beta(y^2 - a^2)^{\frac{1}{2}}] \quad a < y < \infty$                                                       |
| (42) | $x^{\frac{1}{2}} [H_{\frac{1}{4}\nu+\mu}^{(1)}(ax^2) H_{\frac{1}{4}\nu-\mu}^{(1)}(ax^2)$<br>$- H_{\frac{1}{4}\nu+\mu}^{(2)}(ax^2) H_{\frac{1}{4}\nu-\mu}^{(2)}(ax^2)]$<br>$\operatorname{Re} \nu > -\frac{1}{2}$<br>$\operatorname{Re} (\frac{1}{2} \pm \mu + \frac{1}{2} \nu) > 0$                          | $\frac{8\Gamma(\frac{1}{2}-\mu+\frac{1}{4}\nu) \Gamma(\frac{1}{2}+\mu+\frac{1}{4}\nu)}{i\pi [\Gamma(\frac{1}{2}\nu+1)]^2 y^{3/2}}$<br>$\times M_{\mu, \frac{1}{4}\nu} \left( \frac{y^2}{8a} e^{\frac{1}{2}\pi i} \right)$<br>$\times M_{\mu, \frac{1}{4}\nu} \left( \frac{y^2}{8a} e^{-\frac{1}{2}\pi i} \right)$ |

8.13. Modified Bessel functions of argument  $kx$ 

|     | $f(x)$                                                                                                                                          | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                          |
|-----|-------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $x^{\frac{1}{2}} e^{-\beta x^2} I_\nu(ax)$<br>$\operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > -1$                                   | $\frac{y^{\frac{1}{2}}}{2\beta} \exp\left(\frac{a^2 - y^2}{4\beta}\right) J_\nu\left(\frac{ay}{2\beta}\right)$                                                                                                                                                                           |
| (2) | $x^{\frac{1}{2}} K_\nu(ax)$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1$                                                      | $\frac{y^{\nu+\frac{1}{2}}}{a^\nu (y^2 + a^2)}$                                                                                                                                                                                                                                          |
| (3) | $x^{\mu+\nu+\frac{1}{2}} K_\mu(ax)$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re}(\nu+1) >  \operatorname{Re} \mu $                      | $\frac{2^{\nu+\mu} \Gamma(\mu+\nu+1) y^{\nu+\frac{1}{2}}}{a^{-\mu} (y^2 + a^2)^{\mu+\nu+1}}$                                                                                                                                                                                             |
| (4) | $x^{-\lambda-\frac{1}{2}} K_\mu(ax) \quad \operatorname{Re} a > 0$<br>$\operatorname{Re}(\nu - \lambda + 1) >  \operatorname{Re} \mu $          | $\frac{\Gamma[\frac{1}{2}(\nu - \lambda + \mu + 1)] \Gamma[\frac{1}{2}(\nu - \lambda - \mu + 1)]}{2^{\lambda+1} a^{\nu-\lambda+1} \Gamma(\nu+1) y^{-\nu-\frac{1}{2}}}$<br>$\times {}_2F_1\left(\frac{\nu-\lambda+\mu+1}{2}, \frac{\nu-\lambda-\mu+1}{2}; \nu+1; -\frac{y^2}{a^2}\right)$ |
| (5) | $x^\lambda K_\mu(ax) \begin{matrix} \cos(\beta x) \\ \sin(\beta x) \end{matrix}$                                                                | see under Fourier transforms                                                                                                                                                                                                                                                             |
| (6) | $x^{\frac{1}{2}} K_0(ax) J_\nu(\beta x)$<br>$\operatorname{Re} \nu > -1, \quad \operatorname{Re} a >  \operatorname{Im} \beta $                 | $y^{\frac{1}{2}} r_1^{-1} r_2^{-1} (r_2 - r_1)^\nu (r_2 + r_1)^{-\nu}$<br>$r_1 = [a^2 + (\beta - y)^2]^{\frac{1}{2}}$<br>$r_2 = [a^2 + (\beta + y)^2]^{\frac{1}{2}}$                                                                                                                     |
| (7) | $x^{\nu+\frac{1}{2}} J_\nu(\frac{1}{2}ax) K_\nu(\frac{1}{2}ax)$<br>$ \arg a  < \pi/4, \quad \operatorname{Re} \nu > -\frac{1}{2}$               | $\frac{a^{2\nu} 2^\nu \Gamma(\nu+\frac{1}{2}) y^{\nu+\frac{1}{2}}}{\pi^{\frac{1}{2}} (y^4 + a^4)^{\nu+\frac{1}{2}}}$                                                                                                                                                                     |
| (8) | $x^{\nu+\frac{1}{2}} J_\nu(ax) K_\nu(\beta x)$<br>$\operatorname{Re} \beta >  \operatorname{Im} a , \quad \operatorname{Re} \nu > -\frac{1}{2}$ | $\frac{2^{3\nu} (a\beta)^\nu y^{\nu+\frac{1}{2}} \Gamma(\nu+\frac{1}{2})}{\pi^{\frac{1}{2}} [(a^2 + \beta^2 + y^2)^2 - 4a^2 y^2]^{\nu+\frac{1}{2}}}$                                                                                                                                     |

Modified Bessel functions of  $kx$  (cont'd)

|      | $f(x)$                                                                                                                                                                                                         | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                                                                                                                                                            |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (9)  | $x^{\nu+\frac{1}{2}} J_{\nu-1}(ax) K_{\nu-1}(ax)$<br>$ \arg a  < \pi/4, \quad 0 < \operatorname{Re} \nu < \frac{1}{2}$                                                                                         | $\frac{2^{3\nu-1} a^{2\nu-2} \Gamma(\nu+\frac{1}{2}) y^{\nu+5/2}}{\pi^{\frac{1}{2}} (y^4 + a^4)^{\nu+\frac{1}{2}}}$                                                                                                                                                                                                                                                                                                                                                                                        |
| (10) | $x^{\frac{1}{2}} J_\mu(xa \sin \phi)$<br>$\times K_{\nu-\mu}(xa \cos \phi \cos \psi)$<br>$a > 0, \quad 0 < \phi, \psi < \frac{1}{2}\pi$<br>$\operatorname{Re} \mu > -1, \quad \operatorname{Re} \nu > -1$      | $\frac{(\sin \phi)^\mu (\sin \psi)^{\nu+\frac{1}{2}} (\cos \phi)^{\nu-\mu} (\cos \psi)^{\mu-\nu}}{a^{3/2} (1 - \sin^2 \phi \sin^2 \psi)}$<br>$y = a \sin \psi$                                                                                                                                                                                                                                                                                                                                             |
| (11) | $x^{\nu+\frac{1}{2}} J_\mu(xa \sin \psi)$<br>$\times K_\mu(xa \cos \phi \cos \psi)$<br>$a > 0, \quad 0 < \phi, \psi < \frac{1}{2}\pi$<br>$\operatorname{Re} \nu > -1, \quad \operatorname{Re}(\mu + \nu) > -1$ | $\frac{2^\nu \Gamma(\mu + \nu + 1) [\sin \phi \cos^2(\frac{1}{2}a)]^{\nu+\frac{1}{2}}}{a^{\nu+3/2} (\cos \psi)^{2\nu+2}}$<br>$\times P_\nu^{-\mu}(\cos a)$<br>$y = a \sin \phi$<br>$\tan(\frac{1}{2}a) = \tan \psi \cos \phi$                                                                                                                                                                                                                                                                              |
| (12) | $x^{\mu+\frac{1}{2}} J_\nu(\beta x) K_\mu(ax)$<br>$\operatorname{Re} \alpha >  \operatorname{Im} \beta $<br>$\operatorname{Re} \nu > -1, \quad \operatorname{Re}(\mu + \nu) > -1$                              | $(2\pi)^{-\frac{1}{2}} a^\mu \beta^{-\mu-1} y^{-\mu-\frac{1}{2}} e^{-(\mu+\frac{1}{2})\pi i}$<br>$\times (u^2 - 1)^{-\frac{1}{2}\mu-\frac{1}{4}} Q_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(u)$<br>$2\beta y u = a^2 + \beta^2 + y^2$                                                                                                                                                                                                                                                                           |
| (13) | $x^{-\frac{1}{2}} J_\mu(xa \sin \phi)$<br>$\times K_\rho(xa \cos \phi \cos \psi)$<br>$a > 0, \quad 0 < \phi, \psi < \frac{1}{2}\pi$<br>$\operatorname{Re}(\mu + \nu + 1) > \operatorname{Re} \rho$             | $\frac{(\sin \phi)^\mu (\sin \psi)^{\nu+\frac{1}{2}}}{2a^{\frac{1}{2}} (\cos \phi \cos \psi)^\rho}$<br>$\times \frac{\Gamma[\frac{1}{2}(1 + \mu + \nu - \rho)] \Gamma[\frac{1}{2}(1 + \mu + \nu + \rho)]}{\Gamma(1 + \mu) \Gamma(1 + \nu)}$<br>$\times {}_2F_1\left(\frac{1 + \mu + \nu - \rho}{2}, \frac{1 + \mu - \nu - \rho}{2}; \mu + 1; \sin^2 \phi\right)$<br>$\times {}_2F_1\left(\frac{1 + \mu + \nu - \rho}{2}, \frac{1 + \nu - \mu - \rho}{2}; \nu + 1; \sin^2 \psi\right)$<br>$y = a \sin \psi$ |

Modified Bessel functions of  $kx$  (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                                     | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                    |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (14) | $x^{\rho+\nu-\mu+\frac{1}{2}} J_\mu(ax) K_\rho(\beta x)$<br>$\operatorname{Re} \beta >  \operatorname{Im} a , \quad \operatorname{Re} \nu > -1$<br>$\operatorname{Re} \rho > -1, \quad \operatorname{Re} \mu > -1$<br>$\operatorname{Re}(\rho + \nu) > -1$ | $2^{\rho+\nu-\mu-1} [\Gamma(\mu+1)]^{-1} \Gamma(\rho+\nu+1)$<br>$\times \Gamma(\rho+1) \Gamma(\nu+1) a^{\mu-\rho-\nu-2} y^{\frac{1}{2}}$<br>$\times (\cosh \sigma - \cos \theta) P_{\rho+\nu-\mu}^{-\rho}(\cos \theta)$<br>$\times P_{\rho+\nu-\mu}^{-\nu}(\cosh \sigma)$<br>$y + i\beta = i a \operatorname{ctn} [\frac{1}{2}(\theta + i\sigma)]$ |
| (15) | $x^\lambda J_\mu(ax) K_\rho(\beta x)$                                                                                                                                                                                                                      | see Bailey, W. N., 1936: <i>Proc. London Math. Soc.</i> (2), 40, 37-48.                                                                                                                                                                                                                                                                            |
| (16) | $x^{\frac{1}{2}} I_{\frac{1}{2}\nu}(ax) K_{\frac{1}{2}\nu}(ax)$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1$                                                                                                                             | $y^{-\frac{1}{2}} (y^2 + 4a^2)^{-\frac{1}{2}}$                                                                                                                                                                                                                                                                                                     |
| (17) | $x^{\nu+\frac{1}{2}} I_\nu(\frac{1}{2}ax) K_\nu(\frac{1}{2}ax)$<br>$\operatorname{Re} a > 0, \quad  \operatorname{Re} \nu  < \frac{1}{2}$                                                                                                                  | $\frac{2^\nu a^{2\nu} \Gamma(\nu + \frac{1}{2})}{\pi^{\frac{1}{2}} (y^3 + a^2 y)^{\nu+\frac{1}{2}}}$                                                                                                                                                                                                                                               |
| (18) | $x^{\nu+\frac{1}{2}} I_\nu(ax) K_\nu(\beta x)$<br>$\operatorname{Re} \beta > \operatorname{Re} a, \quad \operatorname{Re} \nu > -\frac{1}{2}$                                                                                                              | $\frac{2^{3\nu} (a\beta)^\nu y^{\nu+\frac{1}{2}} \Gamma(\nu + \frac{1}{2})}{\pi^{\frac{1}{2}} [(\beta^2 - a^2 + y^2)^2 + 4a^2 y^2]^{\nu+\frac{1}{2}}}$                                                                                                                                                                                             |
| (19) | $x^{\nu-\frac{1}{2}} I_{\nu-\frac{1}{2}}(\frac{1}{2}ax) K_{\nu-\frac{1}{2}}(\frac{1}{2}ax)$<br>$\operatorname{Re} a > 0, \quad 0 < \operatorname{Re} \nu < 3/2$                                                                                            | $\frac{\Gamma(\nu) (2a)^{\nu-1}}{y^{\nu-\frac{1}{2}}} P_{-\nu} \left[ \frac{2a^2 + y^2}{2a(a^2 + y^2)^{\frac{1}{2}}} \right]$                                                                                                                                                                                                                      |
| (20) | $x^{-\frac{1}{2}} I_\mu(\frac{1}{2}ax) K_\mu(\frac{1}{2}ax)$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1$<br>$\operatorname{Re}(\nu + 2\mu) > -1$                                                                                        | $\frac{e^{\mu\pi i} \Gamma(\frac{1}{2}\nu + \mu + \frac{1}{2})}{\Gamma(\frac{1}{2}\nu - \mu + \frac{1}{2}) y^{\frac{1}{2}}}$<br>$\times P_{\frac{1}{2}\nu-\frac{1}{2}}^{-\mu} [(1 + a^2/y^2)^{\frac{1}{2}}]$<br>$\times Q_{\frac{1}{2}\nu-\frac{1}{2}}^{-\mu} [(1 + a^2/y^2)^{\frac{1}{2}}]$                                                       |
| (21) | $x^{\mu+\frac{1}{2}} I_\nu(\frac{1}{2}ax) K_\mu(\frac{1}{2}ax)$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1$<br>$-\operatorname{Re} \nu - 1 < \operatorname{Re} \mu < \frac{1}{2}$                                                       | $(\frac{1}{2}\pi)^{-\frac{1}{2}} a^{-1} y^{-\mu-\frac{1}{2}} e^{-(\mu-\frac{1}{2}\nu+\frac{1}{2})\pi i}$<br>$\times (1 + y^2/a^2)^{-\frac{1}{2}\mu-\frac{1}{4}} Q_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(iy/a)$                                                                                                                                       |

Modified Bessel functions of  $kx$  (cont'd)

|      | $f(x)$                                                                                                                                                                                                  | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                 |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (22) | $x^{\mu+\frac{1}{2}} I_\nu(ax) K_\mu(\beta x)$<br>$\operatorname{Re} \beta >  \operatorname{Re} a $<br>$\operatorname{Re} \nu > -1, \quad \operatorname{Re}(\mu + \nu) > -1$                            | $(2\pi)^{-\frac{1}{2}} a^{-\mu-1} \beta^\mu y^{-\mu-\frac{1}{2}} e^{-(\mu-\frac{1}{2}\nu+\frac{1}{2})\pi i}$<br>$\times (v^2+1)^{-\frac{1}{2}\mu-\frac{1}{2}} Q_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(iv)$<br>$2ayv = \beta^2 - a^2 + y^2$                        |
| (23) | $x^{\frac{1}{2}} I_{\frac{1}{2}(\nu-\mu)}(\frac{1}{2}ax) K_{\frac{1}{2}(\nu+\mu)}(\frac{1}{2}ax)$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1$<br>$\operatorname{Re}(\nu - \mu) > -2$ | $a^{-\mu} y^{-\frac{1}{2}} (y^2 + a^2)^{-\frac{1}{2}} [y + (y^2 + a^2)^{\frac{1}{2}}]^\mu$                                                                                                                                                                      |
| (24) | $x^{\nu+\frac{1}{2}} I_\mu(\beta x) K_\mu(ax)$<br>$\operatorname{Re} a >  \operatorname{Re} \beta , \quad \operatorname{Re} \nu > -1$<br>$\operatorname{Re}(\mu + \nu) > -1$                            | $(2\pi)^{-\frac{1}{2}} (a\beta)^{-\nu-1} y^{\nu+\frac{1}{2}} e^{-(\nu+\frac{1}{2})\pi i}$<br>$\times (u^2-1)^{-\frac{1}{2}\nu-\frac{1}{2}} Q_{\mu-\frac{1}{2}}^{\nu+\frac{1}{2}}(u)$<br>$2a\beta u = a^2 + \beta^2 + y^2$                                       |
| (25) | $x^\lambda I_\mu(ax) K_\rho(\beta x)$                                                                                                                                                                   | see Bailey, W. N., 1936: <i>Proc. London Math. Soc.</i> (2) 40, 37-48.                                                                                                                                                                                          |
| (26) | $x^{-\nu-\frac{1}{2}} [K_{\nu+\frac{1}{2}}(\frac{1}{2}ax)]^2$<br>$\operatorname{Re} a > 0, \quad -1 < \operatorname{Re} \nu < 0$                                                                        | $\pi^{\frac{1}{2}} (2a)^{-\nu-1} \Gamma(-\nu) y^{\nu+\frac{1}{2}} (a^2 + y^2)^{\frac{1}{2}\nu}$<br>$\times P_\nu \left[ \frac{2a^2 + y^2}{2a(a^2 + y^2)^{\frac{1}{2}}} \right]$                                                                                 |
| (27) | $x^{\frac{1}{2}} [K_\mu(\frac{1}{2}ax)]^2$<br>$\operatorname{Re} a > 0$<br>$\operatorname{Re}(\frac{1}{2}\nu \pm \mu) > -1$                                                                             | $\frac{e^{2\mu\pi i} y^{\frac{1}{2}} \Gamma(1+\frac{1}{2}\nu+\mu)}{(y^2 + a^2)^{\frac{1}{2}} \Gamma(\frac{1}{2}\nu-\mu)}$<br>$\times Q_{\frac{1}{2}\nu}^{-\mu} [(1+a^2/y^2)^{\frac{1}{2}}]$<br>$\times Q_{\frac{1}{2}\nu-1}^{-\mu} [(1+a^2/y^2)^{\frac{1}{2}}]$ |
| (28) | $x^{-\frac{1}{2}} [K_\mu(\frac{1}{2}ax)]^2$<br>$\operatorname{Re} a > 0$<br>$\operatorname{Re}(\frac{1}{2}\nu \pm \mu) > -\frac{1}{2}$                                                                  | $\frac{e^{2\mu\pi i} \Gamma(\frac{1}{2}+\frac{1}{2}\nu+\mu)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\mu) y^{\frac{1}{2}}}$<br>$\times \{Q_{\frac{1}{2}\nu-\frac{1}{2}}^{-\mu} [(1+a^2/y^2)^{\frac{1}{2}}]\}^2$                                                       |



Modified Bessel functions of  $kx$  (cont'd)

|      | $f(x)$                                                                                                                                                                                          | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                               |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (29) | $x^{\frac{1}{2}} K_{\mu-\frac{1}{2}}(\frac{1}{2}ax) K_{\mu+\frac{1}{2}}(\frac{1}{2}ax)$<br>$\text{Re } a > 0, \quad \text{Re } \nu > -1$<br>$ \text{Re } \mu  < 1 + \frac{1}{2} \text{Re } \nu$ | $\frac{e^{2\mu\pi i} \Gamma(\frac{1}{2}\nu + \mu + 1) y^{\frac{1}{2}}}{\Gamma(\frac{1}{2}\nu - \mu) (y^2 + a^2)^{\frac{1}{2}}}$<br>$\times Q_{\frac{1}{2}\nu - \frac{1}{2}}^{-\mu + \frac{1}{2}}[(1 + a^2/y^2)^{\frac{1}{2}}]$<br>$\times Q_{\frac{1}{2}\nu - \frac{1}{2}}^{-\mu - \frac{1}{2}}[(1 + a^2/y^2)^{\frac{1}{2}}]$ |
| (30) | $x^{\nu+\frac{1}{2}} K_\mu(ax) K_\mu(\beta x)$<br>$\text{Re } a > 0, \quad \text{Re } \beta > 0$<br>$\text{Re } (\nu \pm \mu) > -1, \quad \text{Re } \nu > -1$                                  | $\frac{\pi^{\frac{1}{2}} y^{\nu+\frac{1}{2}} \Gamma(\nu + \mu + 1) \Gamma(\nu - \mu + 1)}{2^{3/2} (a\beta)^{\nu+1} (u^2 - 1)^{\nu/2 + 1/4}}$<br>$\times P_{\mu-\frac{1}{2}}^{-\nu-\frac{1}{2}}(u)$<br>$2a\beta u = y^2 + \beta^2 + a^2$                                                                                       |
| (31) | $x^\lambda K_\mu(ax) K_\rho(\beta x)$                                                                                                                                                           | see Bailey, W. N., 1936: <i>Proc. London Math. Soc.</i> (2) 41, 215-220.                                                                                                                                                                                                                                                      |

## 8.14. Modified Bessel functions of other arguments

|     |                                                                                                                                                         |                                                                                                                                               |
|-----|---------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $x^{\frac{1}{2}-\nu} \exp(-\frac{1}{4}a^2x^2)$<br>$\times I_\nu(\frac{1}{4}a^2x^2)$<br>$ \arg a  < \frac{1}{4}\pi, \quad \text{Re } \nu > -\frac{1}{2}$ | $(\frac{1}{2}\pi)^{-\frac{1}{2}} a^{-1} y^{\nu-\frac{1}{2}} \exp\left(-\frac{y^2}{4a^2}\right)$<br>$\times D_{-2\nu}\left(\frac{y}{a}\right)$ |
| (2) | $x^{-\nu-3/2} \exp(-\frac{1}{4}a^2x^2)$<br>$\times I_{\nu+1}(\frac{1}{4}a^2x^2)$<br>$ \arg a  < \frac{1}{4}\pi, \quad \text{Re } \nu > -1$              | $(\frac{1}{2}\pi)^{-\frac{1}{2}} y^{\nu+\frac{1}{2}} \exp\left(-\frac{y^2}{4a^2}\right)$<br>$\times D_{-2\nu-3}\left(\frac{y}{a}\right)$      |
| (3) | $x^{\frac{1}{2}} \exp(-\frac{1}{4}ax^2) I_{\frac{1}{2}\nu}(\frac{1}{4}ax^2)$<br>$\text{Re } a > 0, \quad \text{Re } \nu > -1$                           | $(\frac{1}{2}\pi ay)^{-\frac{1}{2}} \exp\left(-\frac{y^2}{2a}\right)$                                                                         |

## Modified Bessel functions (cont'd)

|      | $f(x)$                                                                                                                                                                            | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                    |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (4)  | $x^{\nu/3+1/6} \exp(-\frac{1}{4}ax^2)$<br>$\times I_{\nu/3+1/6}(\frac{1}{4}ax^2)$<br>$\text{Re } a > 0, \quad -1 < \text{Re } \nu < 5/2$                                          | $\pi^{-1} a^{-\nu/3-2/3} y^{\nu/3+1/6} \exp\left(\frac{y^2}{4a}\right)$<br>$\times K_{\nu/3+1/6}\left(\frac{y^2}{4a}\right)$                                                                                                                                                                       |
| (5)  | $x^{1/6-\nu/3} \exp(-\frac{1}{4}ax^2)$<br>$\times I_{\nu/3-1/6}(\frac{1}{4}ax^2)$<br>$\text{Re } a > 0, \quad \text{Re } \nu > -1$                                                | $a^{\nu/3-2/3} y^{1/6-\nu/3} \exp\left(-\frac{y^2}{4a}\right)$<br>$\times I_{\nu/3-1/6}\left(\frac{y^2}{4a}\right)$                                                                                                                                                                                |
| (6)  | $x^{\frac{1}{2}+2\mu-\nu} \exp(-\frac{1}{4}ax^2)$<br>$\times I_\mu(\frac{1}{4}ax^2) \quad \text{Re } a > 0$<br>$\text{Re } \nu > 2\text{Re } \mu + \frac{1}{2} > -\frac{1}{2}$    | $2^{2\mu-\nu+\frac{1}{2}} (\pi a)^{-\frac{1}{2}} \Gamma(\frac{1}{2}+\mu)$<br>$\times [\Gamma(\frac{1}{2}-\mu+\nu)]^{-1} y^{\nu-2\mu-\frac{1}{2}}$<br>$\times {}_1F_1\left(\frac{1}{2}+\mu; \frac{1}{2}-\mu+\nu; -\frac{y^2}{2a}\right)$                                                            |
| (7)  | $x^{\frac{1}{2}+\nu-2\mu} \exp(-\frac{1}{4}a^2x^2)$<br>$\times I_\mu(\frac{1}{4}a^2x^2) \quad  \arg a  < \frac{1}{4}\pi$<br>$-1 < \text{Re } \nu < 2\text{Re } \mu + \frac{1}{2}$ | $\pi^{-\frac{1}{2}} 2^{\frac{1}{4}(3+2\nu-6\mu)} a^{-\frac{1}{2}-\nu+\mu} y^{\mu-1}$<br>$\times \exp\left(-\frac{y^2}{4a^2}\right) W_{k,m}\left(\frac{y^2}{2a^2}\right)$<br>$2k = \frac{1}{2} + \nu - 3\mu, \quad 2m = -\frac{1}{2} + \mu - \nu$                                                   |
| (8)  | $x^\lambda \exp(-\frac{1}{4}a^2x^2) I_\mu(\frac{1}{4}a^2x^2)$<br>$ \arg a  < \frac{1}{4}\pi$<br>$-3/2 - \text{Re}(2\mu + \nu) < \text{Re } \lambda < 0$                           | $(2\pi)^{-\frac{1}{2}} \left(\frac{2}{y}\right)^{\lambda+1} G_{23}^{21}\left(\frac{y^2}{2a^2} \middle  \begin{matrix} 1-\mu, 1+\mu \\ h, \frac{1}{2}, k \end{matrix}\right)$<br>$h = \frac{3}{4} + \frac{1}{2}\lambda + \frac{1}{2}\nu$<br>$k = \frac{3}{4} + \frac{1}{2}\lambda - \frac{1}{2}\nu$ |
| (9)  | $x^{\frac{1}{2}} K_{\frac{1}{2}\nu}(\frac{1}{4}ax^2)$<br>$\text{Re } a > 0, \quad \text{Re } \nu > -1$                                                                            | $\pi a^{-1} y^{\frac{1}{2}} [I_{\frac{1}{2}\nu}(y^2/a) - \mathbf{L}_{\frac{1}{2}\nu}(y^2/a)]$                                                                                                                                                                                                      |
| (10) | $x^{3/2} K_{\frac{1}{2}\nu+\frac{1}{2}}(\frac{1}{4}ax^2)$<br>$\text{Re } a > 0, \quad \text{Re } \nu > -1$                                                                        | $2\pi a^{-2} y^{3/2} [I_{\frac{1}{2}\nu-\frac{1}{2}}(y^2/a)$<br>$- \mathbf{L}_{\frac{1}{2}\nu-\frac{1}{2}}(y^2/a)]$                                                                                                                                                                                |

## Modified Bessel functions (cont'd)

|      | $f(x)$                                                                                                                                                                        | $\int_0^\infty f(x) J_\nu(xy)(xy)^{1/2} dx \quad y > 0$                                                                                                                                                                                                                                                |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (11) | $x^{\nu/3+1/6} \exp(-\frac{1}{4}ax^2)$<br>$\times K_{\nu/3+1/6}(\frac{1}{4}ax^2)$<br>$\text{Re } a > 0, \quad \text{Re } \nu > -1$                                            | $\pi a^{-\nu/3-2/3} y^{\nu/3+1/6} \exp\left(-\frac{y^2}{4a}\right)$<br>$\times I_{\nu/3+1/6}\left(\frac{y^2}{4a}\right)$                                                                                                                                                                               |
| (12) | $x^{\nu/3+1/6} \exp(\frac{1}{4}ax^2)$<br>$\times K_{\nu/3+1/6}(\frac{1}{4}ax^2)$<br>$-1 < \text{Re } \nu < 5/2$                                                               | $a^{-\nu/3-2/3} y^{\nu/3+1/6} \exp\left(\frac{y^2}{4a}\right)$<br>$\times K_{\nu/3+1/6}\left(\frac{y^2}{4a}\right)$                                                                                                                                                                                    |
| (13) | $x^{2\mu+\nu+1/2} \exp(-\frac{1}{4}a^2x^2)$<br>$\times K_\mu(\frac{1}{4}a^2x^2) \quad  \arg a  < \frac{1}{4}\pi$<br>$\text{Re } \nu > -1, \quad \text{Re}(2\mu+\nu) > -1$     | $\pi^{1/2} 2^\mu a^{-2\mu-2\nu-2} y^{\nu+1/2} \Gamma(1+2\mu+\nu)$<br>$[\Gamma(\mu+\nu+3/2)]^{-1}$<br>$\times {}_1F_1\left(1+2\mu+\nu; \mu+\nu+\frac{3}{2}; -\frac{y^2}{2a^2}\right)$                                                                                                                   |
| (14) | $x^{2\mu+\nu+1/2} \exp(\frac{1}{4}a^2x^2)$<br>$\times K_\mu(\frac{1}{4}a^2x^2)$<br>$\text{Re } a > 0, \quad \text{Re } \nu > -1$<br>$-1 < \text{Re}(2\mu+\nu) < -\frac{1}{2}$ | $\pi^{1/2} \Gamma(1+2\mu+\nu) [\Gamma(\frac{1}{2}-\mu)]^{-1} 2^{3/2-k}$<br>$\times a^{-2m} y^{-\mu-1} \exp\left(\frac{y^2}{4a^2}\right) W_{k,m}\left(\frac{y^2}{2a^2}\right)$<br>$2k = -\frac{1}{2} - 3\mu - \nu, \quad 2m = \frac{1}{2} + \mu + \nu$                                                  |
| (15) | $x^\lambda \exp(-\frac{1}{4}a^2x^2) K_\mu(\frac{1}{4}a^2x^2)$<br>$ \arg a  < \frac{1}{4}\pi$<br>$\text{Re}(\lambda + \nu \pm 2\mu) > -3/2$                                    | $(\frac{1}{2}\pi)^{1/2} \left(\frac{2}{y}\right)^{\lambda+1} G_{23}^{12}\left(\frac{y^2}{2a^2} \middle  \begin{matrix} 1-\mu, 1+\mu \\ h, \frac{1}{2}, k \end{matrix} \right)$<br>$h = \frac{3}{4} + \frac{1}{2}\lambda + \frac{1}{2}\nu, \quad k = \frac{3}{4} + \frac{1}{2}\lambda - \frac{1}{2}\nu$ |
| (16) | $x^\lambda \exp(\frac{1}{4}a^2x^2) K_\mu(\frac{1}{4}a^2x^2)$<br>$\text{Re } a > 0$<br>$-3/2 - \text{Re}(\nu \pm 2\mu) < \text{Re } \lambda < 0$                               | $(2\pi)^{-1/2} \cos(\mu\pi) (2/y)^{\lambda+1}$<br>$\times G_{23}^{22}\left(\frac{y^2}{2a^2} \middle  \begin{matrix} 1-\mu, 1+\mu \\ h, \frac{1}{2}, k \end{matrix} \right)$<br>$h = \frac{3}{4} + \frac{1}{2}\lambda + \frac{1}{2}\nu$<br>$k = \frac{3}{4} + \frac{1}{2}\lambda - \frac{1}{2}\nu$      |

## Modified Bessel functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                             | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                      |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (17) | $x^{\frac{1}{2}} I_{\frac{1}{2}\nu}(\frac{1}{4}ax^2) K_{\frac{1}{2}\nu}(\frac{1}{4}ax^2)$<br>$\operatorname{Re} a > 0, \operatorname{Re} \nu > -1$                                                                 | $\frac{y^{\frac{1}{2}}}{a} I_{\frac{1}{2}\nu}\left(\frac{y^2}{4a}\right) K_{\frac{1}{2}\nu}\left(\frac{y^2}{4a}\right)$                                                                                                                              |
| (18) | $x^{\frac{1}{2}} I_{\frac{1}{2}(\nu-\mu)}(\frac{1}{2}ax^2)$<br>$\times K_{\frac{1}{2}(\nu+\mu)}(\frac{1}{2}ax^2) \quad \operatorname{Re} a > 0$<br>$\operatorname{Re} \nu > -1, \operatorname{Re}(\nu - \mu) > -2$ | $\frac{2\Gamma(\frac{1}{2} + \frac{1}{4}\nu - \frac{1}{4}\mu)}{\Gamma(1 + \frac{1}{2}\nu)} y^{\frac{3}{2}} W_{\frac{1}{4}\mu, \frac{1}{4}\nu}\left(\frac{y^2}{4a}\right)$<br>$\times M_{-\frac{1}{4}\mu, \frac{1}{4}\nu}\left(\frac{y^2}{4a}\right)$ |
| (19) | $x^{-5/2} K_\nu(ax^{-1})$<br>$\operatorname{Re} a > 0,  \operatorname{Re} \nu  < 5/2$                                                                                                                              | $i a^{-1} y^{\frac{1}{2}} [e^{\frac{1}{2}\nu\pi i} K_{2\nu}(2a^{\frac{1}{2}} e^{\frac{1}{4}\pi i} y^{\frac{1}{2}})$<br>$- e^{-\frac{1}{2}\nu\pi i} K_{2\nu}(2a^{\frac{1}{2}} e^{-\frac{1}{4}\pi i} y^{\frac{1}{2}})]$                                |
| (20) | $x^{-2\nu-2} K_{\frac{1}{2}-\nu}(ax^{-1})$<br>$\operatorname{Re} a > 0, -\frac{1}{2} < \operatorname{Re} \nu < 2$                                                                                                  | $(2\pi)^{\frac{1}{2}} a^{-\nu-\frac{1}{2}} y^{\nu+\frac{1}{2}} K_{2\nu}(2^{\frac{1}{2}} a^{\frac{1}{2}} y^{\frac{1}{2}})$<br>$\times J_{2\nu}(2^{\frac{1}{2}} a^{\frac{1}{2}} y^{\frac{1}{2}})$                                                      |
| (21) | $K_{-2\nu-1}(2ax^{\frac{1}{2}})$<br>$\operatorname{Re} a > 0, \operatorname{Re} \nu > -1$                                                                                                                          | $-\frac{1}{4}\pi a \sec(\nu\pi) y^{-3/2} [\mathbf{H}_{-\nu-1}(a^2/y)$<br>$- Y_{-\nu-1}(a^2/y)]$                                                                                                                                                      |
| (22) | $x^{-\frac{1}{2}} K_{2\nu}(2ax^{\frac{1}{2}})$<br>$\operatorname{Re} a > 0, \operatorname{Re} \nu > -\frac{1}{2}$                                                                                                  | $\frac{1}{4}\pi \sec(\nu\pi) y^{-\frac{1}{2}} [\mathbf{H}_{-\nu}(a^2/y)$<br>$- Y_{-\nu}(a^2/y)]$                                                                                                                                                     |
| (23) | $x^{\frac{1}{2}} J_\nu(2a^{\frac{1}{2}}x^{\frac{1}{2}}) K_\nu(2a^{\frac{1}{2}}x^{\frac{1}{2}})$<br>$\operatorname{Re} a > 0, \operatorname{Re} \nu > -1$                                                           | $\frac{1}{2} y^{-3/2} e^{-2a/y}$                                                                                                                                                                                                                     |
| (24) | $x^{\nu+\frac{1}{2}} J_{2\nu}(2a^{\frac{1}{2}}x^{\frac{1}{2}}) K_{2\nu}(2a^{\frac{1}{2}}x^{\frac{1}{2}})$<br>$\operatorname{Re} a > 0, \operatorname{Re} \nu > -\frac{1}{2}$                                       | $\pi^{-\frac{1}{2}} 2^\nu a^{\nu+\frac{1}{2}} y^{-2\nu-2} K_{\frac{1}{2}-\nu}(2a/y)$                                                                                                                                                                 |
| (25) | $x^{-\nu-\frac{1}{2}} J_{2\nu+1}(2a^{\frac{1}{2}}x^{\frac{1}{2}})$<br>$\times K_{2\nu+1}(2a^{\frac{1}{2}}x^{\frac{1}{2}})$<br>$\operatorname{Re} a > 0, \operatorname{Re} \nu > -1$                                | $\pi^{\frac{1}{2}} 2^{-\nu-2} a^{-\nu-\frac{1}{2}} y^{2\nu} [I_{\nu+\frac{1}{2}}(2a/y)$<br>$- \mathbf{L}_{\nu+\frac{1}{2}}(2a/y)]$                                                                                                                   |

## Modified Bessel functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                                                                                    | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                 |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (26) | $x^{-\frac{1}{2}} J_\mu(2a^{\frac{1}{2}} x^{\frac{1}{2}}) K_\mu(2a^{\frac{1}{2}} x^{\frac{1}{2}})$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1$<br>$\operatorname{Re}(\nu + \mu) > -1$                                                                                                  | $\frac{\Gamma(\frac{1}{2} + \frac{1}{2}\mu + \frac{1}{2}\nu) y^{\frac{1}{2}}}{4a \Gamma(1 + \mu)} W_{-\frac{1}{2}\nu, \frac{1}{2}\mu}\left(\frac{2a}{y}\right)$<br>$\times M_{\frac{1}{2}\nu, \frac{1}{2}\mu}\left(\frac{2a}{y}\right)$                                         |
| (27) | $x^{-\frac{1}{2}} [K_{2\nu}(2a^{\frac{1}{2}} x^{\frac{1}{2}})$<br>$- \frac{1}{2}\pi Y_{2\nu}(2a^{\frac{1}{2}} x^{\frac{1}{2}})]$<br>$a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$                                                                                                                   | $-\frac{1}{2}\pi y^{-\frac{1}{2}} Y_\nu(a/y)$                                                                                                                                                                                                                                   |
| (28) | $x^{-\frac{1}{2}} K_\nu(2a^{\frac{1}{2}} x^{\frac{1}{2}}) Y_\nu(2a^{\frac{1}{2}} x^{\frac{1}{2}})$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$                                                                                                                               | $-\frac{1}{4} a^{-1} y^{\frac{1}{2}} W_{\frac{1}{2}\nu, \frac{1}{2}\nu}(2a/y)$<br>$\times W_{-\frac{1}{2}\nu, \frac{1}{2}\nu}(2a/y)$                                                                                                                                            |
| (29) | $x^{-\frac{1}{2}} K_\mu(2a^{\frac{1}{2}} x^{\frac{1}{2}})$<br>$\times \{\sin[\frac{1}{2}(\mu - \nu)\pi] J_\mu(2a^{\frac{1}{2}} x^{\frac{1}{2}})$<br>$+ \cos[\frac{1}{2}(\mu - \nu)\pi] Y_\mu(2a^{\frac{1}{2}} x^{\frac{1}{2}})\}$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re}(\nu \pm \mu) > -1$ | $-\frac{1}{4} a^{-1} y^{\frac{1}{2}} W_{\frac{1}{2}\nu, \frac{1}{2}\mu}(2a/y)$<br>$\times W_{-\frac{1}{2}\nu, \frac{1}{2}\mu}(2a/y)$                                                                                                                                            |
| (30) | $x^{-\frac{1}{2}} K_\mu[(2ax)^{\frac{1}{2}} e^{\frac{1}{4}\pi i}]$<br>$\times K_\mu[(2ax)^{\frac{1}{2}} e^{-\frac{1}{4}\pi i}]$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re}(\nu \pm \mu) > -1$                                                                                                   | $\frac{1}{4} a^{-1} y^{\frac{1}{2}} \Gamma\left(\frac{1+\mu+\nu}{2}\right) \Gamma\left(\frac{1-\mu+\nu}{2}\right)$<br>$\times W_{-\frac{1}{2}\nu, \frac{1}{2}\mu}(ay^{-1} e^{\frac{1}{2}\pi i})$<br>$\times W_{-\frac{1}{2}\nu, \frac{1}{2}\mu}(ay^{-1} e^{-\frac{1}{2}\pi i})$ |
| (31) | $x^{-\nu-\frac{1}{2}} K_{2\nu+1}[(2ax)^{\frac{1}{2}} e^{\frac{1}{4}\pi i}]$<br>$\times K_{2\nu+1}[(2ax)^{\frac{1}{2}} e^{-\frac{1}{4}\pi i}]$<br>$\operatorname{Re} a > 0, \quad -1 < \operatorname{Re} \nu < 0$                                                                                          | $-2^{-5/2} \pi^{3/2} \csc(\nu\pi) a^{-\nu-1/2} y^{2\nu}$<br>$\times [\mathbf{H}_{\nu+\frac{1}{2}}(a/y) - Y_{\nu+\frac{1}{2}}(a/y)]$                                                                                                                                             |
| (32) | $x^{\nu+\frac{1}{2}} (x^2 + \beta^2)^{-\frac{1}{2}\nu-\frac{1}{4}}$<br>$\times K_{\nu+\frac{1}{2}}[a(x^2 + \beta^2)^{\frac{1}{2}}]$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \beta > 0$<br>$\operatorname{Re} \nu > -1$                                                                       | $\pi^{\frac{1}{2}} 2^{-\frac{1}{2}} a^{-\nu-\frac{1}{2}} y^{\nu+\frac{1}{2}} (a^2 + y^2)^{-\frac{1}{2}}$<br>$\times e^{-\beta(a^2 + y^2)^{\frac{1}{2}}}$                                                                                                                        |

## Modified Bessel functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                                                                                   | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (33) | $x^{\nu+\frac{1}{2}}(x^2+\beta^2)^{-\frac{1}{2}\nu-\frac{1}{4}}$<br>$\times K_{\nu+\frac{3}{2}}[a(x^2+\beta^2)^{\frac{1}{2}}]$<br>$\text{Re } a > 0, \quad \text{Re } \beta > 0$<br>$\text{Re } \nu > -1$                                                                                                | $\pi^{1/2} 2^{-1/2} a^{-\nu-3/2} \beta^{-1} y^{\nu+1/2}$<br>$\times e^{-\beta(a^2+y^2)^{\frac{1}{2}}}$                                                         |
| (34) | $x^{\nu+\frac{1}{2}}(x^2+a^2)^{-\frac{1}{2}(\nu+1)}$<br>$\times K_{\frac{1}{2}(\nu+1)}[a(x^2+a^2)^{\frac{1}{2}}]$<br>$\text{Re } a > 0, \quad \text{Re } \nu > -1$                                                                                                                                       | $y^{\nu+\frac{1}{2}}(y^2+a^2)^{-\frac{1}{2}(\nu+1)}$<br>$\times K_{\frac{1}{2}(\nu+1)}[a(y^2+a^2)^{\frac{1}{2}}]$                                              |
| (35) | $x^{\nu+\frac{1}{2}}(x^2+\beta^2)^{-\frac{1}{2}\mu}$<br>$\times K_\mu[a(x^2+\beta^2)^{\frac{1}{2}}] \quad \text{Re } a \geq 0$<br>$\text{Re } \beta > 0, \quad \text{Re } \nu > -1$                                                                                                                      | $a^{-\mu} \beta^{\nu+1-\mu} y^{\nu+\frac{1}{2}} (a^2+y^2)^{\frac{1}{2}\mu-\frac{1}{2}\nu-\frac{1}{2}}$<br>$\times K_{\mu-\nu-1}[\beta(a^2+y^2)^{\frac{1}{2}}]$ |
| (36) | $x^{\nu+\frac{1}{2}}(b^2-x^2)^{\frac{1}{2}\mu} Y_\mu[a(b^2-x^2)^{\frac{1}{2}}]$<br>$0 < x < b$<br>$-2\pi^{-1} x^{\nu+\frac{1}{2}}(x^2-b^2)^{\frac{1}{2}\mu}$<br>$\times K_\mu[a(x^2-b^2)^{\frac{1}{2}}]$<br>$b < x < \infty$<br>$\text{Re } a > 0, \quad \text{Re } \nu > -1, \quad \text{Re } \mu > -1$ | $a^\mu b^{\mu+\nu+1} y^{\nu+\frac{1}{2}} (a^2+y^2)^{-\frac{1}{2}(\mu+\nu+1)}$<br>$\times Y_{\mu+\nu+1}[\beta(a^2+y^2)^{\frac{1}{2}}]$                          |
| (37) | $x^{\frac{1}{2}} I_{\frac{1}{2}\nu}\{\frac{1}{2}\beta[(a^2+x^2)^{\frac{1}{2}}-a]\}$<br>$\times K_{\frac{1}{2}\nu}\{\frac{1}{2}\beta[(a^2+x^2)^{\frac{1}{2}}+a]\}$<br>$\text{Re } a > 0, \quad \text{Re } \beta > 0$<br>$\text{Re } \nu > -1$                                                             | $y^{-\frac{1}{2}}(\beta^2+y^2)^{-\frac{1}{2}} \exp[-a(\beta^2+y^2)^{\frac{1}{2}}]$                                                                             |

## 8.15. Functions related to Bessel functions

|     |                                                                       |                                                                                                                                                 |
|-----|-----------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $H_{\nu-\frac{1}{2}}(ax)$<br>$a > 0, \quad -1 < \text{Re } \nu < 3/2$ | $2^{\frac{1}{2}} \pi^{-\frac{1}{2}} a^{\nu-\frac{1}{2}} y^{\frac{1}{2}-\nu} (a^2-y^2)^{-\frac{1}{2}}$<br>$0 < y < a$<br>$0$<br>$a < y < \infty$ |
|-----|-----------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------|

## Functions related to Bessel functions (cont'd)

|     | $f(x)$                                                                                                                                                                             | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                         |
|-----|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (2) | $x^\lambda \mathbf{H}_\mu(ax) \quad a > 0$<br>$-5/2 - \operatorname{Re} \nu < \operatorname{Re}(\lambda + \mu) < 0$                                                                | $2^{\lambda+\frac{1}{2}} y^{-\lambda-1}$<br>$\times G_{33}^{21} \left( \frac{y^2}{a^2} \left  \begin{matrix} 1-\mu, 1-\frac{\mu}{2}, 1+\frac{\mu}{2} \\ 3+\frac{\lambda+\nu}{2}, \frac{1-\mu}{2}, 3+\frac{\lambda-\nu}{2} \end{matrix} \right. \right)$                                                 |
| (3) | $x^{\frac{1}{2}} \mathbf{H}_{\frac{1}{2}\nu}(\frac{1}{4}ax^2)$<br>$a > 0, \quad -2 < \operatorname{Re} \nu < 3/2$                                                                  | $-2a^{-1}y^{\frac{1}{2}} Y_{\frac{1}{2}\nu}(y^2/a)$                                                                                                                                                                                                                                                     |
| (4) | $x^\lambda \mathbf{H}_\mu(a/x)$<br>$a > 0, \quad \operatorname{Re}(\lambda + \nu) > -2$<br>$-\operatorname{Re} \nu - 5/2 < \operatorname{Re}(\lambda - \mu) < 1$                   | $2^{\lambda+\frac{1}{2}} y^{-\lambda-1}$<br>$\times G_{15}^{21} \left( \frac{a^2 y^2}{16} \left  \begin{matrix} 1+\mu \\ h, \frac{1+\mu}{2}, \frac{\mu}{2}, -\frac{\mu}{2}, k \end{matrix} \right. \right)$<br>$h = \frac{3}{4} + \frac{\lambda+\nu}{2}, \quad k = \frac{3}{4} + \frac{\lambda-\nu}{2}$ |
| (5) | $x^{\frac{1}{2}} [\mathbf{H}_{-\nu}(ax) - Y_{-\nu}(ax)]$<br>$ \arg a  < \pi, \quad -\frac{1}{2} < \operatorname{Re} \nu$                                                           | $2a^{-\nu} \pi^{-1} \cos(\nu\pi) y^{\nu-\frac{1}{2}} (y+a)^{-1}$                                                                                                                                                                                                                                        |
| (6) | $x^\lambda [\mathbf{H}_\mu(ax) - Y_\mu(ax)]$<br>$ \arg a  < \pi, \quad \operatorname{Re}(\lambda + \mu) < 1$<br>$\operatorname{Re}(\lambda + \nu) + 3/2 >  \operatorname{Re} \mu $ | $2^{\lambda+\frac{1}{2}} \pi^{-2} \cos(\mu\pi) y^{-\lambda-1}$<br>$\times G_{33}^{23} \left( \frac{y^2}{a^2} \left  \begin{matrix} 1-\mu, 1-\frac{\mu}{2}, 1+\frac{\mu}{2} \\ 3+\frac{\lambda+\nu}{2}, \frac{1-\mu}{2}, 3+\frac{\lambda-\nu}{2} \end{matrix} \right. \right)$                           |
| (7) | $x^{-\frac{1}{2}} [\mathbf{H}_{-\nu}(ax^{-1}) - Y_{-\nu}(ax^{-1})]$<br>$ \arg a  < \pi, \quad  \operatorname{Re} \nu  < \frac{1}{2}$                                               | $4\pi^{-1} \cos(\nu\pi) y^{-\frac{1}{2}} K_{2\nu}(2a^{\frac{1}{2}}y^{\frac{1}{2}})$                                                                                                                                                                                                                     |

## Functions related to Bessel functions (cont'd)

|      | $f(x)$                                                                                                                                                                           | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                         |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (8)  | $x^{-3/2} [H_{-\nu-1}(ax^{-1}) - Y_{-\nu-1}(ax^{-1})]$<br>$ \arg a  < \pi, \quad  \operatorname{Re} \nu  < \frac{1}{2}$                                                          | $-4\pi^{-1} a^{-\frac{1}{2}} \cos(\nu\pi) K_{-2\nu-1}(2a^{\frac{1}{2}}y^{\frac{1}{2}})$                                                                                                                                                                                                                                                 |
| (9)  | $x^{2\nu} [H_{\nu+\frac{1}{2}}(ax^{-1}) - Y_{\nu+\frac{1}{2}}(ax^{-1})]$<br>$ \arg a  < \pi, \quad -1 < \operatorname{Re} \nu < -1/6$                                            | $-2^{5/2} \pi^{-3/2} a^{\nu+1/2} y^{-\nu-1/2} \sin(\nu\pi)$<br>$\times K_{2\nu+1}(2^{\frac{1}{2}} a^{\frac{1}{2}} e^{\frac{1}{2}\pi i} y^{\frac{1}{2}})$<br>$\times K_{2\nu+1}(2^{\frac{1}{2}} a^{\frac{1}{2}} e^{-\frac{1}{2}\pi i} y^{\frac{1}{2}})$                                                                                  |
| (10) | $x^\lambda [H_\mu(ax^{-1}) - Y_\mu(ax^{-1})]$<br>$ \arg a  < \pi, \quad \operatorname{Re} \lambda < - \operatorname{Re} \mu $<br>$\operatorname{Re}(\nu - \mu + \lambda) > -5/2$ | $2^{\lambda+\frac{1}{2}} \pi^{-2} \cos(\mu\pi) y^{-\lambda-1}$<br>$\times G_{15}^{41} \left( \frac{a^2 y^2}{16} \left  \begin{matrix} \frac{1+\mu}{2} \\ h, \frac{1+\mu}{2}, -\frac{\mu}{2}, \frac{\mu}{2}, k \end{matrix} \right. \right)$<br>$h = \frac{3}{4} + \frac{\lambda+\nu}{2}, \quad k = \frac{3}{4} + \frac{\lambda-\nu}{2}$ |
| (11) | $I_{\nu-\frac{1}{2}}(ax) - L_{\nu-\frac{1}{2}}(ax)$<br>$\operatorname{Re} a > 0, \quad  \operatorname{Re} \nu  < \frac{1}{2}$                                                    | $2^{\frac{1}{2}} \pi^{-\frac{1}{2}} a^{\nu-\frac{1}{2}} y^{\frac{1}{2}-\nu} (y^2 + a^2)^{-\frac{1}{2}}$                                                                                                                                                                                                                                 |
| (12) | $x^{\frac{1}{2}} [I_\nu(ax) - L_\nu(ax)]$<br>$\operatorname{Re} a > 0, \quad -1 < \operatorname{Re} \nu < -\frac{1}{2}$                                                          | $2\pi^{-1} a^{\nu+1} y^{-\nu-\frac{1}{2}} (y^2 + a^2)^{-1}$                                                                                                                                                                                                                                                                             |
| (13) | $x^{\mu-\nu+\frac{1}{2}} [I_\mu(ax) - L_\mu(ax)]$<br>$\operatorname{Re} a > 0$<br>$-1 < 2\operatorname{Re} \mu + 1 < \operatorname{Re} \nu + \frac{1}{2}$                        | $\frac{2^{\mu-\nu+1} a^{\mu-1} y^{\nu-2\mu-\frac{1}{2}}}{\pi^{\frac{1}{2}} \Gamma(\nu-\mu+\frac{1}{2})}$<br>$\times {}_2F_1(1, \frac{1}{2}; \nu-\mu+\frac{1}{2}; -y^2/a^2)$                                                                                                                                                             |
| (14) | $x^{\nu-\mu+\frac{1}{2}} [I_\mu(ax) - L_\mu(ax)]$<br>$\operatorname{Re} a > 0, \quad -1 < \operatorname{Re} \nu < -\frac{1}{2}$                                                  | $\frac{2^{\nu-\mu+1} \Gamma(3/2+\nu) a^{\mu+1}}{\pi \Gamma(3/2+\mu) y^{\nu+5/2}}$<br>$\times {}_2F_1(1, 3/2+\nu; 3/2+\mu; -a^2/y^2)$                                                                                                                                                                                                    |



## Functions related to Bessel functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                              | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (15) | $x^{\nu-\mu-\frac{1}{2}} [I_\mu(ax) - L_\mu(ax)]$<br>$\operatorname{Re} a > 0, \quad  \operatorname{Re} \nu  < \frac{1}{2}$                                                                                         | $\frac{2^{\nu-\mu} \Gamma(\frac{1}{2} + \nu) a^\mu}{\pi^{\frac{1}{2}} \Gamma(1 + \mu) y^{\nu+\frac{1}{2}}}$<br>$\times {}_2F_1(\frac{1}{2} + \nu, \frac{1}{2}; 1 + \mu; -a^2/y^2)$                                                                                             |
| (16) | $x^\lambda [I_\mu(ax) - L_\mu(ax)] \quad \operatorname{Re} a > 0$<br>$-\operatorname{Re} \nu - 3/2 < \operatorname{Re}(\lambda + \mu) < 0$                                                                          | $2^{\lambda+\frac{1}{2}} \pi^{-1} y^{-\lambda-1}$<br>$\times G_{33}^{22} \left( \frac{y^2}{a^2} \middle  \begin{matrix} 1-\frac{\mu}{2}, \frac{1-\mu}{2}, 1+\frac{\mu}{2} \\ 3+\frac{\lambda+\nu}{4}, \frac{1-\mu}{2}, \frac{3}{4}+\frac{\lambda-\nu}{2} \end{matrix} \right)$ |
| (17) | $x^{\frac{1}{2}} [I_\nu(ax) - L_{-\nu}(ax)]$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > \frac{1}{2}$                                                                                                | $2\pi^{-1} a^{1-\nu} y^{\nu-\frac{1}{2}} \cos(\nu\pi) (y^2 + a^2)^{-1}$                                                                                                                                                                                                        |
| (18) | $x^{\mu-\nu+\frac{1}{2}} [I_\mu(ax) - L_{-\mu}(ax)]$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re} \mu > -1$                                                     | $\frac{2^{\mu-\nu+1} a^{-\mu-1} y^{\nu-\frac{1}{2}}}{\Gamma(\frac{1}{2}-\mu) \Gamma(\frac{1}{2}+\nu)}$<br>$\times {}_2F_1(1, \frac{1}{2} + \mu; \frac{1}{2} + \nu; -y^2/a^2)$                                                                                                  |
| (19) | $x^{\nu-\mu+\frac{1}{2}} [I_\mu(ax) - L_{-\mu}(ax)]$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$<br>$\operatorname{Re}(\nu-\mu) > -1, \quad \operatorname{Re}(\nu-2\mu) < \frac{1}{2}$ | $2^{2+\nu-\mu} \pi^{-3/2} \cos(\mu\pi) \Gamma(3/2 + \nu - \mu)$<br>$\times a^{1-\mu} y^{-5/2+2\mu-\nu}$<br>$\times {}_2F_1(3/2 + \nu - \mu, 1; 3/2; -a^2/y^2)$                                                                                                                 |
| (20) | $x^{\mu+\nu-\frac{1}{2}} [I_\mu(ax) - L_{-\mu}(ax)]$<br>$\operatorname{Re} a > 0, \quad -1 < \operatorname{Re} \nu < 3/2$<br>$\operatorname{Re}(\mu + \nu) > -1/2$                                                  | $\frac{2^{\mu+\nu} \Gamma(\frac{1}{2} + \mu + \nu)}{\Gamma(1 + \mu) \Gamma(\frac{1}{2} - \mu)} a^\mu y^{-\frac{1}{2}-2\mu-2\nu}$<br>$\times {}_2F_1(\frac{1}{2} + \mu + \nu, \frac{1}{2} + \mu; 1 + \mu; a^2/y^2)$                                                             |

**Functions related to Bessel functions (cont'd)**

|      | $f(x)$                                                                                                                                                                                            | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                 |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (21) | $x^\lambda [I_\mu(ax) - L_{-\mu}(ax)]$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re}(\mu + \nu + \lambda) > -3/2$<br>$-\operatorname{Re} \nu - 5/2 < \operatorname{Re}(\lambda - \mu) < 1$ | $2^{\lambda+\frac{1}{2}} \pi^{-1} \cos(\mu\pi) y^{-\lambda-1}$<br>$\times G_{33}^{22} \left( \frac{y^2}{a^2} \left  \begin{matrix} \frac{1+\mu}{2}, 1-\frac{\mu}{2}, 1+\frac{\mu}{2} \\ \frac{3}{4} + \frac{\lambda+\nu}{2}, \frac{1+\mu}{2}, \frac{3}{4} + \frac{\lambda-\nu}{2} \end{matrix} \right. \right)$ |
| (22) | $x^{2\nu} [I_{\nu+\frac{1}{2}}(ax^{-1}) - L_{\nu+\frac{1}{2}}(ax^{-1})]$<br>$\operatorname{Re} a > 0, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}$                                             | $2^{3/2} \pi^{-1/2} y^{-\nu-1/2} a^{\nu+1/2}$<br>$\times J_{2\nu+1}[(2ay)^{1/2}] K_{2\nu+1}[(2ay)^{1/2}]$                                                                                                                                                                                                       |
| (23) | $x^{\nu+\frac{1}{2}} U_{\nu+1}(2a^2\beta, ax)$<br>$a > 0, \quad \operatorname{Re} \nu > -1$                                                                                                       | $(2\beta)^{\nu+1} y^{\nu+\frac{1}{2}} \cos[\beta(a^2 - y^2)]$<br>$0 < y < a$<br>$0$<br>$a < y < \infty$                                                                                                                                                                                                         |
| (24) | $x^{\nu+\frac{1}{2}} U_{\nu+2}(2a^2\beta, ax)$<br>$a > 0, \quad \operatorname{Re} \nu > -1$                                                                                                       | $(2\beta)^{\nu+1} y^{\nu+\frac{1}{2}} \sin[\beta(a^2 - y^2)]$<br>$0 < y < a$<br>$0$<br>$a < y < \infty$                                                                                                                                                                                                         |

**8.16. Parabolic cylinder functions**

|     |                                                                                                                                                                  |                                                                                                                         |
|-----|------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------|
| (1) | $x^{\nu-\frac{1}{2}} \exp(-\frac{1}{4}x^2) D_{2\nu-1}(x)$<br>$\operatorname{Re} \nu > -\frac{1}{2}$                                                              | $-\frac{1}{2} \sec(\nu\pi) y^{\nu-\frac{1}{2}} \exp(-\frac{1}{4}y^2)$<br>$\times [D_{2\nu-1}(y) - D_{2\nu-1}(-y)]$      |
| (2) | $x^{\nu-\frac{1}{2}} \exp(-\frac{1}{4}x^2)$<br>$\times \{ [1 - 2 \cos(\nu\pi)] D_{2\nu-1}(x)$<br>$- D_{2\nu-1}(-x) \}$<br>$\operatorname{Re} \nu > -\frac{1}{2}$ | $y^{\nu-\frac{1}{2}} \exp(-\frac{1}{4}y^2)$<br>$\times \{ [1 - 2 \cos(\nu\pi)] D_{2\nu-1}(y)$<br>$- D_{2\nu-1}(-y) \}$  |
| (3) | $x^{\nu-\frac{1}{2}} \exp(-\frac{1}{4}x^2)$<br>$\times \{ [1 + 2 \cos(\nu\pi)] D_{2\nu-1}(x)$<br>$- D_{2\nu-1}(-x) \}$<br>$\operatorname{Re} \nu > -\frac{1}{2}$ | $-y^{\nu-\frac{1}{2}} \exp(-\frac{1}{4}y^2)$<br>$\times \{ [1 + 2 \cos(\nu\pi)] D_{2\nu-1}(y)$<br>$- D_{2\nu-1}(-y) \}$ |

## Parabolic cylinder functions (cont'd)

|      | $f(x)$                                                                                                                                                           | $\int_0^\infty f(x) J_\nu(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                     |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (4)  | $x^{\nu-\frac{1}{2}} \exp(\frac{1}{4}x^2) D_{2\nu-1}(x)$<br>$-\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                                                 | $2^{\frac{1}{2}-\nu} \pi \sin(\nu\pi) \Gamma(2\nu) y^{\frac{1}{2}-\nu}$<br>$\times \exp(\frac{1}{4}y^2) K_\nu(\frac{1}{4}y^2)$                                                       |
| (5)  | $x^{\nu-\frac{1}{2}} \exp(-\frac{1}{4}x^2) D_{2\nu+1}(x)$<br>$\operatorname{Re} \nu > -\frac{1}{2}$                                                              | $\frac{1}{2} \sec(\nu\pi) \exp(-\frac{1}{4}y^2) y^{\nu+\frac{1}{2}}$<br>$\times [D_{2\nu}(y) + D_{2\nu}(-y)]$                                                                        |
| (6)  | $x^{\nu-\frac{1}{2}} \exp(-\frac{1}{4}x^2)$<br>$\times \{ [1 + 2\cos(\nu\pi)] D_{2\nu+1}(x)$<br>$- D_{2\nu+1}(-x) \} \quad \operatorname{Re} \nu > -\frac{1}{2}$ | $y^{\nu+\frac{1}{2}} \exp(-\frac{1}{4}y^2)$<br>$\times \{ [1 + 2\cos(\nu\pi)] D_{2\nu}(x)$<br>$+ D_{2\nu}(-x) \}$                                                                    |
| (7)  | $x^{\nu-\frac{1}{2}} \exp(-\frac{1}{4}x^2)$<br>$\times \{ [1 - 2\cos(\nu\pi)] D_{2\nu+1}(x)$<br>$- D_{2\nu+1}(-x) \} \quad \operatorname{Re} \nu > -\frac{1}{2}$ | $-y^{\nu+\frac{1}{2}} \exp(-\frac{1}{4}y^2)$<br>$\times \{ [1 - 2\cos(\nu\pi)] D_{2\nu}(y)$<br>$+ D_{2\nu}(-y) \}$                                                                   |
| (8)  | $x^{\nu-\frac{1}{2}} \exp(-\frac{1}{4}x^2) D_{-2\nu}(x)$<br>$\operatorname{Re} \nu > -\frac{1}{2}$                                                               | $2^{-\frac{1}{2}} \pi^{\frac{1}{2}} y^{\frac{1}{2}-\nu} \exp(-\frac{1}{4}y^2) I_\nu(\frac{1}{4}y^2)$                                                                                 |
| (9)  | $x^{\nu-\frac{1}{2}} \exp(\frac{1}{4}x^2) D_{-2\nu}(x)$<br>$\operatorname{Re} \nu > -\frac{1}{2}$                                                                | $y^{\nu-\frac{1}{2}} \exp(\frac{1}{4}y^2) D_{-2\nu}(y)$                                                                                                                              |
| (10) | $x^{\nu-\frac{1}{2}} \exp(\frac{1}{4}x^2) D_{-2\nu-2}(x)$<br>$\operatorname{Re} \nu > -\frac{1}{2}$                                                              | $(2\nu+1)^{-1} y^{\nu+\frac{1}{2}} \exp(\frac{1}{4}y^2) D_{-2\nu-1}(y)$                                                                                                              |
| (11) | $x^{\nu-\frac{1}{2}} \exp(-\frac{1}{4}a^2x^2) D_{2\mu}(ax)$<br>$ \arg a  < \frac{1}{4}\pi, \quad \operatorname{Re} \nu > -\frac{1}{2}$                           | $\frac{2^{\mu-\frac{1}{2}} \Gamma(\nu+\frac{1}{2}) y^{\nu+\frac{1}{2}}}{\Gamma(\nu-\mu+1) a^{1+2\nu}}$<br>$\times {}_1F_1\left(\nu+\frac{1}{2}; \nu-\mu+1; -\frac{y^2}{2a^2}\right)$ |

## Parabolic cylinder functions (cont'd)

|      | $f(x)$                                                                                                                                  | $\int_0^\infty f(x) J_\nu(xy) (xy)^{1/2} dx \quad y > 0$                                                                                                                                                          |
|------|-----------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (12) | $x^{\nu-1/2} \exp(1/4 a^2 x^2) D_{2\mu}(ax)$<br>$ \arg a  < 1/4 \pi$<br>$-1/2 < \operatorname{Re} \nu < \operatorname{Re}(1/2 - 2\mu)$  | $\frac{\Gamma(1/2 + \nu) a^{2k} 2^{m+\mu}}{\Gamma(1/2 - \mu) y^{\mu+1}} \exp\left(\frac{y^2}{4a^2}\right)$<br>$\times W_{k,m}\left(\frac{y^2}{2a^2}\right)$<br>$2k = 1/2 + \mu - \nu, \quad 2m = 1/2 + \mu + \nu$ |
| (13) | $x^{\nu+1/2} \exp(-1/4 x^2) D_{2\nu}(x)$<br>$\operatorname{Re} \nu > -1$                                                                | $1/2 \sec(\nu\pi) y^{\nu-1/2} \exp(-1/4 y^2)$<br>$\times [D_{2\nu+1}(y) - D_{2\nu+1}(-y)]$                                                                                                                        |
| (14) | $x^{\nu+1/2} \exp(-1/4 x^2)$<br>$\times \{ [1 + 2 \cos(\nu\pi)] D_{2\nu}(x)$<br>$+ D_{2\nu}(-x) \}$<br>$\operatorname{Re} \nu > -1$     | $y^{\nu-1/2} \exp(-1/4 y^2)$<br>$\times \{ [1 + 2 \cos(\nu\pi)] D_{2\nu+1}(y)$<br>$- D_{2\nu+1}(-y) \}$                                                                                                           |
| (15) | $x^{\nu+1/2} \exp(-1/4 x^2)$<br>$\times \{ [1 - 2 \cos(\nu\pi)] D_{2\nu}(x)$<br>$+ D_{2\nu}(-x) \}$<br>$\operatorname{Re} \nu > -1$     | $-y^{\nu-1/2} \exp(-1/4 y^2)$<br>$\times \{ [1 - 2 \cos(\nu\pi)] D_{2\nu+1}(y)$<br>$- D_{2\nu+1}(-y) \}$                                                                                                          |
| (16) | $x^{\nu+1/2} \exp(-1/4 x^2) D_{2\nu+2}(x)$<br>$\operatorname{Re} \nu > -1$                                                              | $-1/2 \sec(\nu\pi) y^{\nu+1/2} \exp(-1/4 y^2)$<br>$\times [D_{2\nu+2}(y) + D_{2\nu+2}(-y)]$                                                                                                                       |
| (17) | $x^{\nu+1/2} \exp(-1/4 x^2)$<br>$\times \{ [1 - 2 \cos(\nu\pi)] D_{2\nu+2}(x)$<br>$+ D_{2\nu+2}(-x) \}$<br>$\operatorname{Re} \nu > -1$ | $y^{\nu+1/2} \exp(-1/4 y^2)$<br>$\times \{ [1 - 2 \cos(\nu\pi)] D_{2\nu+2}(y)$<br>$+ D_{2\nu+2}(-y) \}$                                                                                                           |
| (18) | $x^{\nu+1/2} \exp(-1/4 x^2)$<br>$\times \{ [1 + 2 \cos(\nu\pi)] D_{2\nu+2}(x)$<br>$+ D_{2\nu+2}(-x) \}$<br>$\operatorname{Re} \nu > -1$ | $-y^{\nu+1/2} \exp(-1/4 y^2)$<br>$\times \{ [1 + 2 \cos(\nu\pi)] D_{2\nu+2}(y)$<br>$+ D_{2\nu+2}(-y) \}$                                                                                                          |
| (19) | $x^{\nu+1/2} \exp(1/4 x^2) D_{2\nu+2}(x)$<br>$-1 < \operatorname{Re} \nu < -5/6$                                                        | $\pi^{-1} \sin(\nu\pi) \Gamma(2\nu+3) y^{-\nu-3/2}$<br>$\times \exp(1/4 y^2) K_{\nu+1}(1/4 y^2)$                                                                                                                  |

## Parabolic cylinder functions (cont'd)

|      | $f(x)$                                                                                                                                               | $\int_0^\infty f(x) J_\nu(xy) (xy)^{1/2} dx \quad y > 0$                                                                                                                                                                                                      |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (20) | $x^{\nu+1/2} \exp(1/4 x^2) D_{-2\nu-1}(x)$<br>$\operatorname{Re} \nu > -1/2$                                                                         | $(2\nu+1) y^{\nu-1/2} \exp(1/4 y^2) D_{-2\nu-2}(y)$                                                                                                                                                                                                           |
| (21) | $x^{\nu+1/2} \exp(-1/4 x^2) D_{-2\nu-3}(x)$<br>$\operatorname{Re} \nu > -1$                                                                          | $2^{-1/2} \pi^{1/2} y^{-\nu-3/2} \exp(-1/4 y^2)$<br>$\times I_{\nu+1}(1/4 y^2)$                                                                                                                                                                               |
| (22) | $x^{\nu+1/2} \exp(1/4 x^2) D_{-2\nu-3}(x)$<br>$\operatorname{Re} \nu > -1$                                                                           | $y^{\nu+1/2} \exp(1/4 y^2) D_{-2\nu-3}(y)$                                                                                                                                                                                                                    |
| (23) | $x^{\nu+1/2} \exp(-1/4 \alpha^2 x^2) D_{2\mu}(\alpha x)$<br>$ \arg \alpha  < 1/4 \pi, \quad \operatorname{Re} \nu > -1$                              | $\frac{2^\mu \Gamma(\nu+3/2) y^{\nu+1/2}}{\Gamma(\nu-\mu+3/2) \alpha^{2\nu+2}}$<br>$\times {}_1F_1\left(\nu+\frac{3}{2}; \nu-\mu+\frac{3}{2}; -\frac{y^2}{2\alpha^2}\right)$                                                                                  |
| (24) | $x^{\nu+1/2} \exp(1/4 \alpha^2 x^2) D_{2\mu}(\alpha x)$<br>$ \arg \alpha  < 3/4 \pi$<br>$-1 < \operatorname{Re} \nu < -1/2 - 2\operatorname{Re} \mu$ | $\frac{\Gamma(3/2+\nu) 2^{1/2+m+\mu} \alpha^{2k+1}}{\Gamma(-\mu) y^{\mu+3/2}}$<br>$\times \exp\left(\frac{y^2}{4\alpha^2}\right) W_{k,m}\left(\frac{y^2}{2\alpha^2}\right)$<br>$2k = \mu - \nu - 1, \quad 2m = \mu + \nu + 1$                                 |
| (25) | $x^\lambda \exp(-1/4 \alpha^2 x^2) D_\mu(\alpha x)$<br>$ \arg \alpha  < 1/4 \pi$<br>$\operatorname{Re}(\lambda + \nu) > -3/2$                        | $\frac{2^{-c-3\nu/2} \pi^{1/2} \Gamma(2b) y^{\nu+1/2}}{\Gamma(\nu+1) \Gamma(c+1/2) \alpha^{2b}}$<br>$\times {}_2F_2\left(b, b+\frac{1}{2}; \nu+1, c+\frac{1}{2}; -\frac{y^2}{2\alpha^2}\right)$<br>$2b = \lambda + \nu + 3/2, \quad 2c = \lambda - \mu + 3/2$ |

## Parabolic cylinder functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                          | $\int_0^\infty f(x) J_\nu(xy) (xy)^{1/2} dx \quad y > 0$                                                                                                                                                                                                                                 |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (26) | $x^\lambda \exp(\frac{1}{4} a^2 x^2) D_\mu(ax)$<br>$ \arg a  < \frac{3}{4} \pi$<br>$\operatorname{Re} \mu < -\operatorname{Re} \lambda < \operatorname{Re} \nu + 3/2$                                           | $\frac{2^{\lambda-1/2} \mu^{-1/2} \pi^{-1/2}}{\Gamma(-\mu) y^{\lambda+1}}$<br>$\times G_{23}^{22} \left( \frac{y^2}{2a^2} \left  \begin{matrix} \frac{1}{2}, 1 \\ \frac{3}{4} + \frac{\lambda+\nu}{2}, -\frac{\mu}{2}, \frac{3}{4} + \frac{\lambda-\nu}{2} \end{matrix} \right. \right)$ |
| (27) | $D_{-\frac{1}{2}-\nu}(ae^{\frac{1}{4}\pi i} x^{\frac{1}{2}})$<br>$\times D_{-\frac{1}{2}-\nu}(ae^{-\frac{1}{4}\pi i} x^{\frac{1}{2}})$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$ | $2^{-\nu} \pi^{\frac{1}{2}} [\Gamma(\nu + \frac{1}{2})]^{-1} y^{-\nu-1/2}$<br>$\times (a^2 + 2y)^{-1/2} [(a^2 + 2y)^{1/2} - a]^{2\nu}$                                                                                                                                                   |
| (28) | $x^{-1/2} D_{-\frac{1}{2}-\nu}(ae^{\frac{1}{4}\pi i} x^{-1/2})$<br>$\times D_{-\frac{1}{2}-\nu}(ae^{-\frac{1}{4}\pi i} x^{-1/2})$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$      | $2^{\frac{1}{2}} \pi^{\frac{1}{2}} [\Gamma(\nu + \frac{1}{2})]^{-1} y^{-1}$<br>$\times \exp[-a(2y)^{1/2}]$                                                                                                                                                                               |

## 8.17. Gauss' hypergeometric function

|     |                                                                                                                                                                                    |                                                                                                                                                                                                                                                                                                           |
|-----|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $x^{2a+\nu-1/2}$<br>$\times {}_2F_1(a-\nu-1/2, a; 2a; -\lambda^2 x^2)$<br>$\operatorname{Re} \nu < -1/2, \quad \operatorname{Re} \lambda > 0$<br>$\operatorname{Re}(a+\nu) > -1/2$ | $\frac{i \Gamma(\frac{1}{2}+a) \Gamma(\frac{1}{2}+a+\nu)}{\pi 2^{1-\nu-2a} \lambda^{2a-1}} y^{-\nu-3/2}$<br>$\times W_{\frac{1}{2}-a, -\frac{1}{2}-\nu}(y/\lambda)$<br>$\times [W_{\frac{1}{2}-a, -\frac{1}{2}-\nu}(e^{-i\pi} y/\lambda)$<br>$- W_{\frac{1}{2}-a, -\frac{1}{2}-\nu}(e^{i\pi} y/\lambda)]$ |
| (2) | $x^{2a-\nu-1/2}$<br>$\times {}_2F_1(\nu+a-1/2, a; 2a; -\lambda^2 x^2)$<br>$\operatorname{Re} a > -1/2, \quad \operatorname{Re} \nu > 1/2$<br>$\operatorname{Re} \lambda > 0$       | $\frac{2^{2a-\nu} \Gamma(\frac{1}{2}+a)}{\lambda^{2a-1} \Gamma(2\nu)} y^{\nu-3/2}$<br>$\times M_{a-1/2, \nu-1/2}(y/\lambda) W_{\frac{1}{2}-a, \nu-1/2}(y/\lambda)$                                                                                                                                        |

## Gauss' hypergeometric function (cont'd)

|     | $f(x)$                                                                                                                                                                                                                                    | $\int_0^\infty f(x) J_\nu(xy) (xy)^{\frac{1}{2}} dx \quad \gamma > 0$                                                                                                                                                                                                   |
|-----|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (3) | $x^{\nu+\frac{1}{2}} {}_2F_1(a, \beta; \nu+1; -\lambda^2 x^2)$<br>$-1 < \operatorname{Re} \nu$<br>$< 2\max(\operatorname{Re} a, \operatorname{Re} \beta) - 3/2$<br>$\operatorname{Re} \lambda > 0$                                        | $\frac{2^{\nu-\alpha-\beta+2} \Gamma(\nu+1)}{\lambda^{\alpha+\beta} \Gamma(a) \Gamma(\beta)} \gamma^{\alpha+\beta-\nu-3/2}$<br>$\times K_{\alpha-\beta}(\gamma/\lambda)$                                                                                                |
| (4) | $x^{\nu+\frac{1}{2}} {}_2F_1[a, \beta; \frac{1}{2}(\beta+\nu)+1; -\lambda^2 x^2]$<br>$-1 < \operatorname{Re} \nu$<br>$< 2\max(\operatorname{Re} a, \operatorname{Re} \beta) - 3/2$                                                        | $\frac{\Gamma(\frac{1}{2}\beta + \frac{1}{2}\nu + 1)}{\pi^{\frac{1}{2}} \Gamma(a) \Gamma(\beta)} \frac{\gamma^{\beta-\frac{1}{2}}}{2^{\beta-1} \lambda^{\nu+\beta+1}}$<br>$\times \left[ K_{\frac{1}{2}(\nu-\beta+1)} \left( \frac{\gamma}{2\lambda} \right) \right]^2$ |
| (5) | $x^{\nu+\frac{1}{2}} {}_2F_1(a, \beta; \gamma; -\lambda^2 x^2)$<br>$-1 < \operatorname{Re} \nu$<br>$< 2\max(\operatorname{Re} a, \operatorname{Re} \beta) - 3/2$<br>$\operatorname{Re} \lambda > 0$                                       | $\frac{2^{\nu+1} \Gamma(\gamma)}{\Gamma(a) \Gamma(\beta)} \gamma^{-\nu-3/2}$<br>$\times G_{13}^{30} \left( \frac{\gamma^2}{4\lambda^2} \middle  \begin{matrix} \gamma \\ \nu+1, a, \beta \end{matrix} \right)$                                                          |
| (6) | $x^{\delta-\frac{1}{2}} {}_2F_1(a, \beta; \gamma; -\lambda^2 x^2)$<br>$-\operatorname{Re} \nu - 1 < \operatorname{Re} \delta$<br>$< 2\max(\operatorname{Re} a, \operatorname{Re} \beta) - \frac{1}{2}$<br>$\operatorname{Re} \lambda > 0$ | $\frac{2^\delta \Gamma(\gamma)}{\Gamma(a) \Gamma(\beta)} \gamma^{-\delta-\frac{1}{2}}$<br>$\times G_{24}^{31} \left( \frac{\gamma^2}{4\lambda^2} \middle  \begin{matrix} 1, \gamma \\ \frac{1+\delta+\nu}{2}, a, \beta, \frac{1+\delta-\nu}{2} \end{matrix} \right)$    |
| (7) | $x^{-2\alpha-3/2}$<br>$\times {}_2F_1 \left( \frac{1}{2} + a, 1+a; 1+2a; -\frac{4\lambda^2}{x^2} \right)$<br>$\operatorname{Re} \nu > -1, \quad \operatorname{Re} \lambda > 0$<br>$\operatorname{Re} a > -\frac{1}{2}$                    | $\lambda^{-2\alpha} \gamma^{\frac{1}{2}} I_{\frac{1}{2}\nu+a}(\lambda\gamma) K_{\frac{1}{2}\nu-a}(\lambda\gamma)$                                                                                                                                                       |
| (8) | $x^{\nu-4\alpha+\frac{1}{2}}$<br>$\times {}_2F_1(a, a+\frac{1}{2}; \nu+1; -\lambda^2 x^{-2})$<br>$\operatorname{Re} a - 1 < \operatorname{Re} \nu < 4\operatorname{Re} a - 3/2$<br>$\operatorname{Re} \lambda > 0$                        | $\frac{\Gamma(\nu)}{\Gamma(2a)} 2^\nu \lambda^{1-2\alpha} \gamma^{2\alpha-\nu-\frac{1}{2}} I_\nu(\frac{1}{2}\lambda\gamma)$<br>$\times K_{2\alpha-\nu-1}(\frac{1}{2}\lambda\gamma)$                                                                                     |

## Gauss' hypergeometric function (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                         | $\int_0^\infty f(x) J_\nu(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                             |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (9)  | $x^{\delta-\frac{1}{2}} {}_2F_1(a, \beta; \gamma; -\lambda^2 x^{-2})$<br>$-1 - \operatorname{Re} \nu - 2 \min(\operatorname{Re} a, \operatorname{Re} \beta)$<br>$< \operatorname{Re} \delta < -\frac{1}{2}$<br>$\operatorname{Re} \lambda > 0$ | $\frac{2^\delta \Gamma(\gamma)}{\Gamma(a) \Gamma(\beta)} y^{-\delta-\frac{1}{2}}$<br>$\times G_{24}^{22} \left( \frac{\lambda^2 y^2}{4} \middle  \begin{matrix} 1-a, 1-\beta \\ 1+\delta+\nu \\ 2, 0, 1-\gamma, \frac{1+\delta-\nu}{2} \end{matrix} \right)$ |
| (10) | $x^{\nu+\frac{1}{2}} (1+x)^{-2a}$<br>$\times {}_2F_1 \left[ a, \nu+\frac{1}{2}; 2\nu+1; \frac{4x}{(1+x)^2} \right]$<br>$-1 < \operatorname{Re} \nu < 2 \operatorname{Re} a - 3/2$                                                              | $\frac{\Gamma(\nu+1) \Gamma(\nu-a+1)}{\Gamma(a)} 2^{2\nu-2a+1}$<br>$\times y^{2a-2\nu-3/2} J_\nu(x)$                                                                                                                                                         |

## 8.18. Confluent hypergeometric functions

|     |                                                                                                                                                                                                          |                                                                                                                                      |
|-----|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $x^{-1} \exp(-\frac{1}{2} x^2)$<br>$\times M_{\frac{1}{2}\nu-\frac{1}{4}, \frac{1}{2}\nu+\frac{1}{4}}(x^2)$<br>$\operatorname{Re} \nu > -\frac{1}{2}$                                                    | $(2\nu+1) 2^{-\nu} y^{\nu-\frac{1}{2}} \operatorname{Erfc}(\frac{1}{2}y)$                                                            |
| (2) | $x^{-3/2} \exp(-\frac{1}{2} x^2)$<br>$\times M_{\frac{1}{2}\nu+\frac{1}{2}, \frac{1}{2}\nu+\frac{1}{2}}(x^2)$<br>$\operatorname{Re} \nu > -1$                                                            | $\frac{\Gamma(\nu+2) y^{\nu+\frac{1}{2}}}{\Gamma(\nu+3/2) 2^\nu} \operatorname{Erfc}(\frac{1}{2}y)$                                  |
| (3) | $x^{2\mu-\nu-\frac{1}{2}} \exp(-\frac{1}{4} x^2)$<br>$\times M_{3\mu-\nu+\frac{1}{2}, \mu}(\frac{1}{2} x^2)$<br>$\operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re}(4\mu-\nu) > -\frac{1}{2}$ | $y^{2\mu-\nu-\frac{1}{2}} \exp(-\frac{1}{4} y^2)$<br>$\times M_{3\mu-\nu+\frac{1}{2}, \mu}(\frac{1}{2} y^2)$                         |
| (4) | $x^{\nu-2\mu-\frac{1}{2}} \exp(-\frac{1}{4} x^2)$<br>$\times M_{\nu-\mu, \mu}(\frac{1}{2} x^2)$<br>$\operatorname{Re} \nu > -\frac{1}{2}$                                                                | $\frac{\Gamma(2\mu+1)}{2^{\nu-\mu} \Gamma(\nu+\frac{1}{2})} y^{\nu-\frac{1}{2}} \exp(-\frac{1}{4} y^2)$<br>$\times D_{2\nu-4\mu}(y)$ |



## Confluent hypergeometric functions (cont'd)

|      | $f(x)$                                                                                                                                                                                   | $\int_0^\infty f(x) J_\nu(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                 |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (5)  | $x^{\nu-2\mu-\frac{1}{2}} \exp(-\frac{1}{4}x^2)$<br>$\times M_{\nu-\mu+1, \mu}(\frac{1}{2}x^2)$<br>$\text{Re } \nu > -1$                                                                 | $\frac{\Gamma(2\mu+1)}{\Gamma(\nu+3/2) 2^{\nu-\mu+1}} y^{\nu+\frac{1}{2}} \exp(-\frac{1}{4}y^2)$<br>$\times D_{2\nu-4\mu+1}(y)$                                                                                                                                                                  |
| (6)  | $x^{-\lambda-1} \exp(-\frac{1}{4}x^2)$<br>$\times M_{\lambda+\mu, \mu}(\frac{1}{2}x^2)$<br>$\lambda = 2\mu - \nu - \frac{1}{2}$<br>$-1 < \text{Re } \nu < 4\text{Re } \mu$               | $\pi^{-\frac{1}{2}} 2^{-5(\mu+\nu/3)} \frac{\Gamma(2\mu+1)}{\Gamma(4\mu-\nu)} y^{\lambda+2\mu}$<br>$\times \exp(-\frac{1}{4}y^2) K_\lambda(\frac{1}{4}y^2)$                                                                                                                                      |
| (7)  | $x^{-\frac{1}{2}} \exp(-\frac{1}{4}x^2)$<br>$\times M_{\kappa, \frac{1}{2}\nu}(\frac{1}{2}x^2)$<br>$\text{Re } \nu > -1, \quad \text{Re } \kappa < \frac{1}{2}$                          | $\frac{2^{-\kappa} \Gamma(\nu+1)}{\Gamma(\kappa + \frac{1}{2}\nu + \frac{1}{2})} y^{2\kappa-\frac{1}{2}}$<br>$\times \exp(-\frac{1}{2}y^2)$                                                                                                                                                      |
| (8)  | $x^{2\mu-\nu-\frac{1}{2}} \exp(-\frac{1}{4}x^2)$<br>$\times M_{\kappa, \mu}(\frac{1}{2}x^2)$<br>$-\frac{1}{2} < \text{Re } \mu < \text{Re}(\kappa + \frac{1}{2}\nu) - \frac{1}{4}$       | $\frac{\Gamma(2\mu+1) 2^{\frac{1}{2}(\frac{1}{2}-\kappa+3\mu-\nu)}}{\Gamma(\frac{1}{2}+\kappa-\mu+\nu)} y^{1-\kappa+\mu}$<br>$\times \exp(-\frac{1}{4}y^2) M_{\alpha, \beta}(\frac{1}{2}y^2)$<br>$2\alpha = \frac{1}{2} + \kappa + 3\mu - \nu$<br>$2\beta = -\frac{1}{2} + \kappa - \mu + \nu$   |
| (9)  | $x^{\nu-2\mu-\frac{1}{2}} \exp(-\frac{1}{4}x^2)$<br>$\times M_{\kappa, \mu}(\frac{1}{2}x^2)$<br>$-1 < \text{Re } \nu < 2\text{Re}(\kappa + \mu) - \frac{1}{2}$                           | $2^{\frac{1}{2}(\frac{1}{2}-\kappa-3\mu+\nu)} \frac{\Gamma(2\mu+1)}{\Gamma(\mu+\kappa+\frac{1}{2})} y^{\kappa+\mu-1}$<br>$\times \exp(-\frac{1}{4}y^2) W_{\alpha, \beta}(\frac{1}{2}y^2)$<br>$2\alpha = \kappa - 3\mu + \nu + \frac{1}{2}$<br>$2\beta = \kappa + \mu - \nu - \frac{1}{2}$        |
| (10) | $x^{2\rho-\frac{1}{2}} \exp(-\frac{1}{2}ax^2) M_{\kappa, \mu}(ax^2)$<br>$-1 - \text{Re}(\frac{1}{2}\nu + \mu) < \text{Re } \rho < \text{Re } \kappa - \frac{1}{4}$<br>$\text{Re } a > 0$ | $\frac{\Gamma(2\mu+1)}{\Gamma(\mu+\kappa+\frac{1}{2})} 2^{2\rho} y^{-2\rho-\frac{1}{2}}$<br>$\times G_{23}^{21} \left( \frac{y^2}{4a} \middle  \begin{matrix} \frac{1}{2}-\mu, \frac{1}{2}+\mu \\ \frac{1}{2}+\rho+\frac{1}{2}\nu, \kappa, \frac{1}{2}+\rho-\frac{1}{2}\nu \end{matrix} \right)$ |

## Confluent hypergeometric functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                                 | $\int_0^\infty f(x) J_\nu(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                           |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (11) | $x^{\nu-2\mu-\frac{1}{2}} \exp(-\frac{1}{4}x^2)$<br>$\times W_{3\mu-\nu-\frac{1}{2}, \pm\mu}(\frac{1}{2}x^2)$<br>$\text{Re } \nu > -1, \quad \text{Re}(\nu - 2\mu) > -1$                                               | $\pi^{\frac{1}{2}} 2^{\nu-3\mu} y^{4\mu-\nu-\frac{1}{2}} \exp(-\frac{1}{4}y^2)$<br>$\times I_{\nu-2\mu+\frac{1}{2}}(\frac{1}{4}y^2)$                                                                                                                                                       |
| (12) | $x^{\nu-2\mu-\frac{1}{2}} \exp(\frac{1}{4}x^2)$<br>$\times W_{\nu-3\mu+\frac{1}{2}, \pm\mu}(\frac{1}{2}x^2)$<br>$\text{Re } \nu > -1, \quad \text{Re}(\nu - 2\mu) > -1$<br>$\text{Re}(3\nu - 8\mu) < -3/2$             | $\pi^{-\frac{1}{2}} 2^{\nu-3\mu} \frac{\Gamma(1+\nu-2\mu)}{\Gamma(4\mu-\nu)} y^{4\mu-\nu-\frac{1}{2}}$<br>$\times \exp(\frac{1}{4}y^2) K_{\nu-2\mu+\frac{1}{2}}(\frac{1}{4}y^2)$                                                                                                           |
| (13) | $x^{\nu-2\mu-\frac{1}{2}} \exp(\frac{1}{4}x^2)$<br>$\times W_{3\mu-\nu-\frac{1}{2}, \pm\mu}(\frac{1}{2}x^2)$<br>$\text{Re } \nu > -1, \quad \text{Re}(\nu - 2\mu) > -1$<br>$\text{Re}(\nu - 4\mu) > -\frac{1}{2}$      | $y^{\nu-2\mu-\frac{1}{2}} \exp(\frac{1}{4}y^2)$<br>$\times W_{3\mu-\nu-\frac{1}{2}, \pm\mu}(\frac{1}{2}y^2)$                                                                                                                                                                               |
| (14) | $x^{\nu-2\mu-\frac{1}{2}} \exp(-\frac{1}{4}x^2)$<br>$\times W_{\kappa, \pm\mu}(\frac{1}{2}x^2)$<br>$\text{Re } \nu > -1, \quad \text{Re}(\nu - 2\mu) > -1$                                                             | $\frac{\Gamma(1+\nu-2\mu)}{\Gamma(1+2\beta)} 2^{\beta-\mu} y^{\kappa+\mu-1}$<br>$\times \exp(-\frac{1}{4}y^2) M_{\alpha\beta}(\frac{1}{2}y^2)$<br>$2\alpha = \frac{1}{2} + \kappa + \nu - 3\mu$<br>$2\beta = \frac{1}{2} - \kappa + \nu - \mu$                                             |
| (15) | $x^{\nu-2\mu-\frac{1}{2}} \exp(\frac{1}{4}x^2)$<br>$\times W_{\kappa, \pm\mu}(\frac{1}{2}x^2)$<br>$\text{Re } \nu > -1, \quad \text{Re}(\nu - 2\mu) > -1$<br>$\text{Re}(\kappa - \mu + \frac{1}{2}\nu) < -\frac{1}{4}$ | $\frac{\Gamma(1+\nu-2\mu)}{\Gamma(\frac{1}{2}+\mu-\kappa)} 2^{\frac{1}{2}(\frac{1}{2}+\kappa-3\mu+\nu)} y^{\mu-\kappa-1}$<br>$\times \exp(\frac{1}{4}y^2) W_{\alpha\beta}(\frac{1}{2}y^2)$<br>$2\alpha = \kappa + 3\mu - \nu - \frac{1}{2}$<br>$2\beta = \kappa - \mu + \nu + \frac{1}{2}$ |

## Confluent hypergeometric functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                    | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                           |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (16) | $x^{2\rho-\frac{1}{2}} \exp(-\frac{1}{2}ax^2)$<br>$\times W_{\kappa,\mu}(ax^2) \quad \text{Re } a > 0$<br>$\text{Re}(\rho \pm \mu + \frac{1}{2}\nu) > -1$                                                 | $\frac{\Gamma(1+\mu+\nu/2+\rho)\Gamma(1-\mu+\nu/2+\rho)}{\Gamma(\nu+1)\Gamma(3/2-\kappa+\nu/2+\rho)}$<br>$\times 2^{-\nu-1} a^{-\frac{1}{2}} \nu^{-\rho-\frac{1}{2}} y^{\nu+\frac{1}{2}}$<br>$\times {}_2F_2\left(\lambda+\mu, \lambda-\mu; \nu+1, \frac{1}{2}-\kappa+\lambda; -\frac{y^2}{4a}\right)$<br>$\lambda = 1 + \frac{1}{2}\nu + \rho$           |
| (17) | $x^{2\rho-\frac{1}{2}} \exp(\frac{1}{2}ax^2)$<br>$\times W_{\kappa,\mu}(ax^2) \quad  \arg a  < \pi$<br>$-1 - \text{Re}(\frac{1}{2}\nu \pm \mu)$<br>$< \text{Re } \rho < -\frac{1}{4} - \text{Re } \kappa$ | $\frac{2^{2\rho} y^{-2\rho-\frac{1}{2}}}{\Gamma(\frac{1}{2}+\mu-\kappa)\Gamma(\frac{1}{2}-\mu-\kappa)}$<br>$\times G_{23}^{22}\left(\frac{y^2}{4a} \middle  \begin{matrix} \frac{1}{2}-\mu, \frac{1}{2}+\mu \\ \frac{1}{2}+\rho+\frac{1}{2}\nu, -\kappa, \frac{1}{2}+\rho-\frac{1}{2}\nu \end{matrix} \right)$                                            |
| (18) | $x^{-\frac{1}{2}} M_{\kappa, \frac{1}{2}\nu}(-iax) M_{-\kappa, \frac{1}{2}\nu}(-iax)$<br>$a > 0, \quad \text{Re } \nu > -1, \quad  \text{Re } \kappa  < \frac{1}{4}$                                      | $\frac{ae^{-\frac{1}{2}(\nu+1)\pi i} [\Gamma(1+\nu)]^2}{\Gamma(\frac{1}{2}+\kappa+\frac{1}{2}\nu)\Gamma(\frac{1}{2}-\kappa+\frac{1}{2}\nu)} y^{-\frac{1}{2}-2\kappa}$<br>$\times (a^2 - y^2)^{-\frac{1}{2}} \{ [a + (a^2 - y^2)^{\frac{1}{2}}]^{2\kappa}$<br>$+ [a - (a^2 - y^2)^{\frac{1}{2}}]^{2\kappa} \} \quad 0 < y < a$<br>$0 \quad a < y < \infty$ |
| (19) | $x^{-\frac{1}{2}} M_{-\frac{1}{2}\mu, \frac{1}{2}\nu}(ax)$<br>$\times W_{\frac{1}{2}\mu, \frac{1}{2}\nu}(ax) \quad \text{Re } \nu > -1$<br>$\text{Re } \mu < \frac{1}{2}, \quad \text{Re } a > 0$         | $a \frac{\Gamma(\nu+1)}{\Gamma(\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu)} y^{-\mu-\frac{1}{2}}$<br>$\times [a + (a^2 + y^2)^{\frac{1}{2}}]^\mu (a^2 + y^2)^{-\frac{1}{2}}$                                                                                                                                                                               |
| (20) | $x^{2\mu-\nu-\frac{1}{2}} W_{\kappa,\mu}(ax) M_{-\kappa,\mu}(ax)$<br>$\text{Re } \mu > -\frac{1}{2}, \quad \text{Re } a > 0$<br>$\text{Re}(2\mu + 2\kappa - \nu) < \frac{1}{2}$                           | $2^{2\mu-\nu+2\kappa} a^{2\kappa} y^{\nu-2\mu-2\kappa-\frac{1}{2}}$<br>$\times \frac{\Gamma(2\mu+1)}{\Gamma(\nu-\kappa-\mu+\frac{1}{2})}$<br>$\times {}_3F_2\left(\frac{1}{2}-\kappa, 1-\kappa, \frac{1}{2}-\kappa+\mu; 1-2\kappa, \frac{1}{2}-\kappa-\mu+\nu; -y^2/a^2\right)$                                                                           |

## Confluent hypergeometric functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                                             | $\int_0^\infty f(x) J_\nu(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                       |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (21) | $x^{2\rho-\nu-5/2} W_{\kappa,\mu}(ax)$<br>$\times M_{-\kappa,\mu}(ax) \quad \text{Re } a > 0$<br>$\text{Re } \rho > 0, \quad \text{Re } (\rho + \mu) > 0$<br>$\text{Re } (2\rho + 2\kappa - \nu) < 5/2$                            | $\frac{2^{2\rho-\nu-2} \Gamma(2\mu+1)}{\pi^{\frac{1}{2}} \Gamma(\frac{1}{2}-\kappa+\mu)} y^{\nu-2\rho+3/2}$<br>$\times G_{44}^{23} \left( \frac{y^2}{a^2} \middle  \begin{matrix} \frac{1}{2}, 0, \frac{1}{2}-\mu, \frac{1}{2}+\mu \\ \rho-\frac{1}{2}, -\kappa, \kappa, \rho-\nu-\frac{1}{2} \end{matrix} \right)$                    |
| (22) | $x^{2\rho-\nu-5/2} W_{\kappa,\mu}(ax)$<br>$\times W_{-\kappa,\mu}(ax)$<br>$\text{Re } \rho >  \text{Re } \mu , \quad \text{Re } a > 0$                                                                                             | $\frac{\Gamma(\rho+\mu) \Gamma(\rho-\mu) \Gamma(2\rho)}{\Gamma(\frac{1}{2}+\kappa+\rho) \Gamma(\frac{1}{2}-\kappa+\rho) \Gamma(1+\nu)}$<br>$\times 2^{-\nu-1} a^{1-2\rho} y^{\nu+\frac{1}{2}}$<br>$\times {}_4F_3(\rho, \rho+\frac{1}{2}, \rho+\mu, \rho-\mu;$<br>$\frac{1}{2}+\kappa+\rho, \frac{1}{2}-\kappa+\rho, 1+\nu; -y^2/a^2)$ |
| (23) | $x^{2\rho-\nu-5/2} W_{\kappa,\mu}(iax)$<br>$\times W_{\kappa,\mu}(-iax) \quad \text{Re } a > 0$<br>$\text{Re } \rho >  \text{Re } \mu $<br>$\text{Re } (2\rho + 2\kappa - \nu) < 5/2$                                              | $\frac{2^{2\rho-\nu-2} y^{\nu-2\rho+3/2}}{\pi^{1/2} \Gamma(\frac{1}{2}-\kappa+\mu) \Gamma(\frac{1}{2}-\kappa-\mu)}$<br>$\times G_{44}^{24} \left( \frac{y^2}{a^2} \middle  \begin{matrix} \frac{1}{2}, 0, \frac{1}{2}-\mu, \frac{1}{2}+\mu \\ \rho-\frac{1}{2}, -\kappa, \kappa, \rho-\nu-\frac{1}{2} \end{matrix} \right)$            |
| (24) | $x^{-3/2} M_{-\mu, \frac{1}{4}\nu}(\frac{1}{2}x^2)$<br>$\times W_{\mu, \frac{1}{4}\nu}(\frac{1}{2}x^2) \quad \text{Re } \nu > -1$                                                                                                  | $\frac{\Gamma(1+\frac{1}{2}\nu) y^{\frac{1}{2}}}{\Gamma(\frac{1}{2}+\frac{1}{4}\nu-\mu)} I_{\frac{1}{4}\nu-\mu}(\frac{1}{4}y^2) K_{\frac{1}{4}\nu+\mu}(\frac{1}{4}y^2)$                                                                                                                                                                |
| (25) | $x^{-3/2} M_{\alpha-\beta, \frac{1}{4}\nu-\gamma}(\frac{1}{2}x^2)$<br>$\times W_{\alpha+\beta, \frac{1}{4}\nu+\gamma}(\frac{1}{2}x^2)$<br>$\text{Re } \beta < 1/8, \quad \text{Re } \nu > -1$<br>$\text{Re } (\nu - 4\gamma) > -2$ | $\frac{\Gamma(1+\frac{1}{2}\nu-2\gamma)}{\Gamma(1+\frac{1}{2}\nu-2\beta)} y^{-3/2} M_{\alpha-\gamma, \frac{1}{4}\nu-\beta}(\frac{1}{2}y^2)$<br>$\times W_{\alpha+\gamma, \frac{1}{4}\nu+\beta}(\frac{1}{2}y^2)$                                                                                                                        |
| (26) | $x^{\frac{1}{2}} M_{\frac{1}{2}\nu, \mu}(2/x) W_{-\frac{1}{2}\nu, \mu}(2/x)$<br>$\text{Re } \nu > -1, \quad \text{Re } \mu > -\frac{1}{4}$                                                                                         | $\frac{4\Gamma(1+2\mu)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\mu)} y^{-\frac{1}{2}} J_{2\mu}(2y^{\frac{1}{2}}) K_{2\mu}(2y^{\frac{1}{2}})$                                                                                                                                                                                                |

## Confluent hypergeometric functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                                                                                                   | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                  |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (27) | $x^{\frac{1}{2}} W_{\frac{1}{2}\nu, \mu}(2/x) W_{-\frac{1}{2}\nu, \mu}(2/x)$<br>$\operatorname{Re}(\nu \pm 2\mu) > -1$                                                                                                                                                                                                   | $-4y^{-\frac{1}{2}} \{ \sin[(\mu - \frac{1}{2}\nu)\pi] J_{2\mu}(2y^{\frac{1}{2}})$<br>$+ \cos[(\mu - \frac{1}{2}\nu)\pi] Y_{2\mu}(2y^{\frac{1}{2}}) \} K_{2\mu}(2y^{\frac{1}{2}})$                               |
| (28) | $x^{\frac{1}{2}} W_{-\frac{1}{2}\nu, \mu}(i a/x)$<br>$\times W_{-\frac{1}{2}\nu, \mu}(-i a/x) \quad \operatorname{Re} a > 0$<br>$ \operatorname{Re} \mu  < \frac{1}{2}, \quad \operatorname{Re} \nu > -1$                                                                                                                | $4 a y^{-\frac{1}{2}} [\Gamma(\frac{1}{2} + \mu + \frac{1}{2}\nu) \Gamma(\frac{1}{2} - \mu + \frac{1}{2}\nu)]^{-1}$<br>$\times K_\mu[(2 i a y)^{\frac{1}{2}}] K_\mu[(-2 i a y)^{\frac{1}{2}}]$                   |
| (29) | $x^{-\frac{1}{2}} M_{-\mu, \frac{1}{2}\nu} \{ a[(\beta^2 + x^2)^{\frac{1}{2}} - \beta] \}$<br>$\times W_{\mu, \frac{1}{2}\nu} \{ a[(\beta^2 + x^2)^{\frac{1}{2}} + \beta] \}$<br>$\operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu < \frac{1}{4}$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \beta > 0$ | $\frac{a \Gamma(1 + \nu) [(a^2 + y^2)^{\frac{1}{2}} + a]^{2\mu}}{\Gamma(\frac{1}{2} + \frac{1}{2}\nu - \mu) y^{\frac{1}{2} + 2\mu} (a^2 + y^2)^{\frac{1}{2}}}$<br>$\times \exp[-\beta(a^2 + y^2)^{\frac{1}{2}}]$ |

## 8.19. Generalized hypergeometric series and miscellaneous functions

|     |                                                                                                                                                                   |                                                                                                                                                                                             |
|-----|-------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $x^{\nu + \frac{1}{2}} {}_1F_1(2a - \nu; a + 1; -\frac{1}{2}x^2)$<br>$\operatorname{Re} \nu > -1, \quad \operatorname{Re}(4a - 3\nu) > \frac{1}{2}$               | $\frac{2^{\nu - a + \frac{1}{2}} \Gamma(a + 1)}{\pi^{\frac{1}{2}} \Gamma(2a - \nu)} y^{2a - \nu - \frac{1}{2}}$<br>$\times \exp(-\frac{1}{4}y^2) K_{a - \nu - \frac{1}{2}}(\frac{1}{4}y^2)$ |
| (2) | $x^{a - \frac{1}{2}} {}_1F_1\left(a; \frac{1 + a + \nu}{2}; -\frac{x^2}{2}\right)$<br>$\operatorname{Re} a > -\frac{1}{2}, \quad \operatorname{Re}(a + \nu) > -1$ | $y^{a - \frac{1}{2}} {}_1F_1\left(a; \frac{1 + a + \nu}{2}; -\frac{y^2}{2}\right)$                                                                                                          |
| (3) | $x^{\nu + \frac{1}{2} - 2a} {}_1F_1(a; 1 + \nu - a; -\frac{1}{2}x^2)$<br>$\operatorname{Re} a - 1 < \operatorname{Re} \nu < 4\operatorname{Re} a - \frac{1}{2}$   | $\frac{\pi^{\frac{1}{2}} \Gamma(1 + \nu - a)}{2^{2a - \nu - \frac{1}{2}} \Gamma(a)} y^{2a - \nu - \frac{1}{2}}$<br>$\times \exp(-\frac{1}{4}y^2) I_{a - \frac{1}{2}}(\frac{1}{4}y^2)$       |

## Miscellaneous functions (cont'd)

|     | $f(x)$                                                                                                                                                                                        | $\int_0^\infty f(x) J_\nu(xy)(xy)^{1/2} dx \quad y > 0$                                                                                                                                                                                       |
|-----|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (4) | $x^{\nu+1/2} {}_1F_1(a; \beta; -\lambda x^2)$<br>$-1 < \operatorname{Re} \nu < 2 \operatorname{Re} a - 1/2$<br>$\operatorname{Re} \lambda > 0$                                                | $\frac{2^{1-a} \Gamma(\beta)}{\Gamma(a) \lambda^{1/2 a + 1/2 \nu}} y^{a-3/2} \exp\left(-\frac{y^2}{8\lambda}\right)$<br>$\times W_{\kappa, \mu}\left(\frac{y^2}{4\lambda}\right)$<br>$2\kappa = a - 2\beta + \nu + 2$<br>$2\mu = a - \nu - 1$ |
| (5) | $x^{2\beta-\nu-3/2} {}_1F_1(a; \beta; -\lambda x^2)$<br>$0 < \operatorname{Re} \beta < 3/4 + \operatorname{Re}(a + 1/2 \nu)$<br>$\operatorname{Re} \lambda > 0$                               | $\frac{2^{2\beta-2a-\nu-1} \Gamma(\beta)}{\Gamma(a - \beta + \nu + 1) \lambda^a} y^{2a-2\beta+\nu+1/2}$<br>$\times {}_1F_1\left(a; 1+a-\beta+\nu; -\frac{y^2}{4\lambda}\right)$                                                               |
| (6) | $x^{2\rho-1/2} {}_1F_1(a; \beta; -\lambda x^2)$<br>$-1 - \operatorname{Re} \nu < 2 \operatorname{Re} \rho < 1/2 + 2 \operatorname{Re} a$<br>$\operatorname{Re} \lambda > 0$                   | $\frac{2^{2\rho} \Gamma(\beta)}{\Gamma(a) y^{2\rho+1/2}}$<br>$\times G_{23}^{21}\left(\frac{y^2}{4\lambda} \left  \begin{matrix} 1, \beta \\ 1/2 + \rho + 1/2 \nu, a, 1/2 + \rho - 1/2 \nu \end{matrix} \right.\right)$                       |
| (7) | $x^{\nu+1/2} {}_1F_2(a; \beta, \nu+1; -1/4 x^2)$<br>$\operatorname{Re} \nu > -1, \quad \operatorname{Re} \beta > \operatorname{Re} a > 0$                                                     | $\frac{\Gamma(\nu+1) \Gamma(\beta)}{\Gamma(a) \Gamma(\beta-a)} 2^{\nu+1} y^{2a-\nu-3/2}$<br>$\times (1-y^2)^{\beta-a-1} \quad 0 < y < 1$<br>$0 \quad 1 < y < \infty$                                                                          |
| (8) | $x^{\nu+1/2} (1-x^2)^{\mu-\nu-1}$<br>$\times {}_1F_2[\mu+1/2; 2\mu+1, \mu-\nu; -a(1-x^2)]$<br>$0 < x < 1$<br>$0 \quad 1 < x < \infty$<br>$\operatorname{Re} \mu > \operatorname{Re} \nu > -1$ | $2^{2\mu-\nu-1} a^{-\mu} \Gamma(1+\mu) \Gamma(\mu-\nu) y^{\nu+1/2}$<br>$\times J_\mu[1/2(y^2+4a^2)^{1/2} + 1/2 y]$<br>$\times J_\mu[1/2(y^2+4a^2)^{1/2} - 1/2 y]$                                                                             |

## Miscellaneous functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                                                                                              | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                     |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (9)  | $x^{\nu+1/2} (1-x^2)^{\mu-3/2}$<br>$\times {}_1F_2 \left[ 1; \frac{\mu}{2}, \frac{\mu+1}{2}; -a^2(1-x^2)^2 \right]$<br>$0 < x < 1$<br>$0 \quad 1 < x < \infty$<br>$\operatorname{Re} \nu > -1, \quad \operatorname{Re} \mu > 0$                                                                                     | $2^{-\mu-2\nu-1} \Gamma(\mu) a^{-\mu-\nu} y^{\nu+1/2} U_{\nu+\mu}(4a, y)$                                                                                                           |
| (10) | $x^{\rho-1/2}$<br>$\times {}_2F_2 \left( \rho, \frac{\rho+m+1}{2}; \right.$<br>$\left. \frac{\rho-m+1}{2}, \frac{\rho+\nu+1}{2}; \frac{x^2}{2} \right)$<br>$\operatorname{Re}(\rho+\nu+1) > 0, \quad \operatorname{Re} \rho > \frac{1}{2}$<br>$\operatorname{Re}(\rho-m+1) > 0$                                     | $(-1)^m y^{\rho-1/2}$<br>$\times {}_2F_2 \left( \rho, \frac{\rho+m+1}{2}; \right.$<br>$\left. \frac{\rho-m+1}{2}, \frac{\rho+\nu+1}{2}; -\frac{y^2}{2} \right)$                     |
| (11) | $x^{\nu+1/2}$<br>$\times {}_3F_2(a, a+\beta, a-\beta;$<br>$\quad a+\frac{1}{2}, \nu+1; -x^2)$<br>$\operatorname{Re} a > \frac{1}{2} \operatorname{Re} \nu + \frac{1}{4} > -\frac{1}{4}$<br>$ \operatorname{Re} \beta  < \operatorname{Re} a - \frac{1}{2} \operatorname{Re} \nu - \frac{1}{4}$                      | $\frac{2^{\nu-2a+2} \Gamma(a+\frac{1}{2}) \Gamma(\nu+1)}{\pi^{\frac{1}{2}} \Gamma(a) \Gamma(a+\beta) \Gamma(a-\beta)}$<br>$\times y^{-2a-\nu-3/2} [K_\beta(\frac{1}{2}y)]^2$        |
| (12) | $x^{\nu+1/2}$<br>$\times {}_3F_2(\beta+\rho, \beta-\rho, 2\beta-\nu-1;$<br>$\quad \beta, \beta+\frac{1}{2}; -\frac{1}{4}x^2)$<br>$ \operatorname{Re} \rho  < \operatorname{Re} \beta - \frac{1}{2} \operatorname{Re} \nu - \frac{1}{4}$<br>$\operatorname{Re} \nu > -1, \quad \operatorname{Re}(4\beta-3\nu) > 5/2$ | $\frac{2^{\nu+3-2\beta} \Gamma(2\beta) y^{2\beta-\nu-3/2}}{\Gamma(\beta+\rho) \Gamma(\beta-\rho) \Gamma(2\beta-\nu-1)}$<br>$\times K_{\nu+1-\beta+\rho}(y) K_{\nu+1-\beta-\rho}(y)$ |

## Miscellaneous functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                                                                                                                                                                                         | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                        |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (13) | $x^{\nu+\frac{1}{2}}$<br>$\times {}_3F_2(a, a+\frac{1}{2}, 2a-\nu-1;$<br>$\quad a+\frac{1}{2}+\kappa, a+\frac{1}{2}-\kappa; -x^2)$<br>$-1 < \operatorname{Re} \nu < 2 \operatorname{Re} a - 1/2$<br>$\operatorname{Re}(4a - 3\nu) > 5/2$                                                                                                                                                                       | $\frac{\Gamma(a+\frac{1}{2}+\kappa) \Gamma(a+\frac{1}{2}-\kappa)}{\Gamma(2a) \Gamma(2a-\nu-1)} 2^{\nu+1}$<br>$\times y^{2a-\nu-5/2} W_{\kappa, a-\nu-1}(y)$<br>$\times W_{-\kappa, a-\nu-1}(y)$                        |
| (14) | $x^{\nu+\frac{1}{2}}$<br>$\times {}_3F_2(\nu+\frac{1}{2}, \nu+\frac{1}{2}+\mu, \nu+\frac{1}{2}-\mu;$<br>$\quad \nu+1+\kappa, \nu+1-\kappa; -x^2)$<br>$\operatorname{Re} \nu > -\frac{1}{2}$<br>$ \operatorname{Re} \mu  < \frac{1}{2} \operatorname{Re} \nu + \frac{1}{4}$                                                                                                                                     | $\frac{\Gamma(\nu+\kappa+1) \Gamma(\nu-\kappa+1) \pi^{1/2} 2^{1-\nu}}{\Gamma(\nu+\frac{1}{2}) \Gamma(\nu+\frac{1}{2}+\mu) \Gamma(\nu+\frac{1}{2}-\mu)}$<br>$\times y^{\nu-3/2} W_{\kappa, \mu}(y) W_{-\kappa, \mu}(y)$ |
| (15) | $x^{\nu+\frac{1}{2}}$<br>$\times {}_3F_2\left(\nu+\frac{3}{2}, \nu+1+\mu, \nu+1-\mu;$<br>$\quad \nu+\frac{3}{2}+\kappa, \nu+\frac{3}{2}-\kappa; -x^2\right)$<br>$\operatorname{Re} \nu > -1$<br>$ \operatorname{Re} \mu  < \frac{1}{2} \operatorname{Re} \nu + \frac{3}{4}$                                                                                                                                    | $\frac{\Gamma(\nu+\kappa+3/2) \Gamma(\nu-\kappa+3/2) \pi^{1/2} 2^{-\nu}}{\Gamma(\nu+3/2) \Gamma(\nu+1+\mu) \Gamma(\nu+1-\mu)}$<br>$\times y^{\nu-1/2} W_{\kappa, \mu}(y) W_{-\kappa, \mu}(y)$                          |
| (16) | $x^{\nu+\frac{1}{2}}$<br>$\times {}_4F_3(a+\beta, a-\beta, a+\gamma, a-\gamma;$<br>$\quad a, a+\frac{1}{2}, \nu+1; -x^2)$<br>$-1 < \operatorname{Re} \nu < 2 \operatorname{Re} a - \frac{1}{2}$<br>$ \operatorname{Re} \beta  < \operatorname{Re} a - \frac{1}{2} \operatorname{Re} \nu - \frac{1}{4}$<br>$ \operatorname{Re} \gamma  < \operatorname{Re} a - \frac{1}{2} \operatorname{Re} \nu - \frac{1}{4}$ | $\frac{2^{\nu-4a+3} \Gamma(2a) \Gamma(\nu+1) y^{2a-\nu-3/2}}{\Gamma(a+\beta) \Gamma(a-\beta) \Gamma(a-\gamma) \Gamma(a+\gamma)}$<br>$\times K_{\beta+\gamma}(\frac{1}{2}y) K_{\beta-\gamma}(\frac{1}{2}y)$             |



## Miscellaneous functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                                                                                                                                                                            | $\int_0^\infty f(x) J_\nu(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                               |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (17) | $x^{\nu+\frac{1}{2}}$<br>$\times {}_4F_3(a, a+\frac{1}{2}, a+\mu, a-\mu;$<br>$\frac{1}{2}+a+\kappa, \frac{1}{2}+a-\kappa; -x^2)$<br>$-1 < \operatorname{Re} \nu < 2 \operatorname{Re} a - \frac{1}{2}$<br>$ \operatorname{Re} \mu  < \operatorname{Re} a - \frac{1}{2} \operatorname{Re} \nu - \frac{1}{4}$                                                                                       | $\frac{\Gamma(\frac{1}{2}+a+\kappa) \Gamma(\frac{1}{2}+a-\kappa) \Gamma(\nu+1)}{\Gamma(2a) \Gamma(a+\mu) \Gamma(a-\mu)}$ $\times 2^{\nu+1} y^{2a-\nu-5/2} W_{\kappa, \mu}(y) W_{-\kappa, \mu}(y)$                                                                                                                                                                              |
| (18) | $x^{2\rho-\frac{1}{2}}$<br>$\times {}_pF_p(a_1, \dots, a_p;$<br>$\beta_1, \dots, \beta_p; -\lambda x^2)$<br>$-1 - \operatorname{Re} \nu < 2 \operatorname{Re} \rho < \frac{1}{2} + 2 \operatorname{Re} a_r$<br>$\operatorname{Re} \lambda > 0, \quad r = 1, \dots, p$                                                                                                                             | $\frac{\Gamma(\beta_1) \dots \Gamma(\beta_p)}{\Gamma(a_1) \dots \Gamma(a_p)} 2^{2\rho} y^{-2\rho-\frac{1}{2}}$ $\times G_{p+1, p+2}^{\rho+1, 1} \left( \frac{y^2}{4\lambda} \middle  \begin{matrix} 1, \beta_1, \dots, \beta_p \\ h, a_1, \dots, a_p, k \end{matrix} \right)$ $h = \frac{1}{2} + \rho + \frac{1}{2} \nu, \quad k = \frac{1}{2} + \rho - \frac{1}{2} \nu$       |
| (19) | $x^{2\rho-\frac{1}{2}}$<br>$\times {}_{m+1}F_m(a_1, \dots, a_{m+1};$<br>$\beta_1, \dots, \beta_m; -\lambda^2 x^2)$<br>$\operatorname{Re}(2\rho+\nu) > -1, \quad \operatorname{Re} \lambda > 0$<br>$\operatorname{Re}(\rho - a_r) < \frac{1}{4}, \quad r = 1, \dots, m+1$                                                                                                                          | $\frac{2^{2\rho} \Gamma(\beta_1) \dots \Gamma(\beta_m)}{\Gamma(a_1) \dots \Gamma(a_{m+1})} y^{2\rho+\frac{1}{2}}$ $\times G_{m+1, m+3}^{m+2, 1} \left( \frac{y^2}{4\lambda^2} \middle  \begin{matrix} 1, \beta_1, \dots, \beta_m \\ h, a_1, \dots, a_{m+1}, k \end{matrix} \right)$ $h = \frac{1}{2} + \rho + \frac{1}{2} \nu, \quad k = \frac{1}{2} + \rho - \frac{1}{2} \nu$ |
| (20) | $x^{2\rho-\frac{1}{2}} G_{pq}^{mn} \left( \lambda x^2 \middle  \begin{matrix} a_1, \dots, a_p \\ \beta_1, \dots, \beta_q \end{matrix} \right)$<br>$p+q < 2(m+n)$<br>$ \arg \lambda  < (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi$<br>$\operatorname{Re}(\beta_j + \rho + \frac{1}{2}\nu) > -\frac{1}{2}$<br>$j = 1, \dots, m$<br>$\operatorname{Re}(a_j + \rho) < \frac{3}{4}, \quad j = 1, \dots, n$ | $\frac{2^{2\rho}}{y^{2\rho+\frac{1}{2}}}$ $\times G_{p+2, q}^{m, n+1} \left( \frac{4\lambda}{y^2} \middle  \begin{matrix} h, a_1, \dots, a_p, k \\ \beta_1, \dots, \beta_q \end{matrix} \right)$ $h = \frac{1}{2} - \rho - \frac{1}{2} \nu, \quad k = \frac{1}{2} - \rho + \frac{1}{2} \nu$                                                                                    |

## Miscellaneous functions (cont'd)

|      | $f(x)$                                                                                                                    | $\int_0^\infty f(x) J_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                              |
|------|---------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (21) | $x^{\nu+2n-1} [B(a+x, a-x)]^{-1}$<br>$-1 < \operatorname{Re} \nu < 2a - 2n - 7/2$                                         | 0<br>$\pi \leq y < \infty$                                                                                                                                                                                                   |
| (22) | $x^{\nu+\frac{1}{2}} \operatorname{Erfc}(ax)$<br>$ \arg a  < \frac{1}{4}\pi, \quad \operatorname{Re} \nu > -1$            | $a^{-\nu} \frac{\Gamma(\nu+3/2)}{\Gamma(\nu+2)} y^{-3/2} \exp\left(-\frac{y^2}{8a^2}\right)$<br>$\times M_{\frac{1}{2}\nu+\frac{1}{2}, \frac{1}{2}\nu+\frac{1}{2}}\left(\frac{y^2}{4a^2}\right)$                             |
| (23) | $x^{\nu-\frac{1}{2}} \operatorname{Erfc}(ax)$<br>$ \arg a  < \frac{1}{4}\pi, \quad \operatorname{Re} \nu > -\frac{1}{2}$  | $2^{-\frac{1}{2}} a^{\frac{1}{2}-\nu} \frac{\Gamma(\nu+1/2)}{\Gamma(\nu+3/2)} y^{-1} \exp\left(-\frac{y^2}{8a^2}\right)$<br>$\times M_{\frac{1}{2}\nu-\frac{1}{2}, \frac{1}{2}\nu+\frac{1}{2}}\left(\frac{y^2}{4a^2}\right)$ |
| (24) | $x^{-\mu-\frac{1}{2}} s_{\nu+\mu, -\nu+\mu+1}(x)$<br>$\operatorname{Re} \mu > -1, \quad -1 < \operatorname{Re} \nu < 3/2$ | $2^{\nu-1} \Gamma(\nu) y^{\frac{1}{2}-\nu} (1-y^2)^\mu$<br>0<br>$0 < y < 1$<br>$1 < y < \infty$                                                                                                                              |

## CHAPTER IX

### Y-TRANSFORMS

We call

$$\mathfrak{Y}_\nu\{f(x); y\} = \int_0^\infty f(x) Y_\nu(xy) (xy)^{1/2} dx$$

the *Y-transform of order  $\nu$*  of  $f(x)$  and regard  $y$  as a positive real variable. The inversion formula 9.1(1) has been given by Titchmarsh (1937, p. 215). The reciprocal transform is the **H**-transform (see chapter XI).

From the transform pairs given in this chapter further transform pairs may be derived by means of the methods indicated in the introduction to volume I, and also by the general formulas of sec. 9.1. Moreover, *Y*-transforms and Hankel transforms are connected by the relation

$$\mathfrak{Y}_\nu\{f(x); y\} = \cot(\nu\pi) \mathfrak{H}_\nu\{f(x); y\} - \csc(\nu\pi) \mathfrak{H}_{-\nu}\{f(x); y\}$$

which is an immediate consequence of the relation between Bessel functions of the first and second kind and may be used to evaluate *Y*-transforms by means of the table of Hankel transforms given in chapter VIII.

### REFERENCE

Titchmarsh, E.C., 1937: *Introduction to the theory of Fourier integrals*, Oxford.



## Y-TRANSFORMS

### 9.1. General formulas

|     |                                                                                                                       |                                                                                                                                |
|-----|-----------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------|
|     | $f(x)$                                                                                                                | $\int_0^\infty f(x) Y_\nu(xy) (xy)^{\frac{1}{2}} dx$<br>$= g(y; \nu)$ <span style="float: right;"><math>y &gt; 0</math></span> |
| (1) | $\int_0^\infty g(y) \mathbf{H}_\nu(xy) (xy)^{\frac{1}{2}} dy$<br>$-\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$ | $g(y)$                                                                                                                         |
| (2) | $f(ax)$ <span style="float: right;"><math>a &gt; 0</math></span>                                                      | $a^{-1} g(a^{-1} y; \nu)$                                                                                                      |
| (3) | $x^m f(x)$ <span style="float: right;"><math>m = 0, 1, 2, \dots</math></span>                                         | $y^{\frac{1}{2}-\nu} \left( \frac{d}{y dy} \right)^m [y^{\nu-\frac{1}{2}+m} g(y; \nu + m)]$                                    |
| (4) | $x^m f(x)$ <span style="float: right;"><math>m = 0, 1, 2, \dots</math></span>                                         | $(-1)^m y^{\frac{1}{2}+\nu} \left( \frac{d}{y dy} \right)^m$<br>$\times [y^{-\frac{1}{2}+m-\nu} g(y; \nu - m)]$                |
| (5) | $2\nu x^{-1} f(x)$                                                                                                    | $y g(y; \nu - 1) + y g(y; \nu + 1)$                                                                                            |
| (6) | $2\nu f'(x)$                                                                                                          | $(\nu - \frac{1}{2}) y g(y; \nu + 1)$<br>$- (\nu + \frac{1}{2}) y g(y; \nu - 1)$                                               |

**General formulas (cont'd)**

|      |                                                                                                                                                       |                                                                                                                                                                     |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------|
|      | $f(x)$                                                                                                                                                | $\int_0^\infty f(x) Y_\nu(xy) (xy)^{\frac{1}{2}} dx$<br>$= g(y; \nu)$<br>$y > 0$                                                                                    |
| (7)  | $x^{\frac{1}{2}-\nu} \left( \frac{d}{x dx} \right)^m$<br>$\times [x^{\nu+m-\frac{1}{2}} f(x)]$<br>$m = 0, 1, 2, \dots$                                | $y^m g(y; \nu + m)$                                                                                                                                                 |
| (8)  | $x^{\frac{1}{2}+\nu} \left( \frac{d}{x dx} \right)^m$<br>$\times [x^{m-\nu-\frac{1}{2}} f(x)]$<br>$m = 0, 1, 2, \dots$                                | $(-y)^m g(y; \nu - m)$                                                                                                                                              |
| (9)  | $x^{\frac{1}{2}-\nu} \int_0^x \xi^{\nu-\mu+\frac{1}{2}} (x^2 - \xi^2)^{\mu-1}$<br>$\times f(\xi) d\xi$<br>$\text{Re } \nu + 3/2 > \text{Re } \mu > 0$ | $2^{\mu-1} \Gamma(\mu) y^{-\mu} g(y; \nu - \mu)$                                                                                                                    |
| (10) | $x^{-\mu} f(x)$<br>$\text{Re } \mu > 0, \quad \text{Re } \nu > -3/2$                                                                                  | $2^{1-\mu} [\Gamma(\mu)]^{-1} y^{\nu+\frac{1}{2}}$<br>$\times \int_y^\infty \eta^{\frac{1}{2}-\mu-\nu} (\eta^2 - y^2)^{\mu-1}$<br>$\times g(\eta; \nu + \mu) d\eta$ |

**9.2. Algebraic functions and powers with an arbitrary index**

|     |                                                 |                                                                 |
|-----|-------------------------------------------------|-----------------------------------------------------------------|
|     | $f(x)$                                          | $\int_0^\infty f(x) Y_\nu(xy) (xy)^{\frac{1}{2}} dx$<br>$y > 0$ |
| (1) | $x^{-\frac{1}{2}}$<br>$-1 < \text{Re } \nu < 1$ | $-\tan(\frac{1}{2}\nu\pi) y^{-\frac{1}{2}}$                     |

## Algebraic functions (cont'd)

|     | $f(x)$                                                                                                                   | $\int_0^\infty f(x) Y_\nu(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                |
|-----|--------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (2) | $0 \quad 0 < x < a$<br>$x^{\frac{1}{2}+\nu} \quad a < x < \infty$<br>$\operatorname{Re} \nu < -\frac{1}{2}$              | $-a^{\nu+1} y^{-\frac{1}{2}} Y_{\nu+1}(ay)$                                                                                                                                                                     |
| (3) | $x^\mu \quad  \operatorname{Re} \nu  - 3/2 < \mu < 0$                                                                    | $2^{\mu+\frac{1}{2}} \cotn[\frac{1}{2}(\nu + \frac{1}{2} - \mu)\pi] y^{-\mu-1}$<br>$\times \frac{\Gamma(\frac{3}{4} + \frac{1}{2}\nu + \frac{1}{2}\mu)}{\Gamma(\frac{1}{4} + \frac{1}{2}\nu - \frac{1}{2}\mu)}$ |
| (4) | $0 \quad 0 < x < a$<br>$x^\mu \quad a < x < \infty$<br>$\operatorname{Re} \mu < 0$                                       | $ay^{-\mu} [Y_{\nu-1}(ay) S_{\mu+\frac{1}{2},\nu}(ay)$<br>$- (\mu + \nu - \frac{1}{2}) Y_\nu(ay)$<br>$\times S_{\mu-\frac{1}{2},\nu-1}(ay)]$                                                                    |
| (5) | $x^{-\frac{1}{2}} (x+a)^{-1}$<br>$ \arg a  < \pi, \quad -1 < \operatorname{Re} \nu < 1$<br>$\nu \neq 0, \pm \frac{1}{2}$ | $\pi y^{\frac{1}{2}} \csc(\nu\pi) [\mathbf{E}_\nu(ay) + Y_\nu(ay)]$<br>$+ 2 \cotn(\nu\pi) [\mathbf{J}_\nu(ay) - J_\nu(ay)]$                                                                                     |
| (6) | $x^{\nu-\frac{1}{2}} (x+a)^{-1} \quad  \arg a  < \pi$<br>$-1/2 < \operatorname{Re} \nu < 3/2$                            | $-2^{\nu+1} \pi^{-1} a^\nu y^{\frac{1}{2}} \Gamma(\nu+1)$<br>$\times S_{-\nu-1,\nu}(ay)$                                                                                                                        |
| (7) | $x^{-\nu-\frac{1}{2}} (x+a)^{-1} \quad  \arg a  < \pi$<br>$-3/2 < \operatorname{Re} \nu < 1/2$                           | $a^{-\nu} y^{\frac{1}{2}} \{\frac{1}{2} \pi \tan(\nu\pi) [Y_\nu(ay)$<br>$- \mathbf{H}_\nu(ay)] - 2^{1-\nu} \pi^{-1} \cos(\nu\pi)$<br>$\times \Gamma(1-\nu) S_{\nu-1,\nu}(ay)\}$                                 |

**Algebraic functions (cont'd)**

|      | $f(x)$                                                                                                                            | $\int_0^\infty f(x) Y_\nu(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                                             |
|------|-----------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (8)  | $x^{\mu-\frac{1}{2}} (x+a)^{-1} \quad  \arg a  < \pi$<br>$\operatorname{Re}(\mu \pm \nu) > -1, \quad \operatorname{Re} \mu < 3/2$ | $(2a)^\mu \pi^{-1} y^{\frac{1}{2}}$<br>$\times \{ \sin[\frac{1}{2}\pi(\mu-\nu)] \Gamma(\frac{1}{2} + \frac{1}{2}\mu + \frac{1}{2}\nu)$<br>$\times \Gamma(\frac{1}{2} + \frac{1}{2}\mu - \frac{1}{2}\nu) S_{-\mu, \nu}(ay)$<br>$- 2 \cos[\frac{1}{2}\pi(\mu-\nu)] \Gamma(1 + \frac{1}{2}\mu + \frac{1}{2}\nu)$<br>$\times \Gamma(1 + \frac{1}{2}\mu - \frac{1}{2}\nu) S_{-\mu-1, \nu}(ay) \}$ |
| (9)  | $x^{-\frac{1}{2}} (x-a)^{-1}$<br>$a > 0, \quad -1 < \operatorname{Re} \nu < 1$                                                    | $\pi y^{\frac{1}{2}} \{ \operatorname{ctn}(\nu\pi) [Y_\nu(ay) + \mathbf{E}_\nu(ay)]$<br>$+ \mathbf{J}_\nu(ay) + 2[\operatorname{ctn}(\nu\pi)]^2$<br>$\times [\mathbf{J}_\nu(ay) - J_\nu(ay)] \}$<br>The integral is a Cauchy<br>Principal Value.                                                                                                                                             |
| (10) | $x^{\nu-\frac{1}{2}} (x-a)^{-1}$<br>$a > 0, \quad -1/2 < \operatorname{Re} \nu < 3/2$                                             | $a^\nu y^{\frac{1}{2}} [\pi J_\nu(ay) - 2^{\nu+1} \pi^{-1} \Gamma(\nu+1)$<br>$\times S_{-\nu-1, \nu}(ay)]$<br>The integral is a Cauchy<br>Principal Value.                                                                                                                                                                                                                                   |
| (11) | $x^{-\nu-\frac{1}{2}} (x-a)^{-1}$<br>$a > 0, \quad -3/2 < \operatorname{Re} \nu < 1/2$                                            | $a^{-\nu} y^{\frac{1}{2}} \{ \frac{1}{2}\pi \tan(\nu\pi) [\mathbf{H}_\nu(ay)$<br>$- Y_\nu(ay)] + \pi J_\nu(ay)$<br>$- 2^{1-\nu} \pi^{-1} \cos(\nu\pi) \Gamma(1-\nu)$<br>$\times S_{\nu-1, \nu}(ay) \}$<br>The integral is a Cauchy<br>Principal Value.                                                                                                                                       |



## Algebraic functions (cont'd)

|      | $f(x)$                                                                                                                   | $\int_0^\infty f(x) Y_\nu(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                                                                                                                                   |
|------|--------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (12) | $x^{\mu-\frac{1}{2}} (x-a)^{-1} \quad a > 0$<br>$\operatorname{Re}(\mu \pm \nu) > -1, \quad \operatorname{Re} \mu > 3/2$ | $\pi a^\mu y^{\frac{1}{2}} J_\nu(ay) - (2a)^\mu \pi^{-1} y^{\frac{1}{2}}$<br>$\times \{ \sin[\frac{1}{2}\pi(\mu-\nu)] \Gamma(\frac{1}{2} + \frac{1}{2}\mu + \frac{1}{2}\nu)$<br>$\times \Gamma(\frac{1}{2} + \frac{1}{2}\mu - \frac{1}{2}\nu) S_{-\mu, \nu}(ay)$<br>$+ 2 \cos[\frac{1}{2}\pi(\mu-\nu)] \Gamma(1 + \frac{1}{2}\mu + \frac{1}{2}\nu)$<br>$\times \Gamma(1 + \frac{1}{2}\mu - \frac{1}{2}\nu) S_{-\mu-1, \nu}(ay) \}$<br>The integral is a Cauchy<br>Principal Value. |
| (13) | $x^{-\frac{1}{2}} (x^2 + a^2)^{-1}$<br>$\operatorname{Re} a > 0, \quad -1 < \operatorname{Re} \nu < 1$                   | $\frac{y^{\frac{1}{2}}}{\cos \frac{1}{2}\nu\pi} \left[ -\frac{\pi}{2a} \tan\left(\frac{\nu\pi}{2}\right) I_\nu(ay) \right.$<br>$\left. - \frac{1}{a} K_\nu(ay) + \frac{y \sin(\frac{1}{2}\nu\pi)}{1-\nu^2} \right.$<br>$\left. \times {}_1F_2\left(1; \frac{3-\nu}{2}, \frac{3+\nu}{2}; \frac{a^2 y^2}{4}\right) \right]$                                                                                                                                                          |
| (14) | $x^{\nu-\frac{1}{2}} (x^2 + a^2)^{-1}$<br>$\operatorname{Re} a > 0, \quad -1/2 < \operatorname{Re} \nu < 5/2$            | $-a^{\nu-1} y^{\frac{1}{2}} K_\nu(ay)$                                                                                                                                                                                                                                                                                                                                                                                                                                             |
| (15) | $x^{\nu+3/2} (x^2 + a^2)^{-1}$<br>$\operatorname{Re} a > 0, \quad -3/2 < \operatorname{Re} \nu < 1/2$                    | $a^{\nu+1} y^{\frac{1}{2}} K_\nu(ay)$                                                                                                                                                                                                                                                                                                                                                                                                                                              |
| (16) | $x^{-\nu-\frac{1}{2}} (x^2 + a^2)^{-1} \quad \operatorname{Re} a > 0$<br>$-5/2 < \operatorname{Re} \nu < 1/2$            | $a^{-\nu-1} y^{\frac{1}{2}} \{ \frac{1}{2}\pi \tan(\nu\pi) [L_\nu(ay)$<br>$- I_\nu(ay)] - \sec(\nu\pi) K_\nu(ay) \}$                                                                                                                                                                                                                                                                                                                                                               |

## Algebraic functions (cont'd)

|      | $f(x)$                                                                                                                | $\int_0^\infty f(x) Y_\nu(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                                                                                                                   |
|------|-----------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (17) | $x^{\mu-3/2} (x^2 + a^2)^{-1} \quad \text{Re } a > 0$<br>$ \text{Re } \nu  < \text{Re } \mu < 7/2$                    | $2^{\mu-3} \pi^{-1} y^{5/2-\mu} \cos [\frac{1}{2}\pi(\mu-\nu)]$<br>$\times \Gamma(\frac{1}{2}\mu + \frac{1}{2}\nu - 1) \Gamma(\frac{1}{2}\mu - \frac{1}{2}\nu - 1)$<br>$\times {}_1F_2\left(1; 2 - \frac{\mu+\nu}{2}, 2 - \frac{\mu-\nu}{2}; \frac{a^2 y^2}{4}\right)$<br>$- \frac{1}{2}\pi a^{\mu-2} y^{1/2} \csc[\frac{1}{2}\pi(\mu+\nu)]$<br>$\times \cotn[\frac{1}{2}\pi(\mu-\nu)] I_\nu(ay)$<br>$- a^{\mu-2} y^{1/2} \csc[\frac{1}{2}\pi(\mu-\nu)] K_\nu(ay)$ |
| (18) | $x^{-1/2} (x^2 + a^2)^{-1/2}$<br>$\text{Re } a > 0, \quad -1 < \text{Re } \nu < 1$                                    | $-\pi^{-1} y^{1/2} \sec(\frac{1}{2}\nu\pi) K_{\frac{1}{2}\nu}(\frac{1}{2}ay)$<br>$\times [K_{\frac{1}{2}\nu}(\frac{1}{2}ay) + \pi \sin(\frac{1}{2}\nu\pi)$<br>$\times I_{\frac{1}{2}\nu}(\frac{1}{2}ay)]$                                                                                                                                                                                                                                                          |
| (19) | $x^{\frac{1}{2}+\nu} (x^2 + a^2)^\mu \quad \text{Re } a > 0$<br>$-1 < \text{Re } \nu < -2 \text{Re } \mu$             | $2^{\nu-1} \pi^{-1} a^{2\mu+2} (1+\mu)^{-1} \Gamma(\nu) y^{\frac{1}{2}-\nu}$<br>$\times {}_1F_2(1; 1-\nu, 2+\mu; \frac{1}{4}a^2 y^2)$<br>$- 2^\mu a^{\mu+\nu+1} (\sin \nu\pi)^{-1} \Gamma(\mu+1)$<br>$\times y^{-\frac{1}{2}-\mu} [I_{\mu+\nu+1}(ay)$<br>$- 2 \cos(\mu\pi) K_{\mu+\nu+1}(ay)]$                                                                                                                                                                     |
| (20) | $x^{\frac{1}{2}-\nu} (x^2 + a^2)^\mu \quad \text{Re } a > 0$<br>$\frac{1}{2} + 2 \text{Re } \mu < \text{Re } \nu < 1$ | $2^\mu a^{\mu-\nu+1} y^{-\frac{1}{2}-\mu} \{\pi^{-1} \cos(\nu\pi)$<br>$\times \Gamma(\mu+1) \Gamma(\nu) I_{\nu-\mu-1}(ay)$<br>$- 2 \csc(\nu\pi) [\Gamma(-\mu)]^{-1} K_{\nu-\mu-1}(ay)\}$<br>$- \frac{a^{2\mu+2} \cotn(\nu\pi) y^{\frac{1}{2}+\nu}}{2^{\nu+1} (\mu+1) \Gamma(\nu+1)}$<br>$\times {}_1F_2(1; \nu+1, \mu+2; \frac{1}{4}a^2 y^2)$                                                                                                                      |

## Algebraic functions (cont'd)

|      | $f(x)$                                                                                                 | $\int_0^\infty f(x) Y_\nu(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                 |
|------|--------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (21) | $x^{-\frac{1}{2}} (x^2 - a^2)^{-1}$<br>$a > 0, \quad -1 < \operatorname{Re} \nu < 1$                   | $\frac{1}{2} \pi a^{-1} y^{\frac{1}{2}} [J_\nu(ay)$<br>$+ \tan(\frac{1}{2} \nu \pi) \{ \tan(\frac{1}{2} \nu \pi) [J_\nu(ay)$<br>$- J_\nu(ay)] - E_\nu(ay) - Y_\nu(ay) \}]$<br>The integral is a Cauchy<br>Principal Value.                                                                       |
| (22) | $x^{\nu-\frac{1}{2}} (x^2 - a^2)^{-1}$<br>$a > 0, \quad -1/2 < \operatorname{Re} \nu < 5/2$            | $\frac{1}{2} \pi a^{\nu-1} y^{\frac{1}{2}} J_\nu(ay)$<br>The integral is a Cauchy<br>Principal Value.                                                                                                                                                                                            |
| (23) | $x^{\nu+\frac{1}{2}} (x^2 - a^2)^{-1}$<br>$a > 0, \quad -1 < \operatorname{Re} \nu < 3/2$              | $\frac{1}{2} \pi a^\nu y^{\frac{1}{2}} J_\nu(ay)$<br>$- 2^{\nu+1} \pi^{-1} \Gamma(\nu+1) a^\nu y^{\frac{1}{2}}$<br>$\times S_{-\nu-1, \nu}(ay)$<br>The integral is a Cauchy<br>Principal Value.                                                                                                  |
| (24) | $x^{-\nu-\frac{1}{2}} (x^2 - a^2)^{-1}$<br>$a > 0, \quad -5/2 < \operatorname{Re} \nu < 1/2$           | $\frac{1}{2} \pi a^{-\nu-1} y^{\frac{1}{2}} \sec(\nu \pi) [J_{-\nu}(ay)$<br>$+ \sin(\nu \pi) H_\nu(ay)]$<br>The integral is a Cauchy<br>Principal Value                                                                                                                                          |
| (25) | $x^{\mu-3/2} (x^2 - a^2)^{-1}$<br>$a > 0, \quad  \operatorname{Re} \nu  < \operatorname{Re} \mu < 7/2$ | $\frac{1}{2} \pi a^{\mu-2} y^{\frac{1}{2}} J_\nu(ay)$<br>$+ 2^{\mu-1} \pi^{-1} a^{\mu-2} y^{\frac{1}{2}} \cos[\frac{1}{2} \pi(\mu-\nu)]$<br>$\times \Gamma\left(\frac{\mu-\nu}{2}\right) \Gamma\left(\frac{\mu+\nu}{2}\right) S_{1-\mu, \nu}(ay)$<br>The integral is a Cauchy<br>Principal Value |

**Algebraic functions (cont'd)**

|      | $f(x)$                                                                                                                             | $\int_0^\infty f(x) Y_\nu(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                         |
|------|------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (26) | $x^{-\frac{1}{2}} (a^2 - x^2)^{-\frac{1}{2}} \quad 0 < x < a$<br>$0 \quad a < x < \infty$<br>$\nu = 0$                             | $\frac{1}{2} \pi y^{-\frac{1}{2}} J_0(\frac{1}{2} ay) Y_0(\frac{1}{2} ay)$                                                                                               |
| (27) | $0 \quad 0 < x < a$<br>$x^{-\frac{1}{2}} (x^2 - a^2)^{-\frac{1}{2}} \quad a < x < \infty$                                          | $\frac{1}{4} \pi y^{\frac{1}{2}} \{ [J_{\frac{1}{2}\nu}(\frac{1}{2} ay)]^2$<br>$- [Y_{\frac{1}{2}\nu}(\frac{1}{2} ay)]^2 \}$                                             |
| (28) | $x^{\nu+\frac{1}{2}} (a^2 - x^2)^{-\frac{1}{2}} \quad 0 < x < a$<br>$0 \quad a < x < \infty$<br>$\text{Re } \nu > -1$              | $(\frac{1}{2}\pi)^{\frac{1}{2}} a^{\nu+\frac{1}{2}} \csc(\nu\pi)$<br>$\times [\cos(\nu\pi) J_{\nu+\frac{1}{2}}(ay)$<br>$- \mathbf{H}_{-\nu-\frac{1}{2}}(y)]$             |
| (29) | $0 \quad 0 < x < a$<br>$x^{\nu+\frac{1}{2}} (x^2 - a^2)^{-\frac{1}{2}} \quad a < x < \infty$<br>$\text{Re } \nu < \frac{1}{2}$     | $(\frac{1}{2}\pi)^{\frac{1}{2}} a^{\nu+\frac{1}{2}} J_{\nu+\frac{1}{2}}(ay)$                                                                                             |
| (30) | $x^{\frac{1}{2}-\nu} (a^2 - x^2)^{-\frac{1}{2}} \quad 0 < x < a$<br>$0 \quad a < x < \infty$<br>$\text{Re } \nu < 1$               | $(\frac{1}{2}\pi)^{\frac{1}{2}} a^{\frac{1}{2}-\nu} \{ \text{ctn}(\nu\pi) [\mathbf{H}_{\nu-\frac{1}{2}}(ay)$<br>$- Y_{\nu-\frac{1}{2}}(ay)] - J_{\nu-\frac{1}{2}}(y) \}$ |
| (31) | $x^{\nu-\frac{1}{2}} (a^2 - x^2)^{\nu-\frac{1}{2}} \quad 0 < x < a$<br>$0 \quad a < x < \infty$<br>$\text{Re } \nu > -\frac{1}{2}$ | $2^{\nu-1} \pi^{\frac{1}{2}} a^{2\nu} y^{\frac{1}{2}-\nu} \Gamma(\nu + \frac{1}{2})$<br>$\times J_\nu(\frac{1}{2} ay) Y_\nu(\frac{1}{2} ay)$                             |

## Algebraic functions (cont'd)

|      | $f(x)$                                                                                                                                             | $\int_0^\infty f(x) Y_\nu(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                    |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (32) | $0 \quad 0 < x < a$<br>$x^{\nu-\frac{1}{2}} (x^2 - a^2)^{\nu-\frac{1}{2}} \quad a < x < \infty$<br>$-\frac{1}{2} < \text{Re } \nu < \frac{1}{2}$   | $2^{\nu-2} \pi^{\frac{1}{2}} a^{2\nu} y^{\frac{1}{2}-\nu} \Gamma(\nu + \frac{1}{2})$<br>$\times [J_\nu(\frac{1}{2}ay) J_{-\nu}(\frac{1}{2}ay)$<br>$- Y_\nu(\frac{1}{2}ay) Y_{-\nu}(\frac{1}{2}ay)]$ |
| (33) | $0 \quad 0 < x < a$<br>$x^{\frac{1}{2}-\nu} (x^2 - a^2)^{\nu-\frac{1}{2}} \quad a < x < \infty$<br>$-\frac{1}{2} < \text{Re } \nu < \frac{1}{2}$   | $2^\nu \pi^{-\frac{1}{2}} y^{-\frac{1}{2}-\nu} \Gamma(\nu + \frac{1}{2}) \sin(ay)$                                                                                                                  |
| (34) | $0 \quad 0 < x < a$<br>$x^{-\nu-\frac{1}{2}} (x^2 - a^2)^{-\nu-\frac{1}{2}} \quad a < x < \infty$<br>$-\frac{1}{2} < \text{Re } \nu < \frac{1}{2}$ | $2^{-\nu-2} \pi^{\frac{1}{2}} a^{-2\nu} y^{\nu+\frac{1}{2}} \Gamma(\frac{1}{2} - \nu)$<br>$\times \{ [J_\nu(\frac{1}{2}ay)]^2 - [Y_\nu(\frac{1}{2}ay)]^2 \}$                                        |
| (35) | $x^{\nu+\frac{1}{2}} (a^2 - x^2)^\mu \quad 0 < x < a$<br>$0 \quad a < x < \infty$<br>$\text{Re } \mu > -1, \quad \text{Re } \nu > -1$              | $a^{\mu+\nu+1} y^{-\mu-\frac{1}{2}} [2^\mu \Gamma(\mu+1)$<br>$\times Y_{\mu+\nu+1}(ay) + 2^{\nu+1} \pi^{-1} \Gamma(\nu+1)$<br>$\times S_{\mu-\nu, \mu+\nu+1}(ay)]$                                  |
| (36) | $0 \quad 0 < x < a$<br>$x^{\nu+\frac{1}{2}} (x^2 - a^2)^\mu \quad a < x < \infty$<br>$-2 < 2 \text{Re } \mu < -\frac{1}{2} - \text{Re } \nu$       | $-2^\mu a^{\mu+\nu+1} y^{-\mu-\frac{1}{2}} \Gamma(\mu+1)$<br>$\times [\sin(\mu\pi) J_{\mu+\nu+1}(ay)$<br>$+ \cos(\mu\pi) Y_{\mu+\nu+1}(ay)]$                                                        |

## Algebraic functions (cont'd)

|      | $f(x)$                                                                                                                                                                                             | $\int_0^\infty f(x) Y_\nu(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                     |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (37) | $x^{\frac{1}{2}-\nu} (a^2 - x^2)^\mu \quad 0 < x < a$<br>$0 \quad a < x < \infty$<br>$\operatorname{Re} \mu > -1, \quad \operatorname{Re} \nu < 1$                                                 | $a^{\mu-\nu+1} y^{-\mu-\frac{1}{2}} [2^{1-\nu} \pi^{-1} \cos(\nu\pi)$<br>$\times \Gamma(1-\nu) s_{\mu+\nu, \mu-\nu+1}(ay)$<br>$- 2^\mu \csc(\nu\pi) \Gamma(\mu+1)$<br>$\times J_{\mu-\nu+1}(ay)]$                                                                                                    |
| (38) | $0 \quad 0 < x < a$<br>$x^{\frac{1}{2}-\nu} (x^2 - a^2)^\mu \quad a < x < \infty$<br>$-1 < \operatorname{Re} \mu < \frac{1}{2} \operatorname{Re} \nu - \frac{1}{4}$                                | $2^\mu a^{\mu-\nu+1} y^{-\mu-\frac{1}{2}} \Gamma(\mu+1)$<br>$\times Y_{\nu-\mu-1}(ay)$                                                                                                                                                                                                               |
| (39) | $x^{2n+\nu+4\mu-\frac{1}{2}} (x^4 + a^4)^{-\mu-1}$                                                                                                                                                 | see Watson, G.N., 1922:<br><i>Bessel Functions</i> , p. 432,<br>Cambridge.                                                                                                                                                                                                                           |
| (40) | $x^{-\frac{1}{2}} (x^2 + a^2)^{-\frac{1}{2}}$<br>$\times [(x^2 + a^2)^{\frac{1}{2}} - x]^\mu$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \mu > -3/2$<br>$-1 < \operatorname{Re} \nu < 1$ | $a^\mu y^{\frac{1}{2}} [\operatorname{ctn}(\nu\pi) I_{\frac{1}{2}\mu+\frac{1}{2}\nu}(\frac{1}{2}ay)$<br>$\times K_{\frac{1}{2}\mu-\frac{1}{2}\nu}(\frac{1}{2}ay)$<br>$- \csc(\nu\pi) I_{\frac{1}{2}\mu-\frac{1}{2}\nu}(\frac{1}{2}ay)$<br>$\times K_{\frac{1}{2}\mu+\frac{1}{2}\nu}(\frac{1}{2}ay)]$ |
| (41) | $x^{-\frac{1}{2}} (x^2 + a^2)^{-\frac{1}{2}}$<br>$\times \{[(x^2 + a^2)^{\frac{1}{2}} + x]^\mu$<br>$+ [(x^2 + a^2)^{\frac{1}{2}} - x]^\mu\} \quad \nu = 0$<br>$-3/2 < \operatorname{Re} \mu < 3/2$ | $-2\pi^{-1} a^\mu y^{\frac{1}{2}} \cos(\frac{1}{2}\mu\pi)$<br>$\times [K_{\frac{1}{2}\mu}(\frac{1}{2}ay)]^2$                                                                                                                                                                                         |

## Algebraic functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                                    | $\int_0^\infty f(x) Y_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                              |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (42) | $x^{-\frac{1}{2}}(x^2 + a^2)^{-\frac{1}{2}}$<br>$\times [(x^2 + a^2)^{\frac{1}{2}} - a]^{2k}$<br>$\text{Re } a > 0, \quad  \text{Re } \nu  < \frac{1}{2} + \text{Re } k$                                                  | $-a^{-1} y^{-\frac{1}{2}} W_{-k, \frac{1}{2}\nu}(ay)$<br>$\times \left\{ \frac{\Gamma(\frac{1}{2} + \frac{1}{2}\nu + k)}{\Gamma(\nu + 1)} \tan[(\frac{1}{2}\nu - k)\pi] \right.$<br>$\times M_{k, \frac{1}{2}\nu}(ay) + \sec[(\frac{1}{2}\nu - k)\pi]$<br>$\left. \times W_{k, \frac{1}{2}\nu}(ay) \right\}$ |
| (43) | $0 \quad 0 < x < a$<br>$x^{-\frac{1}{2}}(x^2 - a^2)^{-\frac{1}{2}}$<br>$\times \{ [x + (x^2 - a^2)^{\frac{1}{2}}]^\mu$<br>$+ [x - (x^2 - a^2)^{\frac{1}{2}}]^\mu \}$<br>$a < x < \infty$<br>$-3/2 < \text{Re } \mu < 3/2$ | $\frac{1}{2}\pi a^\mu y^{\frac{1}{2}} [J_{\frac{1}{2}\nu + \frac{1}{2}\mu}(\frac{1}{2}ay)$<br>$\times J_{\frac{1}{2}\nu - \frac{1}{2}\mu}(\frac{1}{2}ay) - Y_{\frac{1}{2}\nu + \frac{1}{2}\mu}(\frac{1}{2}ay)$<br>$\times Y_{\frac{1}{2}\nu - \frac{1}{2}\mu}(\frac{1}{2}ay)]$                               |

## 9.3. Other elementary functions

|     |                                                                                 |                                                                                                                                                                                          |
|-----|---------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $x^{-\frac{1}{2}} e^{-ax}$<br>$\text{Re } a > 0, \quad -1 < \text{Re } \nu < 1$ | $y^{\frac{1}{2}}(y^2 + a^2)^{-\frac{1}{2}} \csc(\nu\pi)$<br>$\times \{ y^\nu [(y^2 + a^2)^{\frac{1}{2}} + a]^{-\nu} \cos(\nu\pi)$<br>$- y^{-\nu} [(y^2 + a^2)^{\frac{1}{2}} + a]^\nu \}$ |
| (2) | $x^{\mu-3/2} e^{-ax}$<br>$a > 0, \quad \text{Re } \mu >  \text{Re } \nu $       | $-2\pi^{-1} \Gamma(\mu + \nu) y^{\frac{1}{2}}(y^2 + a^2)^{-\frac{1}{2}\mu}$<br>$\times Q_{\mu-1}^{-\nu} [a(y^2 + a^2)^{-\frac{1}{2}}]$                                                   |

## Elementary functions (cont'd)

|     | $f(x)$                                                                                                                                                                                                    | $\int_0^\infty f(x) Y_\nu(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                                                                                   |
|-----|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (3) | $x^{-\frac{1}{2}} e^{-ax^2}$<br>$\operatorname{Re} a > 0, \quad -1 < \operatorname{Re} \nu < 1$                                                                                                           | $-\frac{1}{2} \left( \frac{\pi y}{a} \right)^{\frac{1}{2}} \exp \left( -\frac{y^2}{8a} \right) \\ \times \left[ \tan \frac{\nu\pi}{2} I_{\frac{1}{2}\nu} \left( \frac{y^2}{8a} \right) \right. \\ \left. + \pi^{-1} \sec \frac{\nu\pi}{2} K_{\frac{1}{2}\nu} \left( \frac{y^2}{8a} \right) \right]$                                                                                                                                |
| (4) | $x^{\mu-\frac{1}{2}} e^{-ax^2}$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \mu >  \operatorname{Re} \nu  - 1$                                                                                   | $-a^{-\frac{1}{2}\mu} y^{-\frac{1}{2}} \sec [\tfrac{1}{2}(\nu - \mu)\pi] \\ \times \exp \left( -\frac{y^2}{8a} \right) \\ \times \left\{ \frac{\Gamma(\frac{1}{2} + \frac{1}{2}\mu + \frac{1}{2}\nu)}{\Gamma(1 + \nu)} \sin [\tfrac{1}{2}(\nu - \mu)\pi] \right. \\ \times M_{\frac{1}{2}\mu, \frac{1}{2}\nu} \left( \frac{y^2}{4a} \right) \\ \left. + W_{\frac{1}{2}\mu, \frac{1}{2}\nu} \left( \frac{y^2}{4a} \right) \right\}$ |
| (5) | $x^{-3/2} e^{-a/x} \quad \operatorname{Re} a > 0$                                                                                                                                                         | $2y^{\frac{1}{2}} Y_\nu [(2ay)^{\frac{1}{2}}] K_\nu [(2ay)^{\frac{1}{2}}]$                                                                                                                                                                                                                                                                                                                                                         |
| (6) | $x^{-\frac{1}{2}} (x^2 + \beta^2)^{-\frac{1}{2}}$<br>$\times \exp [-a(x^2 + \beta^2)^{-\frac{1}{2}}]$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \beta > 0$<br>$-1 < \operatorname{Re} \nu < 1$ | $-y^{\frac{1}{2}} \sec (\tfrac{1}{2}\nu\pi) \\ \times K_{\frac{1}{2}\nu} \{ \tfrac{1}{2}\beta [(y^2 + a^2)^{\frac{1}{2}} + a] \} \\ \times (\pi^{-1} K_{\frac{1}{2}\nu} \{ \tfrac{1}{2}\beta [(y^2 + a^2)^{\frac{1}{2}} - a] \} \\ + \sin (\tfrac{1}{2}\nu\pi) \\ \times I_{\frac{1}{2}\nu} \{ \tfrac{1}{2}\beta [(y^2 + a^2)^{\frac{1}{2}} - a] \})$                                                                              |



## Elementary functions (cont'd)

|     | $f(x)$                                                                                                                                                                                                                                                                                     | $\int_0^\infty f(x) Y_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                                       |
|-----|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (7) | $x^{-\frac{1}{2}} \sin ax^2$<br>$a > 0, \quad -3 < \operatorname{Re} \nu < 3$                                                                                                                                                                                                              | $-\frac{1}{4} \left( \frac{\pi y}{a} \right)^{\frac{1}{2}} \sec \left( \frac{\nu \pi}{2} \right)$ $\times \left[ \cos \left( \frac{y^2}{8a} - \frac{3\nu + 1}{4} \pi \right) \right.$ $\times J_{\frac{1}{2}\nu} \left( \frac{y^2}{8a} \right) - \sin \left( \frac{y^2}{8a} + \frac{\nu - 1}{4} \pi \right)$ $\times Y_{\frac{1}{2}\nu} \left( \frac{y^2}{8a} \right) \left. \right]$ |
| (8) | $x^{-\frac{1}{2}} \cos ax^2$<br>$a > 0, \quad -1 < \operatorname{Re} \nu < 1$                                                                                                                                                                                                              | $\frac{1}{4} \left( \frac{\pi y}{a} \right)^{\frac{1}{2}} \sec \left( \frac{\nu \pi}{2} \right)$ $\times \left[ \sin \left( \frac{y^2}{8a} - \frac{3\nu + 1}{4} \pi \right) \right.$ $\times J_{\frac{1}{2}\nu} \left( \frac{y^2}{8a} \right) + \cos \left( \frac{y^2}{8a} + \frac{\nu - 1}{4} \pi \right)$ $\times Y_{\frac{1}{2}\nu} \left( \frac{y^2}{8a} \right) \left. \right]$  |
| (9) | $x^{-\frac{1}{2}} (a^2 - x^2)^{-\frac{1}{2}}$<br>$\times \sin [b(a^2 - x^2)^{\frac{1}{2}}]$<br>$0 < x < a$<br><br>$-x^{-\frac{1}{2}} (x^2 - a^2)^{-\frac{1}{2}}$<br>$\times \exp [-b^2(x^2 - a^2)^{\frac{1}{2}}]$<br>$a < x < \infty$<br><br>$b > 0, \quad -1 < \operatorname{Re} \nu < 1$ | $\frac{1}{4} \pi y^{\frac{1}{2}} Y_{\frac{1}{2}\nu}^2 \left\{ \frac{1}{2} a [(y^2 + b^2)^{\frac{1}{2}} + b] \right\}$ $- \frac{1}{4} \pi y^{\frac{1}{2}} J_{\frac{1}{2}\nu}^2 \left\{ \frac{1}{2} a [(y^2 + b^2)^{\frac{1}{2}} - b] \right\}$                                                                                                                                         |
|     | For other transforms containing trigonometric functions see the tables of Fourier transforms.                                                                                                                                                                                              |                                                                                                                                                                                                                                                                                                                                                                                       |

## 9.4. Higher transcendental functions

|     | $f(x)$                                                                                                                                                                                                                       | $\int_0^\infty f(x) Y_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                              |
|-----|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $x^{\frac{1}{2}} P_n(1-2x^2) \quad 0 < x < 1$<br>$0 \quad 1 < x < \infty$<br>$n = 0, 1, 2, \dots, \quad \nu = 0$                                                                                                             | $\pi^{-1} y^{-\frac{1}{2}} [S_{2n+1}(y) + \pi Y_{2n+1}(y)]$                                                                                                                                                                  |
| (2) | $x^{\nu-2\mu+2n+3/2} \exp(x^2) \Gamma(\mu, x^2)$<br>$n$ integer<br>$\operatorname{Re}(\nu - \mu + n) > -3/2$<br>$\operatorname{Re}(-\mu + n) > -3/2$<br>$\operatorname{Re} \nu < \frac{1}{2} - 2n$                           | $(-1)^n \frac{\Gamma(3/2 - \mu + \nu + n) \Gamma(3/2 - \mu + n)}{y^{\frac{1}{2}} \Gamma(1 - \mu)}$<br>$\times \exp\left(\frac{y^2}{8}\right) W_{\mu-\frac{1}{2}, \nu-n-1, \frac{1}{2}\nu}\left(\frac{y^2}{4}\right)$         |
| (3) | $0 \quad 0 < x < a$<br>$P_{\nu-\frac{1}{2}}(a^{-1}x) \quad a < x < \infty$<br>$\operatorname{Re} \nu < \frac{1}{2}$                                                                                                          | $\left(\frac{a}{2y}\right)^{\frac{1}{2}} [\cos(\frac{1}{2}ay) J_\nu(\frac{1}{2}ay)$<br>$- \sin(\frac{1}{2}ay) Y_\nu(\frac{1}{2}ay)]$                                                                                         |
| (4) | $0 \quad 0 < x < a$<br>$x^{-\mu}(x^2 - a^2)^{-\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^\mu(x/a)$<br>$a < x < \infty$<br>$-\frac{1}{4} < \operatorname{Re} \mu < 1$<br>$\operatorname{Re}(2\mu - \nu) > -\frac{1}{2}$              | $2^{-3/2} \pi^{1/2} a^{1-\mu} y^\mu [J_\nu(\frac{1}{2}ay)$<br>$\times J_{\mu-\frac{1}{2}}(\frac{1}{2}ay) - Y_\nu(\frac{1}{2}ay)$<br>$\times Y_{\mu-\frac{1}{2}}(\frac{1}{2}ay)]$                                             |
| (5) | $0 \quad 0 < x < a$<br>$(x^2 - a^2)^{\frac{1}{2}\nu-\frac{1}{2}} P_{\mu}^{-\nu}(2a^{-2}x^2-1)$<br>$a < x < \infty$<br>$\operatorname{Re} \nu > -\frac{1}{2}$<br>$\operatorname{Re} \nu +  2\operatorname{Re} \mu + 1  < 3/2$ | $\pi^{\frac{1}{2}} 2^{\nu-2} ay^{\frac{1}{2}-\nu} [J_{\mu+\frac{1}{2}}(\frac{1}{2}ay)$<br>$\times J_{-\mu-\frac{1}{2}}(\frac{1}{2}ay) - Y_{\mu+\frac{1}{2}}(\frac{1}{2}ay)$<br>$\times Y_{-\mu-\frac{1}{2}}(\frac{1}{2}ay)]$ |

## Higher transcendental functions (cont'd)

|      |                                                                                                          |                                                                                                                                                                                                                                                                                                                                |
|------|----------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
|      | $f(x)$                                                                                                   | $\int_0^\infty f(x) Y_\nu(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                               |
| (6)  | $x^\lambda J_\mu(ax)$                                                                                    | see under Mellin transforms                                                                                                                                                                                                                                                                                                    |
| (7)  | $\sin ax J_{\nu+\frac{1}{2}}(ax)$<br>$a > 0, \quad \operatorname{Re} \nu > -3/2$                         | $(\pi \cos \theta)^{-\frac{1}{2}} (2a \sin \theta)^{-1}$<br>$\times \cos [(\nu + 1) \theta]$<br>$y = 2a \cos \theta, \quad 0 < \theta < \frac{1}{2}\pi$<br>0 <span style="float: right;"><math>2a &lt; y &lt; \infty</math></span>                                                                                             |
| (8)  | $x^{\nu+\frac{1}{2}} [J_\nu(ax)]^2$<br>$a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$ | 0 <span style="float: right;"><math>0 &lt; y &lt; 2a</math></span><br>$\frac{2^{3\nu+1} a^{2\nu}}{\pi^{\frac{1}{2}} \Gamma(\frac{1}{2} - \nu)} y^{-\nu-\frac{1}{2}} (y^2 - 4a^2)^{-\nu-\frac{1}{2}}$<br><span style="float: right;"><math>2a &lt; y &lt; \infty</math></span>                                                  |
| (9)  | $x^{\frac{1}{2}} J_{\frac{1}{2}\nu}(ax^2)$<br>$a > 0, \quad \operatorname{Re} \nu > -1$                  | $\frac{y^{\frac{1}{2}}}{4a} \left[ Y_{\frac{1}{2}\nu} \left( \frac{y^2}{4a} \right) \right.$<br>$\left. - \tan \left( \frac{\nu\pi}{2} \right) J_{\frac{1}{2}\nu} \left( \frac{y^2}{4a} \right) \right.$<br>$\left. + \sec \left( \frac{\nu\pi}{2} \right) \mathbf{H}_{-\frac{1}{2}\nu} \left( \frac{y^2}{4a} \right) \right]$ |
| (10) | $x^{5/2} J_{\frac{1}{2}\nu-\frac{1}{2}}(ax^2)$<br>$a > 0, \quad \operatorname{Re} \nu > -3/2$            | $a^{-2} y^{\frac{1}{2}} J_{\frac{1}{2}\nu+\frac{1}{2}} \left( \frac{y^2}{4a} \right)$                                                                                                                                                                                                                                          |

**Higher transcendental functions (cont'd)**

|      | $f(x)$                                                                                                                | $\int_0^\infty f(x) Y_\nu(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                                                                  |
|------|-----------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (11) | $x^{\frac{1}{2}} J_{\frac{1}{4}\nu}(ax^2) J_{-\frac{1}{4}\nu}(ax^2)$<br>$a > 0, \quad -2 < \operatorname{Re} \nu < 2$ | $\frac{y^{\frac{1}{2}} \sec(\frac{1}{4}\nu\pi)}{16a} \left\{ [1 + 2 \cos(\frac{1}{2}\nu\pi)] \right.$ $\times \left[ J_{\frac{1}{4}\nu} \left( \frac{y^2}{16a} \right) \right]^2 + 2 \sin(\frac{1}{2}\nu\pi)$ $\times J_{\frac{1}{4}\nu} \left( \frac{y^2}{16a} \right) Y_{\frac{1}{4}\nu} \left( \frac{y^2}{16a} \right)$ $\left. - \left[ Y_{\frac{1}{4}\nu} \left( \frac{y^2}{16a} \right) \right]^2 \right\}$ |
| (12) | $x^{-\frac{1}{2}} J_\nu(a^2 x^{-1})$<br>$a > 0, \quad -1/2 < \operatorname{Re} \nu < 3/2$                             | $y^{-\frac{1}{2}} [Y_{2\nu}(2ay^{\frac{1}{2}})$ $+ 2\pi^{-1} K_{2\nu}(2ay^{\frac{1}{2}})]$                                                                                                                                                                                                                                                                                                                        |
| (13) | $x^{-5/2} J_\nu(a^2 x^{-1})$<br>$a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                     | $a^{-2} y^{\frac{1}{2}} [Y_{2\nu}(2ay^{\frac{1}{2}})$ $- 2\pi^{-1} K_{2\nu}(2ay^{\frac{1}{2}})]$                                                                                                                                                                                                                                                                                                                  |
| (14) | $x^{-\frac{1}{2}} Y_\nu(a^2 x^{-1})$<br>$a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$             | $-y^{-\frac{1}{2}} J_{2\nu}(2ay^{\frac{1}{2}})$                                                                                                                                                                                                                                                                                                                                                                   |
| (15) | $x^{-5/2} Y_\nu(a^2 x^{-1})$<br>$a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                     | $-a^{-2} y^{\frac{1}{2}} J_{2\nu}(2ay^{\frac{1}{2}})$                                                                                                                                                                                                                                                                                                                                                             |
| (16) | $x^{-3/2} Y_{\nu+1}(a^2 x^{-1})$<br>$a > 0, \quad -3/2 < \operatorname{Re} \nu < 1/2$                                 | $-a^{-1} J_{2\nu+1}(2ay^{\frac{1}{2}})$                                                                                                                                                                                                                                                                                                                                                                           |

## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                                                                          | $\int_0^\infty f(x) Y_\nu(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                        |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (17) | $J_{2\nu-1}(ax^{\frac{1}{2}})$<br>$a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$                                                                                                                                                                                                           | $-\frac{a}{2y^{3/2}} \mathbf{H}_{\nu-1} \left( \frac{a^2}{4y} \right)$                                                                                                                                                                                                  |
| (18) | $x^{-\frac{1}{2}} J_{2\nu}(ax^{\frac{1}{2}})$<br>$a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$                                                                                                                                                                                            | $-y^{-\frac{1}{2}} \mathbf{H}_\nu \left( \frac{a^2}{4y} \right)$                                                                                                                                                                                                        |
| (19) | $x^{-\frac{1}{2}} Y_{2\nu}(ax^{\frac{1}{2}})$<br>$a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                                                                                                                                                                              | $\frac{1}{2} y^{-\frac{1}{2}} \left[ \sec(\nu\pi) J_{-\nu} \left( \frac{a^2}{4y} \right) \right.$<br>$\left. + \csc(\nu\pi) \mathbf{H}_{-\nu} \left( \frac{a^2}{4y} \right) \right.$<br>$\left. - 2 \cot(2\nu\pi) \mathbf{H}_\nu \left( \frac{a^2}{4y} \right) \right]$ |
| (20) | $x^{\nu+2n-\frac{1}{2}} (x^2 + \lambda^2)^{-1}$<br>$\times (x^2 + a^2)^{-\frac{1}{2}\mu}$<br>$\times J_\mu[b(x^2 + a^2)^{\frac{1}{2}}] \quad b > 0$<br>$\operatorname{Re} \lambda > 0, \quad n = 0, 1, 2, \dots$<br>$-\frac{1}{2} - n < \operatorname{Re} \mu < 3 - 2n + \operatorname{Re} \nu$ | $(-1)^{n+1} \lambda^{\nu+2n-1} y^{\frac{1}{2}} K_\nu(\lambda y)$<br>$\times (\lambda^2 - a^2)^{-\frac{1}{2}\mu} I_\mu[b(\lambda^2 - a^2)^{\frac{1}{2}}]$<br>$y > b$                                                                                                     |
| (21) | 0<br>$x^{\nu+\frac{1}{2}} (x^2 - a^2)^{\frac{1}{2}\mu}$<br>$\times J_\mu[b(x^2 - a^2)^{\frac{1}{2}}] \quad a < x < \infty$<br>$b > 0$<br>$-1 < \operatorname{Re} \mu < -\operatorname{Re} \nu$                                                                                                  | $-2\pi^{-1} a^{\mu+\nu+1} b^\mu y^{\nu+\frac{1}{2}} \cos(\nu\pi)$<br>$\times (b^2 - y^2)^{-\frac{1}{2}(\mu+\nu+1)}$<br>$\times K_{\mu+\nu+1}[a(b^2 - y^2)^{\frac{1}{2}}]$<br>$0 < y < b$                                                                                |

## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                             | $\int_0^\infty f(x) Y_\nu(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                                                            |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (22) | $0 \quad 0 < x < a$<br>$x^{\nu+\frac{1}{2}} (x^2 - a^2)^{\frac{1}{2}\mu}$<br>$\times J_\mu [b(x^2 - a^2)^{\frac{1}{2}}] \quad a < x < \infty$<br>$b > 0$<br>$-1 < \operatorname{Re} \mu < -\operatorname{Re} \nu$  | $-a^{\mu+\nu+1} b^\mu y^{\nu+\frac{1}{2}} (y^2 - b^2)^{-\frac{1}{2}(\mu+\nu+1)}$<br>$\times \{ \sin(\mu\pi) J_{\mu+\nu+1} [a(y^2 - b^2)^{\frac{1}{2}}]$<br>$+ \cos(\mu\pi) Y_{\mu+\nu+1} [a(y^2 - b^2)^{\frac{1}{2}}] \}$<br>$b < y < \infty$                                                                                                                                                               |
| (23) | $0 \quad 0 < x < a$<br>$x^{\frac{1}{2}-\nu} (x^2 - a^2)^{\frac{1}{2}\mu}$<br>$\times J_\mu [b(x^2 - a^2)^{\frac{1}{2}}] \quad a < x < \infty$<br>$b > 0, \quad -1 < \operatorname{Re} \mu < \operatorname{Re} \nu$ | $-2\pi^{-1} a^{\mu-\nu+1} b^\mu y^{\frac{1}{2}-\nu}$<br>$\times (b^2 - y^2)^{-\frac{1}{2}(\mu-\nu+1)}$<br>$\times K_{\mu-\nu+1} [a(b^2 - y^2)^{\frac{1}{2}}] \quad 0 < y < b$                                                                                                                                                                                                                               |
| (24) | $0 \quad 0 < x < a$<br>$x^{\frac{1}{2}-\nu} (x^2 - a^2)^{\frac{1}{2}\mu}$<br>$\times J_\mu [b(x^2 - a^2)^{\frac{1}{2}}] \quad a < x < \infty$<br>$b > 0, \quad -1 < \operatorname{Re} \mu < \operatorname{Re} \nu$ | $a^{\mu-\nu+1} b^\mu y^{\frac{1}{2}-\nu} (y^2 - b^2)^{\frac{1}{2}(\nu-\mu-1)}$<br>$\times Y_{\nu-\mu-1} [a(y^2 - b^2)^{\frac{1}{2}}] \quad b < y < \infty$                                                                                                                                                                                                                                                  |
| (25) | $x^{\frac{1}{2}} K_{\frac{1}{2}\nu}(ax^2)$<br>$\operatorname{Re} a > 0, \quad -1 < \operatorname{Re} \nu < 1$                                                                                                      | $\frac{\pi y^{\frac{1}{2}}}{4a} \left[ \csc(\nu\pi) \mathbf{L}_{-\frac{1}{2}\nu} \left( \frac{y^2}{4a} \right) \right.$<br>$- \operatorname{ctn}(\nu\pi) \mathbf{L}_{\frac{1}{2}\nu} \left( \frac{y^2}{4a} \right)$<br>$- \tan(\frac{1}{2}\nu\pi) I_{\frac{1}{2}\nu} \left( \frac{y^2}{4a} \right)$<br>$\left. - \pi^{-1} \sec(\frac{1}{2}\nu\pi) K_{\frac{1}{2}\nu} \left( \frac{y^2}{4a} \right) \right]$ |

## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                                                                  | $\int_0^\infty f(x) Y_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                           |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (26) | $x^{-\frac{1}{2}} \exp(\frac{1}{2}ax^2) K_0(\frac{1}{2}ax^2)$<br>$\nu = 0$                                                                              | $-\frac{1}{2}\pi a^{-\frac{1}{2}} y^{\frac{1}{2}} \exp\left(\frac{y^2}{8a}\right) K_0\left(\frac{y^2}{8a}\right)$                                                                                         |
| (27) | $x^{-\frac{1}{2}-2\mu} \exp(\frac{1}{2}ax^2)$<br>$\times K_\mu(\frac{1}{2}ax^2)$<br>$\nu = 0, \quad -\frac{3}{4} < \operatorname{Re} \mu < \frac{1}{4}$ | $-\frac{a^\mu \pi^{\frac{1}{2}} [\Gamma(\frac{1}{2} - 2\mu)]^2}{y^{\frac{1}{2}} \Gamma(1 - 2\mu)}$<br>$\times \exp\left(\frac{y^2}{8a}\right) W_{2\mu, 0}\left(\frac{y^2}{4a}\right)$                     |
| (28) | $x^{-\frac{1}{2}} K_\nu(ax^{-1})$<br>$\operatorname{Re} a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                                | $-2y^{-\frac{1}{2}} [\sin(3\nu\pi/2) \ker_{2\nu}(2a^{\frac{1}{2}}y^{\frac{1}{2}})$<br>$+ \cos(3\nu\pi/2) \operatorname{kei}_{2\nu}(2a^{\frac{1}{2}}y^{\frac{1}{2}})]$                                     |
| (29) | $x^{-5/2} K_\nu(ax^{-1})$<br>$\operatorname{Re} a > 0, \quad -5/2 < \operatorname{Re} \nu < 5/2$                                                        | $2a^{-1} y^{\frac{1}{2}} [\sin(3\pi\nu/2) \operatorname{kei}_{2\nu}(2a^{\frac{1}{2}}y^{\frac{1}{2}})$<br>$- \cos(3\pi\nu/2) \ker_{2\nu}(2a^{\frac{1}{2}}y^{\frac{1}{2}})]$                                |
| (30) | $x^{-2\nu} K_{\nu-\frac{1}{2}}(ax^{-1})$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > 1/6$                                                | $(2\pi)^{\frac{1}{2}} a^{\frac{1}{2}-\nu} y^{\nu-\frac{1}{2}} Y_{2\nu-1}[(2ay)^{\frac{1}{2}}]$<br>$\times K_{2\nu-1}[(2ay)^{\frac{1}{2}}]$                                                                |
| (31) | $x^{-2\nu-2} K_{\nu-\frac{1}{2}}(ax^{-1})$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$                                     | $(2\pi)^{\frac{1}{2}} a^{-\frac{1}{2}-\nu} y^{\frac{1}{2}+\nu} Y_{2\nu}[(2ay)^{\frac{1}{2}}]$<br>$\times K_{2\nu}[(2ay)^{\frac{1}{2}}]$                                                                   |
| (32) | $x^{2\nu-2} K_{\nu+\frac{1}{2}}(ax^{-1})$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu < \frac{1}{2}$                                       | $(\frac{1}{2}\pi)^{\frac{1}{2}} \csc(\nu\pi) a^{\nu-\frac{1}{2}} y^{\frac{1}{2}-\nu}$<br>$\times K_{2\nu}[(2ay)^{\frac{1}{2}}] \{ J_{2\nu}[(2ay)^{\frac{1}{2}}]$<br>$- J_{-2\nu}[(2ay)^{\frac{1}{2}}] \}$ |

**Higher transcendental functions (cont'd)**

|      | $f(x)$                                                                                                                                               | $\int_0^\infty f(x) Y_\nu(xy) (xy)^\frac{1}{2} dx \quad y > 0$                                                                                                                                                                                      |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (33) | $x^{-\frac{1}{2}} [K_\mu(a^2 x^{-1})]^2$<br>$ \arg a  < \frac{1}{4}\pi$<br>$-\frac{1}{4} < \operatorname{Re} \mu < \frac{1}{4}, \quad \nu = 0$       | $2\pi y^{-\frac{1}{2}} [\cos(\mu\pi) J_{2\mu}(2ay^\frac{1}{2})$<br>$- \sin(\mu\pi) Y_{2\mu}(2ay^\frac{1}{2})]$                                                                                                                                      |
| (34) | $x^{-\frac{1}{2}} K_{2\nu}(ax^\frac{1}{2})$<br>$\operatorname{Re} a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                   | $-\frac{1}{4}\pi y^{-\frac{1}{2}} \left[ \sec(\nu\pi) J_{-\nu}\left(\frac{a^2}{4y}\right) \right.$<br>$- \csc(\nu\pi) \mathbf{H}_{-\nu}\left(\frac{a^2}{4y}\right)$<br>$\left. + 2 \csc(2\nu\pi) \mathbf{H}_\nu\left(\frac{a^2}{4y}\right) \right]$ |
| (35) | $x^{\nu-\frac{1}{2}} J_{2\nu-1}(ax^\frac{1}{2}) K_{2\nu-1}(ax^\frac{1}{2})$<br>$ \arg a  < \frac{1}{4}\pi, \quad \operatorname{Re} \nu > 0$          | $2^{-\nu-1} \pi^\frac{1}{2} a^{2\nu-1} y^{-2\nu} \csc(\nu\pi)$<br>$\times \left[ \mathbf{L}_{\frac{1}{2}-\nu}\left(\frac{a^2}{2y}\right) - I_{\nu-\frac{1}{2}}\left(\frac{a^2}{2y}\right) \right]$                                                  |
| (36) | $x^{-\frac{1}{2}} \mathbf{H}_{\nu-1}(ax)$<br>$a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                                       | $-a^{\nu-1} y^{\frac{1}{2}-\nu} \quad 0 < y < a$<br>$0 \quad a < y < \infty$                                                                                                                                                                        |
| (37) | $x^{\nu-\mu+\frac{1}{2}} \mathbf{H}_\mu(ax)$<br>$a > 0, \quad \operatorname{Re} \mu > \operatorname{Re} \nu$<br>$-3/2 < \operatorname{Re} \nu < 1/2$ | $\frac{2^{\nu+1-\mu} y^{\nu+\frac{1}{2}}}{a^\mu \Gamma(\mu-\nu)} (a^2 - y^2)^{\mu-\nu-1}$<br>$0 < y < a$<br>$0 \quad a < y < \infty$                                                                                                                |
| (38) | $x^{-5/2} S_{-\nu-3,\nu}(a^2 x^{-1}) \quad \operatorname{Re} a > 0$<br>$-3/2 < \operatorname{Re} \nu < 1/2$                                          | $\pi 2^{-\nu-2} a^{-2} y^\frac{1}{2} K_{2\nu}(2ay^\frac{1}{2})$<br>$\times [\Gamma(\nu+2)]^{-1}$                                                                                                                                                    |



## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                                   | $\int_0^\infty f(x) Y_\nu(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                           |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (39) | $x^{\nu-\frac{1}{2}} \exp(\frac{1}{4} a^2 x^2)$<br>$\times D_{\frac{1}{2}\nu-\frac{1}{2}}(ax)$<br>$ \arg a  < \frac{3}{4}\pi$<br>$-1/2 < \operatorname{Re} \nu < 2/3$                                                    | $-\pi^{-1} 2^{\frac{3}{4}\nu+\frac{3}{4}} a^{-\nu} y^{-\frac{1}{2}} \Gamma(\nu+1)$<br>$\times \exp\left(\frac{y^2}{4a^2}\right) W_{-\frac{1}{2}\nu-\frac{1}{2}, \frac{1}{2}\nu}\left(\frac{y^2}{2a^2}\right)$                                                                                                              |
| (40) | $D_{\nu-\frac{1}{2}}(ax^{-\frac{1}{2}}) D_{-\nu-\frac{1}{2}}(ax^{-\frac{1}{2}})$<br>$ \arg a  < \frac{1}{4}\pi$                                                                                                          | $y^{-1} \exp(-ay^{\frac{1}{2}})$<br>$\times \sin[ay^{\frac{1}{2}} - \frac{1}{2}(\nu - \frac{1}{2})\pi]$                                                                                                                                                                                                                    |
| (41) | $x^{\nu-m} \exp(-\frac{1}{4}x^2)$<br>$\times M_{\kappa, \frac{1}{2}\nu-\frac{1}{2}m}(\frac{1}{2}x^2) \quad m \text{ integer}$<br>$\operatorname{Re}(2\kappa - \nu) > -m \geq -1$<br>$\operatorname{Re} \nu > m - 3/2$    | $\frac{(-1)^m 2^{\frac{1}{2}\nu-\frac{1}{2}m} \Gamma(3/2 - m)}{\Gamma(3/4 + \kappa - m/2)} (\frac{1}{2}y^2)^\lambda$<br>$\times \exp(-\frac{1}{4}y^2) W_{\alpha, \beta}(\frac{1}{2}y^2)$<br>$\alpha = \kappa/2 + m/4 + \nu/2 + 5/8$<br>$\beta = \kappa/2 + m/4 - \nu/2 - 3/8$<br>$\lambda = \kappa/2 + m/4 - 5/8$          |
| (42) | $x^{-m} \exp(-\frac{1}{4}x^2)$<br>$\times M_{\kappa, \frac{1}{2}\nu-\frac{1}{2}m+\frac{1}{4}}(\frac{1}{2}x^2)$<br>$m \text{ integer}, \quad 2\operatorname{Re} \kappa > -m \geq -1$<br>$\operatorname{Re} \nu > m - 3/2$ | $\frac{(-1)^m \Gamma(\nu - m + 3/2) 2^{-\frac{1}{2}m}}{\Gamma(\kappa + \nu/2 - m/2 + 3/4)} (\frac{1}{2}y^2)^\lambda$<br>$\times \exp(-\frac{1}{4}y^2) W_{\alpha, \beta}(\frac{1}{2}y^2)$<br>$\alpha = \kappa/2 - 3m/4 + \nu/4 + 5/8$<br>$\beta = \kappa/2 + m/4 + \nu/4 - 3/8$<br>$\lambda = \kappa/2 + m/4 - \nu/4 - 5/8$ |

## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                                                                                                    | $\int_0^\infty f(x) Y_\nu(xy) (xy)^{1/2} dx \quad y > 0$                                                                                                                                                                                                                                                                                                                                                                         |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (43) | $x^{2\mu+\nu-1/2} \exp(-1/4 x^2)$<br>$\times M_{\kappa, \mu}(1/2 x^2)$<br>$-1 < 2 \operatorname{Re} \mu < \operatorname{Re}(2\kappa - \nu) + 1/2$<br>$\operatorname{Re}(2\mu + \nu) > -1$ | $\pi^{-1} 2^{\mu+\beta} y^{\kappa-\mu-1} \Gamma(2\mu+1)$<br>$\times \Gamma(1/2 - \mu - \kappa) \exp(-1/4 y^2)$<br>$\times \left\{ \cos(2\mu\pi) \frac{\Gamma(2\mu+\nu+1)}{\Gamma(\mu+\nu-\kappa+3/2)} \right.$<br>$\times M_{\alpha, \beta}(1/2 y^2) + \sin[(\mu - \kappa)\pi]$<br>$\left. \times W_{\alpha, \beta}(1/2 y^2) \right\}$<br>$2\alpha = 3\mu + \nu + \kappa + 1/2$<br>$2\beta = \mu + \nu - \kappa + 1/2$           |
| (44) | $x^{2\mu-\nu-1/2} \exp(-1/4 x^2)$<br>$\times M_{\kappa, \mu}(1/2 x^2)$<br>$-1 < 2 \operatorname{Re} \mu < \operatorname{Re}(2\kappa + \nu) + 1/2$<br>$\operatorname{Re}(2\mu - \nu) > -1$ | $\pi^{-1} 2^{\mu+\beta} y^{\kappa-\mu-1} \exp(-1/4 y^2)$<br>$\times \Gamma(2\mu+1) \Gamma(1/2 - \kappa - \mu)$<br>$\times \left\{ \cos[(\nu - 2\mu)\pi] \frac{\Gamma(2\mu - \nu - 1)}{\Gamma(2\beta + 1)} \right.$<br>$\times M_{\alpha, \beta}(1/2 y^2) - \sin[(\nu + \kappa - \mu)\pi]$<br>$\left. \times W_{\alpha, \beta}(1/2 y^2) \right\}$<br>$2\alpha = 3\mu - \nu + \kappa + 1/2$<br>$2\beta = \mu - \nu - \kappa + 1/2$ |
| (45) | $x^{2\lambda} \exp(-1/4 x^2) M_{\kappa, \mu}(1/2 x^2)$<br>$\operatorname{Re}(\kappa - \lambda) > 0$<br>$\operatorname{Re}(2\lambda + 2\mu \pm \nu) > -5/2$                                | $\frac{2^\lambda \Gamma(2\mu+1)}{\Gamma(1/2 + \kappa + \mu)}$<br>$\times G_{34}^{31} \left( \frac{y^2}{2} \left  \begin{matrix} -\mu - \lambda, \mu - \lambda, l \\ h, k, \kappa - \lambda - 1/2, l \end{matrix} \right. \right)$<br>$h = 1/4 + 1/2 \nu, \quad k = 1/4 - 1/2 \nu$<br>$l = -1/4 - 1/2 \nu$                                                                                                                        |

## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                                                                                    | $\int_0^\infty f(x) Y_\nu(xy) (xy)^{1/2} dx \quad y > 0$                                                                                                                                                                                                                                                                                                                                   |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (46) | $x^{2\lambda} \exp(-\frac{1}{4}x^2) W_{\kappa, \mu}(\frac{1}{2}x^2)$<br>$\operatorname{Re}(2\lambda \pm 2\mu \pm \nu) > -5/2$                                             | $(-1)^m 2^\lambda$<br>$\times G_{34}^{22} \left( \frac{y^2}{2} \left  \begin{matrix} -\mu - \lambda, \mu - \lambda, l \\ h, k, \kappa - \lambda - \frac{1}{2}, l \end{matrix} \right. \right)$<br>$h = \frac{1}{4} + \frac{1}{2}\nu, \quad k = \frac{1}{4} - \frac{1}{2}\nu$<br>$l = -\frac{1}{4} - \frac{1}{2}\nu$                                                                        |
| (47) | $x^{2\lambda} \exp(\frac{1}{4}x^2) W_{\kappa, \mu}(\frac{1}{2}x^2)$<br>$\operatorname{Re}(\kappa + \lambda) < 0$<br>$\operatorname{Re}(2\lambda \pm 2\mu \pm \nu) > -5/2$ | $2^\lambda [\Gamma(\frac{1}{2} - \kappa + \mu) \Gamma(\frac{1}{2} - \kappa - \mu)]^{-1}$<br>$\times G_{34}^{32} \left( \frac{y^2}{2} \left  \begin{matrix} -\mu - \lambda, \mu - \lambda, l \\ h, k, -\frac{1}{2} - \kappa - \lambda, l \end{matrix} \right. \right)$<br>$h = \frac{1}{4} + \frac{1}{2}\nu, \quad k = \frac{1}{4} - \frac{1}{2}\nu$<br>$l = -\frac{1}{4} - \frac{1}{2}\nu$ |
| (48) | $x^{1/2} W_{\frac{1}{2}\nu, \mu}(2/x) W_{-\frac{1}{2}\nu, \mu}(2/x)$<br>$-\frac{1}{4} < \operatorname{Re} \mu < \frac{1}{4}$                                              | $4y^{-1/2} K_{2\mu}(2y^{1/2})$<br>$\times \{ \cos[(\mu - \frac{1}{2}\nu)\pi] J_{2\mu}(2y^{1/2})$<br>$- \sin[(\mu - \frac{1}{2}\nu)\pi] Y_{2\mu}(2y^{1/2}) \}$                                                                                                                                                                                                                              |
| (49) | $x^{\nu+3/2}$<br>$\times {}_2F_1\left(\frac{1}{2}, \frac{1}{2} - \nu; \frac{3}{2}; -a^2x^2\right)$<br>$\operatorname{Re} a > 0$<br>$-3/2 < \operatorname{Re} \nu < -1/2$  | $\frac{2^\nu y^{-\nu-1/2}}{\pi^{1/2} a^2 \Gamma(\frac{1}{2} - \nu)} K_\nu\left(\frac{y}{2a}\right)$<br>$\times K_{\nu+1}\left(\frac{y}{2a}\right)$                                                                                                                                                                                                                                         |
| (50) | $x^{\nu+3/2}$<br>$\times {}_2F_1(1, 2\nu+3/2; \nu+2; -a^2x^2)$<br>$\operatorname{Re} a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                     | $\pi^{-1/2} 2^{-\nu} a^{-2\nu-3} \frac{\Gamma(\nu+2)}{\Gamma(2\nu+3/2)}$<br>$\times y^{\nu+1/2} \left[ K_\nu\left(\frac{y}{2a}\right) \right]^2$                                                                                                                                                                                                                                           |

## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                                                             | $\int_0^\infty f(x) Y_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (51) | $x^{\nu+3/2}$<br>$\times {}_2F_1(1, \mu+\nu+3/2; 3/2; -a^2x^2)$<br>$\operatorname{Re} a > 0, \quad -3/2 < \operatorname{Re} \nu < 1/2$<br>$\operatorname{Re}(2\mu+\nu) > -3/2$                                                                                                     | $\frac{\pi^{\frac{1}{2}} 2^{-\mu-\nu-1} a^{-\mu-2\nu-3}}{\Gamma(\mu+\nu+3/2)}$ $\times y^{\mu+\nu+\frac{1}{2}} K_\mu\left(\frac{y}{a}\right)$                                                                                                                                                                                                                                                                                                                                                                                                                 |
| (52) | $x^\sigma {}_2F_1(a, \beta; \gamma; -\lambda^2 x^2)$<br>$\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \sigma >  \operatorname{Re} \nu  - 3/2$<br>$\operatorname{Re} \sigma < 2\operatorname{Re} a, \quad \operatorname{Re} \sigma < 2\operatorname{Re} \beta$            | $\frac{\lambda^{-\sigma-1} \Gamma(\gamma)}{2^{\frac{1}{2}} \Gamma(a) \Gamma(\beta)}$ $\times G_{35}^{41}\left(\frac{y^2}{4\lambda^2} \middle  \begin{matrix} 1-p, \gamma-p, l \\ h, k, \alpha-p, \beta-p, l \end{matrix}\right)$ $h = \frac{1}{4} + \frac{1}{2}\nu, \quad k = \frac{1}{4} - \frac{1}{2}\nu$ $l = -\frac{1}{4} - \frac{1}{2}\nu, \quad p = \frac{1}{2} + \frac{1}{2}\sigma$                                                                                                                                                                    |
| (53) | $x^{\sigma-3/2}$<br>$\times {}_pF_{p-1}(a_1, \dots, a_p;$<br>$\quad \beta_1, \dots, \beta_{p-1}; -\lambda x^2)$<br>$ \arg \lambda  < \pi, \quad \operatorname{Re} \sigma >  \operatorname{Re} \nu $<br>$\operatorname{Re} a_j > \frac{1}{2}\operatorname{Re} \sigma - \frac{3}{4}$ | $\frac{y^{\frac{1}{2}} \Gamma(\beta_1) \dots \Gamma(\beta_{p-1})}{2\lambda^{\frac{1}{2}\sigma} \Gamma(a_1) \dots \Gamma(a_p)}$ $\times G_{p+2, p+3}^{p+2, 1}\left(\frac{y^2}{4\lambda} \middle  \begin{matrix} \beta_0^*, \dots, \beta_{p-1}^*, l \\ h, k, \alpha_1^*, \dots, \alpha_p^*, l \end{matrix}\right)$ $\alpha_j^* = a_j - \frac{\sigma}{2}, \quad j = 1, \dots, p$ $\beta_0^* = 1 - \frac{\sigma}{2},$ $\beta_j^* = \beta_j - \frac{\sigma}{2}, \quad j = 1, \dots, p-1$ $h = \frac{\nu}{2}, \quad k = -\frac{\nu}{2}, \quad l = -\frac{1+\nu}{2}$ |

## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                                                                                                                                    | $\int_0^\infty f(x) Y_\nu(xy) (xy)^{1/2} dx \quad y > 0$                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (54) | $x^{\sigma-3/2}$<br>$\times {}_pF_p(a_1, \dots, a_p; \beta_1, \dots, \beta_p; -\lambda x^2)$<br>$\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \sigma >  \operatorname{Re} \nu $<br>$\operatorname{Re} a_j > \frac{1}{2} \operatorname{Re} \sigma - \frac{3}{4}$<br>$j = 1, \dots, p$                                                            | $\frac{y^{1/2}}{2\lambda^{1/2}\sigma} \frac{\Gamma(\beta_1) \dots \Gamma(\beta_p)}{\Gamma(a_1) \dots \Gamma(a_p)}$<br>$\times G_{p+2, p+3}^{p+2, 1} \left( \frac{y^2}{4\lambda} \left  \begin{matrix} \beta_0^*, \dots, \beta_p^*, l \\ h, k, a_1^*, \dots, a_p^*, l \end{matrix} \right. \right)$<br>$\beta_0^* = 1 - \frac{\sigma}{2}, \quad a_j^* = a_j - \frac{\sigma}{2}$<br>$\beta_j^* = \beta_j - \frac{\sigma}{2}, \quad j = 1, \dots, p$<br>$h = \frac{\nu}{2}, \quad k = -\frac{\nu}{2}, \quad l = -\frac{1+\nu}{2}$ |
| (55) | $x^{\sigma-3/2}$<br>$\times {}_pF_q(a_1, \dots, a_p; \beta_1, \dots, \beta_q; -\lambda x^2)$<br>$p \leq q-1, \quad \operatorname{Re} \sigma >  \operatorname{Re} \nu $                                                                                                                                                                                    | $-\pi^{-1} 2^{\sigma-1} y^{1/2-\sigma} \cos[\frac{1}{2}\pi(\sigma-\nu)]$<br>$\times \Gamma\left(\frac{\sigma+\nu}{2}\right) \Gamma\left(\frac{\sigma-\nu}{2}\right)$<br>$\times {}_{p+2}F_q\left(a_1, \dots, a_p, \frac{\sigma+\nu}{2}, \frac{\sigma-\nu}{2}; \beta_1, \dots, \beta_q; -\frac{4\lambda}{y^2}\right)$                                                                                                                                                                                                           |
| (56) | $G_{pq}^{mn} \left( \lambda x^2 \left  \begin{matrix} a_1, \dots, a_p \\ \beta_1, \dots, \beta_q \end{matrix} \right. \right)$<br>$p+q < 2(m+n)$<br>$ \arg \lambda  < (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi$<br>$\operatorname{Re} a_j < 1 \quad j = 1, \dots, n$<br>$\operatorname{Re}(\beta_j \pm \frac{1}{2}\nu) > -\frac{3}{4}$<br>$j = 1, \dots, m$ | $(2\lambda)^{-1/2}$<br>$\times G_{q+1, p+3}^{n+2, n} \left( \frac{y^2}{4\lambda} \left  \begin{matrix} \frac{1}{2}-\beta_1, \dots, \frac{1}{2}-\beta_q, l \\ h, k, \frac{1}{2}-a_1, \dots, \frac{1}{2}-a_p, l \end{matrix} \right. \right)$<br>$h = \frac{1}{4} + \frac{1}{2}\nu, \quad k = \frac{1}{4} - \frac{1}{2}\nu$<br>$l = -\frac{1}{4} - \frac{1}{2}\nu$                                                                                                                                                               |



## CHAPTER X

### K-TRANSFORMS

We call

$$\mathfrak{R}_\nu \{f(x); y\} = \int_0^\infty f(x) K_\nu(xy) (xy)^{\frac{1}{2}} dx$$

the *K-transform of order  $\nu$*  of  $f(x)$  and regard  $y$  as a complex variable. This transformation was introduced by C.S. Meijer (1940) who gave the inversion formula 10.1 (1) and representation theorems: the transformation was further investigated by Boas (1942 a, 1942 b) and Erdélyi (1950-51).

By virtue of the connection between Bessel functions of the first and second kinds, and the modified Bessel function of the third kind  $K_\nu$ , the  $\mathfrak{R}_\nu$  transform may be expressed as a linear combination of any two of the transforms  $\mathfrak{H}_\nu$ ,  $\mathfrak{H}_{-\nu}$ ,  $\mathfrak{Y}_\nu$ ,  $\mathfrak{Y}_{-\nu}$ . However, the variable  $y$  in the Hankel and  $Y$ -transforms occurring in these expressions is negative, and very few Hankel or  $Y$ -transforms converge for negative (or complex) values of  $y$ . Conversely,

$$\begin{aligned} \pi \mathfrak{H}_\nu \{f(x); y\} &= e^{\frac{1}{2}i(\nu+\frac{1}{2})\pi} \mathfrak{R}_\nu \{f(x); iy\} \\ &+ e^{-\frac{1}{2}i(\nu+\frac{1}{2})\pi} \mathfrak{R}_\nu \{f(x); -iy\} \end{aligned}$$

$$\begin{aligned} \pi \mathfrak{Y}_\nu \{f(x); y\} &= -e^{\frac{1}{2}i(\nu+\frac{1}{2})\pi} \mathfrak{R}_\nu \{f(x); iy\} \\ &- e^{-\frac{1}{2}i(\nu-\frac{1}{2})\pi} \mathfrak{R}_\nu \{f(x); -iy\}, \end{aligned}$$

and these relations enable us to evaluate Hankel and  $Y$ -transforms by means of a table of  $K$ -transforms, although in many cases the  $K$ -transforms involved are to be taken on the boundary of the half-plane of convergence, and additional restrictions on the parameters must be introduced to

secure convergence. If  $\nu = \pm \frac{1}{2}$ , the  $K$ -transform reduces to the Laplace transform,

$$\mathfrak{R}_{\pm \frac{1}{2}} \{f(x); y\} = (\frac{1}{2}\pi)^{\frac{1}{2}} \mathfrak{L} \{f(x); y\}$$

and the above relations become the expressions of Fourier's sine and cosine transforms in terms of Laplace integrals.

From the transform pairs given in this chapter further transform pairs may be derived by means of the methods indicated in the introduction to volume I, and also by the general formulas of sec. 10.1. The connection with the Laplace transformation in either of the two forms

$$\begin{aligned} \mathfrak{R}_{\nu} \{f(x); y\} &= \frac{\pi^{\frac{1}{2}} 2^{-\nu} y^{\nu+\frac{1}{2}}}{\Gamma(\nu + \frac{1}{2})} \\ &\times \mathfrak{L} \left\{ \int_0^x (x^2 - t^2)^{\nu-\frac{1}{2}} t^{\frac{1}{2}-\nu} f(t) dt; y \right\} \end{aligned} \quad \text{Re } \nu > -\frac{1}{2}$$

$$\begin{aligned} \mathfrak{R}_{\nu} \{f(x); y\} &= \frac{\pi^{\frac{1}{2}} 2^{-\nu} y^{\frac{1}{2}-\nu}}{\Gamma(\nu + \frac{1}{2})} \\ &\times \int_y^{\infty} (t^2 - y^2)^{\nu-\frac{1}{2}} \mathfrak{L} \{x^{\frac{1}{2}+\nu} f(x); t\} dt \end{aligned} \quad \text{Re } \nu > -\frac{1}{2}$$

may be used to evaluate  $K$ -transforms by means of the tables of Laplace transforms given in chapter IV.



## REFERENCES

- Boas, R.P., 1942a: *Proc. Nat. Acad. Sci. U.S.A.* 28, 21-24.
- Boas, R.P., 1942b: *Bull. Amer. Math. Soc.* 48, 286-294.
- Erdélyi, Arthur, 1950-51: *Rend. Sem. Mat. Univ. Torino* 10, 217-234.
- Meijer, C.S., 1940: *Proc. Amsterdam Akad. Wet.* 43, 599-608 and 702-711.



## K-TRANSFORMS

### 10.1. General formulas

|     |                                                                                     |                                                                                            |
|-----|-------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------|
|     | $f(x)$                                                                              | $\int_0^\infty f(x) K_\nu(xy) (xy)^{\frac{1}{2}} dx$ $= g(y; \nu)$                         |
| (1) | $\frac{1}{\pi i} \int_{c-i\infty}^{c+i\infty} g(y) I_\nu(xy) (xy)^{\frac{1}{2}} dy$ | $g(y)$                                                                                     |
| (2) | $f(ax) \quad a > 0$                                                                 | $a^{-1} g(y/a; \nu)$                                                                       |
| (3) | $x^m f(x) \quad m = 0, 1, 2, \dots$                                                 | $y^{\frac{1}{2}-\nu} \left( -\frac{d}{y dy} \right)^m [y^{m+\nu-\frac{1}{2}} g(y; \nu+m)]$ |
| (4) | $x^m f(x) \quad m = 0, 1, 2, \dots$                                                 | $y^{\frac{1}{2}+\nu} \left( -\frac{d}{y dy} \right)^m [y^{m-\nu-\frac{1}{2}} g(y; \nu-m)]$ |
| (5) | $2\nu x^{-1} f(x)$                                                                  | $y g(y; \nu+1) - y g(y; \nu-1)$                                                            |
| (6) | $x^{-1} f(x)$                                                                       | $y^{\nu+\frac{1}{2}} \int_y^\infty \eta^{-\nu-\frac{1}{2}} g(\eta; \nu+1) d\eta$           |

## General formulas (cont'd)

|      |                                                                                                                                        |                                                                                                                                                                     |
|------|----------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------|
|      | $f(x)$                                                                                                                                 | $\int_0^\infty f(x) K_\nu(xy) (xy)^{\frac{1}{2}} dx$<br>$= g(y; \nu)$                                                                                               |
| (7)  | $x^{-\mu} f(x) \quad \operatorname{Re} \mu > 0$                                                                                        | $2^{1-\mu} [\Gamma(\mu)]^{-1} y^{\nu+\frac{1}{2}}$<br>$\times \int_y^\infty \eta^{\frac{1}{2}-\mu-\nu} (\eta^2 - y^2)^{\mu-1}$<br>$\times g(\eta; \nu + \mu) d\eta$ |
| (8)  | $2\nu f'(x)$                                                                                                                           | $(\nu - \frac{1}{2}) y g(y; \nu + 1)$<br>$+ (\nu + \frac{1}{2}) y g(y; \nu - 1)$                                                                                    |
| (9)  | $x^{\frac{1}{2}-\nu} \left( \frac{d}{x dx} \right)^m$<br>$\times [x^{m+\nu-\frac{1}{2}} f(x)]$<br>$m = 0, 1, 2, \dots$                 | $y^m g(y; \nu + m)$                                                                                                                                                 |
| (10) | $x^{\frac{1}{2}+\nu} \left( \frac{d}{x dx} \right)^m$<br>$\times [x^{m-\nu-\frac{1}{2}} f(x)]$<br>$m = 0, 1, 2, \dots$                 | $y^m g(y; \nu - m)$                                                                                                                                                 |
| (11) | $x^{\frac{1}{2}-\nu} \int_0^x \xi^{\nu-\mu+\frac{1}{2}} (x^2 - \xi^2)^{\mu-1}$<br>$\times f(\xi) d\xi \quad \operatorname{Re} \mu > 0$ | $2^{\mu-1} \Gamma(\mu) y^{-\mu} g(y; \nu - \mu)$                                                                                                                    |

## 10.2. Elementary functions

|     | $f(x)$                                                                               | $\int_0^\infty f(x) K_\nu(xy) (xy)^{\frac{1}{2}} dx$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             |
|-----|--------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $x^{\rho-1}$ $\text{Re } \rho >  \text{Re } \nu  - \frac{1}{2}$                      | $2^{\rho-3/2} y^{-\rho} \Gamma\left(\frac{\rho}{2} + \frac{\nu}{2} + \frac{1}{4}\right) \times \Gamma\left(\frac{\rho}{2} - \frac{\nu}{2} + \frac{1}{4}\right) \quad \text{Re } y > 0$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           |
| (2) | $0$ $0 < x < a$<br>$x^{\nu+1/2}$ $a < x < \infty$                                    | $a^{\nu+1} y^{-\frac{1}{2}} K_{\nu+1}(ay) \quad \text{Re } y > 0$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                |
| (3) | $0$ $0 < x < a$<br>$x^{\sigma+1/2}$ $a < x < \infty$                                 | $ay^{-\frac{1}{2}-\sigma} e^{-\frac{1}{2}\pi\sigma i} [K_{\nu-1}(ay) S_{\sigma+1, \nu}(iay) + i(\nu + \sigma) K_\nu(ay) S_{\sigma, \nu-1}(iay)] \quad \text{Re } y > 0$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |
| (4) | $x^{\mu-1/2} (x+a)^{-1}$ $ \arg a  < \pi$<br>$\text{Re } \mu >  \text{Re } \nu  - 1$ | $2^{\mu-2} \Gamma\left(\frac{\mu}{2} + \frac{\nu}{2}\right) \Gamma\left(\frac{\mu}{2} - \frac{\nu}{2}\right) y^{\frac{1}{2}-\mu} \times {}_1F_2\left(1; 1 - \frac{\mu}{2} - \frac{\nu}{2}, 1 - \frac{\mu}{2} + \frac{\nu}{2}; \frac{a^2 y^2}{4}\right) - 2^{\mu-3} \Gamma\left(\frac{\mu}{2} - \frac{\nu}{2} - \frac{1}{2}\right) \Gamma\left(\frac{\mu}{2} + \frac{\nu}{2} - \frac{1}{2}\right) ay^{3/2-\mu} \times {}_1F_2\left(1; \frac{3}{2} - \frac{\mu}{2} - \frac{\nu}{2}, \frac{3}{2} - \frac{\mu}{2} + \frac{\nu}{2}; \frac{a^2 y^2}{4}\right) - \pi a^\mu y^{\frac{1}{2}} \csc[\pi(\mu - \nu)] \{K_\nu(ay) + \pi \cos(\mu\pi) \csc[\pi(\nu + \mu)] I_\nu(ay)\} \quad \text{Re } y > 0$ |

## Elementary functions (cont'd)

|     | $f(x)$                                                                                                              | $\int_0^\infty f(x) K_\nu(xy) (xy)^{\frac{1}{2}} dx$                                                                                                                                                                                                                                                                                                                                                                    |
|-----|---------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (5) | $x^{-\frac{1}{2}} (x+a)^{-1}$<br>$ \arg a  < \pi, \quad -1 < \operatorname{Re} \nu < 1$                             | $\frac{1}{2} \pi^2 [\csc(\nu\pi)]^2 y^{\frac{1}{2}} [I_\nu(ay) + I_{-\nu}(ay) - e^{-\frac{1}{2}i\nu\pi} \mathbf{J}_\nu(iay) - e^{\frac{1}{2}i\nu\pi} \mathbf{J}_{-\nu}(iay)]$<br>$\operatorname{Re} y > 0$                                                                                                                                                                                                              |
| (6) | $x^{-\frac{1}{2}} (a^2 + x^2)^{-\frac{1}{2}}$<br>$\operatorname{Re} a > 0, \quad -1 < \operatorname{Re} \nu < 1$    | $\frac{1}{8} \pi^2 \sec(\frac{1}{2}\nu\pi) y^{\frac{1}{2}} \{[J_{\frac{1}{2}\nu}(\frac{1}{2}ay)]^2 + [Y_{\frac{1}{2}\nu}(\frac{1}{2}ay)]^2\}$<br>$\operatorname{Re} y > 0$                                                                                                                                                                                                                                              |
| (7) | $x^{-\frac{1}{2}-\nu} (x^2 + a^2)^{-1}$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu < \frac{1}{2}$     | $\frac{1}{4} \pi^2 \sec(\nu\pi) a^{-\nu-1} y^{\frac{1}{2}} [\mathbf{H}_\nu(ay) - Y_\nu(ay)]$<br>$\operatorname{Re} y > 0$                                                                                                                                                                                                                                                                                               |
| (8) | $x^{\frac{1}{2}+\nu} (x^2 + a^2)^\mu$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1$                | $2^\nu \Gamma(\nu+1) a^{\nu+\mu+1} y^{-\frac{1}{2}-\mu} \times S_{\mu-\nu, \mu+\nu+1}(ay)$<br>$\operatorname{Re} y > 0$                                                                                                                                                                                                                                                                                                 |
| (9) | $x^{\rho-3/2} (x^2 + a^2)^\mu$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \rho >  \operatorname{Re} \nu $ | $\frac{a^{\rho+2\mu} y^{\frac{1}{2}}}{4 \Gamma(-\mu)} [f(\nu) + f(-\nu)] + 2^{2\mu+\rho-2} \times \Gamma\left(\frac{\rho}{2} + \mu - \frac{\nu}{2}\right) \Gamma\left(\frac{\rho}{2} + \mu + \frac{\nu}{2}\right) y^{\frac{1}{2}-\rho-2\mu} \times {}_1F_2\left(-\mu; 1-\mu, \frac{\rho}{2} - \frac{\nu}{2}, 1-\mu - \frac{\rho}{2} + \frac{\nu}{2}; -\frac{a^2 y^2}{4}\right)$<br><br>Continued on the following page. |

## Elementary functions (cont'd)

|      | $f(x)$                                                                                                                      | $\int_0^\infty f(x) K_\nu(xy) (xy)^{1/2} dx$                                                                                                                                                                                                                                                                                   |
|------|-----------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (9)  | Continued from the preceding page.                                                                                          | $f(\nu) = (\frac{1}{2}a)^\nu \Gamma(-\nu) \Gamma\left(\frac{\rho}{2} + \frac{\nu}{2}\right)$ $\times \Gamma\left(-\frac{\nu}{2} - \frac{\rho}{2} - \mu\right) y^\nu$ $\times {}_1F_2\left(\frac{\rho}{2} + \frac{\nu}{2}; \frac{\rho}{2} + \mu + 1 + \frac{\nu}{2}, 1 + \nu; -\frac{a^2 y^2}{4}\right) \quad \text{Re } y > 0$ |
| (10) | $[x(a^2 - x^2)]^{\nu-1/2} \quad 0 < x < a$<br>$0 \quad a < x < \infty$<br>$\text{Re } \nu > -1/2$                           | $\pi^{1/2} 2^{\nu-1} a^{2\nu} y^{1/2-\nu} \Gamma(\nu + 1/2)$ $\times I_\nu(\frac{1}{2}ay) K_\nu(\frac{1}{2}ay)$                                                                                                                                                                                                                |
| (11) | $0 \quad 0 < x < a$<br>$[x(x^2 - a^2)]^{\nu-1/2} \quad a < x < \infty$<br>$\text{Re } \nu > -1/2$                           | $\pi^{-1/2} 2^{\nu-1} a^{2\nu} y^{1/2-\nu} \Gamma(\nu + 1/2)$ $\times [K_\nu(\frac{1}{2}ay)]^2 \quad \text{Re } y > 0$                                                                                                                                                                                                         |
| (12) | $x^{1/2-\nu}(a^2 - x^2)^\mu \quad 0 < x < a$<br>$0 \quad a < x < \infty$<br>$\text{Re } \mu > -1, \quad \text{Re } \nu < 1$ | $2^{-\nu-2} a^{2\mu+2} y^{\nu+1/2} (\mu + 1)^{-1} \Gamma(-\nu)$ $\times {}_1F_2(1; \nu + 1, \mu + 2; \frac{1}{4}a^2 y^2)$ $+ \pi 2^{\mu-1} a^{\mu-\nu+1} y^{-\mu-1/2} \csc(\nu\pi)$ $\times \Gamma(\mu + 1) I_{\mu-\nu+1}(ay)$                                                                                                 |
| (13) | $0 \quad 0 < x < a$<br>$x^{1/2-\nu}(x^2 - a^2)^\mu \quad a < x < \infty$<br>$\text{Re } \mu > -1$                           | $2^\mu a^{\mu-\nu+1} y^{-\mu-1/2} \Gamma(\mu + 1)$ $\times K_{\mu-\nu+1}(ay) \quad \text{Re } y > 0$                                                                                                                                                                                                                           |

## Elementary functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                       | $\int_0^\infty f(x) K_\nu(xy) (xy)^{\frac{1}{2}} dx$                                                                                                                                                                                                                            |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (14) | $x^{-\frac{1}{2}} (x^2 + a^2)^{-\frac{1}{2}}$ $\times [(x^2 + a^2)^{\frac{1}{2}} + x]^{-2\mu}$ $\text{Re } a > 0, \quad \nu = 0$                                                                                                             | $-\frac{1}{4} \pi a^{-2\mu} y^{\frac{1}{2}} \left\{ J_\mu(\frac{1}{2} a y) \frac{\partial}{\partial \mu} \right.$ $\times [Y_\mu(\frac{1}{2} a y)] - Y_\mu(\frac{1}{2} a y) \frac{\partial}{\partial \mu}$ $\left. \times [J_\mu(\frac{1}{2} a y)] \right\}$ $\text{Re } y > 0$ |
| (15) | $x^{-\frac{1}{2}} (x^2 + a^2)^{-\frac{1}{2}}$ $\times [(x^2 + a^2)^{\frac{1}{2}} + x]^{-2\mu}$ $\text{Re } a > 0$                                                                                                                            | $\frac{1}{4} \pi^2 a^{-2\mu} y^{\frac{1}{2}} \csc(\nu \pi)$ $\times [J_{\mu+\frac{1}{2}\nu}(\frac{1}{2} a y) Y_{\mu-\frac{1}{2}\nu}(\frac{1}{2} a y)$ $- Y_{\mu+\frac{1}{2}\nu}(\frac{1}{2} a y) J_{\mu-\frac{1}{2}\nu}(\frac{1}{2} a y)]$ $\text{Re } y > 0$                   |
| (16) | $x^{-\frac{1}{2}} (x^2 + a^2)^{-\frac{1}{2}}$ $\times \{[(x^2 + a^2)^{\frac{1}{2}} + x]^{2\mu}$ $+ [(x^2 + a^2)^{\frac{1}{2}} - x]^{2\mu}\}$ $\text{Re } a > 0, \quad \nu = 0$                                                               | $\frac{1}{4} \pi^2 a^{2\mu} y^{\frac{1}{2}} \{[J_\mu(\frac{1}{2} a y)]^2$ $+ [Y_\mu(\frac{1}{2} a y)]^2\}$ $\text{Re } y > 0$                                                                                                                                                   |
| (17) | $x^{-\frac{1}{2}} (x^2 + a^2)^{-\frac{1}{2}}$ $\times \{[(x^2 + a^2)^{\frac{1}{2}} + x]^{2\mu}$ $+ \cos[(\frac{1}{2}\nu - \mu)\pi]$ $+ [(x^2 + a^2)^{\frac{1}{2}} - x]^{2\mu}$ $\times \cos[(\frac{1}{2}\nu + \mu)\pi]\}$ $\text{Re } a > 0$ | $\frac{1}{4} \pi^2 a^{2\mu} y^{\frac{1}{2}} [J_{\frac{1}{2}\nu+\mu}(\frac{1}{2} a y) J_{\frac{1}{2}\nu-\mu}(\frac{1}{2} a y)$ $+ Y_{\frac{1}{2}\nu+\mu}(\frac{1}{2} a y) Y_{\frac{1}{2}\nu-\mu}(\frac{1}{2} a y)]$ $\text{Re } y > 0$                                           |
| (18) | $x^{-\frac{1}{2}-2\mu} (x^2 + a^2)^{-\frac{1}{2}}$ $\times [(x^2 + a^2)^{\frac{1}{2}} + a]^{2\mu} \text{Re } a > 0$ $2 \text{Re } \mu +  \text{Re } \nu  < 1$                                                                                | $\frac{1}{2} a^{-1} y^{-\frac{1}{2}} \Gamma\left(\frac{1+\nu}{2} - \mu\right) \Gamma\left(\frac{1-\nu}{2} - \mu\right)$ $\times W_{\mu, \frac{1}{2}\nu}(i a y) W_{\mu, \frac{1}{2}\nu}(-i a y)$ $\text{Re } y > 0$                                                              |



## Elementary functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                | $\int_0^\infty f(x) K_\nu(xy)(xy)^{\frac{1}{2}} dx$                                                                                                                                                                                                                                   |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (19) | $0 \quad 0 < x < a$<br>$x^{-\frac{1}{2}}(x^2 - a^2)^{-\frac{1}{2}}$<br>$\times \{ [x + (x^2 - a^2)^{\frac{1}{2}}]^{2\mu}$<br>$+ [x - (x^2 - a^2)^{\frac{1}{2}}]^{2\mu} \}$<br>$a < x < \infty$        | $a^{2\mu} y^{\frac{1}{2}} K_{\frac{1}{2}\nu + \mu}(\frac{1}{2}ay)$<br>$\times K_{\frac{1}{2}\nu - \mu}(\frac{1}{2}ay) \quad \text{Re } y > 0$                                                                                                                                         |
| (20) | $0 \quad 0 < x < a$<br>$x^{-\frac{1}{2}-2\mu}(x^2 - a^2)^{-\frac{1}{2}}$<br>$\times \{ [a + i(x^2 - a^2)^{\frac{1}{2}}]^{2\mu}$<br>$+ [a - i(x^2 - a^2)^{\frac{1}{2}}]^{2\mu} \}$<br>$a < x < \infty$ | $\pi a^{-1} y^{-\frac{1}{2}} W_{\mu, \frac{1}{2}\nu}(ay) W_{-\mu, \frac{1}{2}\nu}(ay)$<br>$\text{Re } y > 0$                                                                                                                                                                          |
| (21) | $x^{-\frac{1}{2}} e^{-ax} \quad \nu = 0$                                                                                                                                                              | $y^{\frac{1}{2}}(y^2 - a^2)^{-\frac{1}{2}} \cos^{-1}(a/y)$<br>$\text{Re}(a + y) > 0$<br>$y(y^2 - a^2)^{-\frac{1}{2}} \cos^{-1}(a/y) \rightarrow \frac{1}{2}\pi$<br>$\text{as } y \rightarrow \infty.$                                                                                 |
| (22) | $x^{-\frac{1}{2}} e^{-ax} \quad -1 < \text{Re } \nu < 1$                                                                                                                                              | $\frac{\pi y^{-\frac{1}{2}}}{\sin(\nu\pi)} \frac{\sin(\nu\theta)}{\sin \theta} \quad \text{Re}(a + y) > 0$<br>$\cos \theta = a/y, \quad \theta \rightarrow \frac{1}{2}\pi \text{ as } y \rightarrow \infty$                                                                           |
| (23) | $x^{\mu-1} e^{-ax} \quad \text{Re } \mu >  \text{Re } \nu  - \frac{1}{2}$                                                                                                                             | $\frac{\pi^{\frac{1}{2}} 2^\nu y^{\nu+\frac{1}{2}}}{(a+y)^{\mu+\nu+\frac{1}{2}}} \frac{\Gamma(\mu+\nu+\frac{1}{2})\Gamma(\mu-\nu+\frac{1}{2})}{\Gamma(\mu+1)}$<br>$\times {}_2F_1\left(\mu+\nu+\frac{1}{2}, \nu+\frac{1}{2}; \mu+1; \frac{a-y}{a+y}\right)$<br>$\text{Re}(a + y) > 0$ |

## Elementary functions (cont'd)

|      | $f(x)$                                                                                                                                                                                              | $\int_0^\infty f(x) K_\nu(xy)(xy)^{\frac{1}{2}} dx$                                                                                                                                                                                                                                                                                                                                                                                  |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (24) | $x^{-\frac{1}{2}} \exp(-ax^2)$<br>$\operatorname{Re} a > 0, \quad -1 < \operatorname{Re} \nu < 1$                                                                                                   | $\frac{1}{4} \sec(\frac{1}{2}\nu\pi) \left(\frac{\pi y}{a}\right)^{\frac{1}{2}}$<br>$\times \exp\left(\frac{y^2}{8a}\right) K_{\frac{1}{2}\nu}\left(\frac{y^2}{8a}\right)$                                                                                                                                                                                                                                                           |
| (25) | $x^{-\frac{1}{2}-2\mu} \exp(-ax^2) \quad \operatorname{Re} a > 0$<br>$2\operatorname{Re} \mu < 1 -  \operatorname{Re} \nu $                                                                         | $\frac{1}{2} a^\mu y^{-\frac{1}{2}} \Gamma\left(\frac{1+\nu}{2} - \mu\right) \Gamma\left(\frac{1-\nu}{2} - \mu\right)$<br>$\times \exp\left(\frac{y^2}{8a}\right) W_{\mu, \frac{1}{2}\nu}\left(\frac{y^2}{4a}\right)$                                                                                                                                                                                                                |
| (26) | $x^{-\frac{1}{2}} (x^2 + a^2)^{-\frac{1}{2}}$<br>$\times \exp[-\beta(x^2 + a^2)^{\frac{1}{2}}]$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \beta > 0$<br>$-1 < \operatorname{Re} \nu < 1$ | $\frac{1}{2} y^{\frac{1}{2}} \sec(\frac{1}{2}\nu\pi)$<br>$\times K_{\frac{1}{2}\nu} \{ \frac{1}{2} a [\beta + (\beta^2 - y^2)^{\frac{1}{2}}] \}$<br>$\times K_{\frac{1}{2}\nu} \{ \frac{1}{2} a [\beta - (\beta^2 - y^2)^{\frac{1}{2}}] \}$<br>$\operatorname{Re} (y + \beta) > 0$                                                                                                                                                   |
| (27) | $x^{-1} \cos(ax^{\frac{1}{2}})$<br>$-\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                                                                                                             | $\frac{1}{2} \pi \sec(\nu\pi) \left[ D_{\nu-\frac{1}{2}} \left( \frac{a}{2^{\frac{1}{2}} y^{\frac{1}{2}}} \right) \right.$<br>$\times D_{-\nu-\frac{1}{2}} \left( -\frac{a}{2^{\frac{1}{2}} y^{\frac{1}{2}}} \right)$<br>$\left. + D_{\nu-\frac{1}{2}} \left( -\frac{a}{2^{\frac{1}{2}} y^{\frac{1}{2}}} \right) D_{-\nu-\frac{1}{2}} \left( \frac{a}{2^{\frac{1}{2}} y^{\frac{1}{2}}} \right) \right]$<br>$\operatorname{Re} y > 0$ |

## Elementary functions (cont'd)

|      | $f(x)$                                                                                                                                                                        | $\int_0^\infty f(x) K_\nu(xy) (xy)^{\frac{1}{2}} dx$                                                                                                                                                                                                                          |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (28) | $x^{-1} \exp(-ax^{\frac{1}{2}})$<br>$\times \cos(ax^{\frac{1}{2}} + \frac{1}{4}\pi - \frac{1}{2}\nu\pi)$<br>$-\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$              | $(\frac{1}{2}\pi)^{\frac{1}{2}} \Gamma(\frac{1}{2} - \nu) D_{\nu-\frac{1}{2}}(ay^{-\frac{1}{2}}e^{\frac{1}{4}\pi i})$<br>$\times D_{\nu-\frac{1}{2}}(ay^{-\frac{1}{2}}e^{-\frac{1}{4}\pi i}) \quad \operatorname{Re} y > 0$                                                   |
| (29) | $0 \quad 0 < x < a$<br>$x^{\frac{1}{2}-\nu} \sin[\beta(x^2 - a^2)^{\frac{1}{2}}]$<br>$a < x < \infty$                                                                         | $(\frac{1}{2}\pi)^{1/2} a^{3/2-\nu} \beta y^{1/2-\nu} (y^2 + \beta^2)^{\nu/2-3/4}$<br>$\times K_{\nu-3/2}[a(y^2 + \beta^2)^{1/2}]$<br>$\operatorname{Re} y >  \operatorname{Im} \beta $                                                                                       |
| (30) | $x^{-\frac{1}{2}}(a^2 - x^2)^{-\frac{1}{2}}$<br>$\times \cos[\beta(a^2 - x^2)^{-\frac{1}{2}}]$<br>$0 < x < a$<br>$0 \quad a < x < \infty$<br>$-1 < \operatorname{Re} \nu < 1$ | $-\frac{1}{4}\pi^2 y^{\frac{1}{2}} \csc(\nu\pi) [J_{\frac{1}{2}\nu}(u) J_{\frac{1}{2}\nu}(v)$<br>$- J_{-\frac{1}{2}\nu}(u) J_{-\frac{1}{2}\nu}(v)]$<br>$u = \frac{1}{2}a[\beta + (\beta^2 - y^2)^{\frac{1}{2}}]$<br>$v = \frac{1}{2}a[\beta - (\beta^2 - y^2)^{\frac{1}{2}}]$ |
| (31) | $0 \quad 0 < x < a$<br>$x^{\frac{1}{2}-\nu}(x^2 - a^2)^{-\frac{1}{2}}$<br>$\times \cos[\beta(x^2 - a^2)^{\frac{1}{2}}]$<br>$a < x < \infty$                                   | $(\frac{1}{2}\pi)^{\frac{1}{2}} a^{\frac{1}{2}-\nu} y^{\frac{1}{2}-\nu} (y^2 + \beta^2)^{\frac{1}{2}\nu-\frac{1}{4}}$<br>$\times K_{\nu-\frac{1}{2}}[a(y^2 + \beta^2)^{\frac{1}{2}}]$<br>$\operatorname{Re} y >  \operatorname{Im} \beta $                                    |
| (32) | $x^{-\frac{1}{2}} \sinh(ax) \quad -2 < \operatorname{Re} \nu < 2$                                                                                                             | $\frac{1}{2}\pi y^{\frac{1}{2}} (y^2 - a^2)^{-\frac{1}{2}} \csc(\frac{1}{2}\nu\pi)$<br>$\times \sin[\nu \sin^{-1}(a/y)]$<br>$\operatorname{Re} y >  \operatorname{Re} a $                                                                                                     |

## Elementary functions (cont'd)

|      | $f(x)$                                                                                                                                                                         | $\int_0^\infty f(x) K_\nu(xy) (xy)^{\frac{1}{2}} dx$                                                                                                                                                                                                                                         |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (33) | $x^{-\frac{1}{2}} \cosh(ax)$<br>$-1 < \operatorname{Re} \nu < 1$                                                                                                               | $\frac{\pi y^{\frac{1}{2}} \cos[\nu \sin^{-1}(a/y)]}{2(y^2 - a^2)^{\frac{1}{2}} \cos(\frac{1}{2}\nu\pi)}$<br>$\operatorname{Re} y >  \operatorname{Re} a $                                                                                                                                   |
| (34) | $x^{-3/2} \sinh(ax)$<br>$-1 < \operatorname{Re} \nu < 1$                                                                                                                       | $\frac{\pi \sec(\frac{1}{2}\nu\pi)}{2\nu y^{\frac{1}{2}}} \sin[\nu \sin^{-1}(a/y)]$<br>$\operatorname{Re} y \geq  \operatorname{Re} a $                                                                                                                                                      |
| (35) | $x^{-\frac{1}{2}} (a^2 - x^2)^{-\frac{1}{2}}$<br>$\times \cosh[\beta(a^2 - x^2)^{\frac{1}{2}}]$<br>$0 < x < a$<br>$0 \quad a < x < \infty$<br>$-1 < \operatorname{Re} \nu < 1$ | $\frac{1}{4} \pi^2 y^{\frac{1}{2}} \csc(\frac{1}{2}\nu\pi) [I_{-\frac{1}{2}\nu}(u) I_{-\frac{1}{2}\nu}(v)$<br>$- I_{\frac{1}{2}\nu}(u) I_{\frac{1}{2}\nu}(v)]$<br>$u = \frac{1}{2} a [(\beta^2 + y^2)^{\frac{1}{2}} + \beta]$<br>$v = \frac{1}{2} a [(\beta^2 + y^2)^{\frac{1}{2}} - \beta]$ |

## 10.3. Higher transcendental functions

|     |                                                                                                                           |                                                                                                                                                                |
|-----|---------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $x^{\frac{1}{2}} P_n(1-2x^2) \quad 0 < x < 1$<br>$0 \quad 1 < x < \infty$<br>$\nu = 0, \quad n = 0, 1, 2, \dots$          | $y^{-\frac{1}{2}} [(-1)^{n+1} K_{2n+1}(y) + \frac{1}{2} i S_{2n+1}(iy)]$                                                                                       |
| (2) | $0 \quad 0 < x < a$<br>$x^{-\frac{1}{2}} (x^2 - a^2)^{-\frac{1}{2}} T_n(a/x)$<br>$a < x < \infty$<br>$n = 0, 1, 2, \dots$ | $\frac{1}{2} \pi a^{-1} y^{-\frac{1}{2}} W_{\frac{1}{2}n, \frac{1}{2}\nu}(ay)$<br>$\times W_{-\frac{1}{2}n, \frac{1}{2}\nu}(ay) \quad \operatorname{Re} y > 0$ |

## Higher transcendental functions (cont'd)

|     | $f(x)$                                                                                                                                   | $\int_0^\infty f(x) K_\nu(xy)(xy)^{\frac{1}{2}} dx$                                                                                      |
|-----|------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------|
| (3) | $0 < x < a$<br>$x^\mu (x^2 - a^2)^{-\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^\mu(x/a)$<br>$a < x < \infty$<br>$\operatorname{Re} \mu < 1$     | $(\frac{1}{2}\pi)^{\frac{1}{2}} y^{-1} e^{-\frac{1}{2}ay} W_{\mu, \nu}(ay)$<br>$\operatorname{Re} y > 0$                                 |
| (4) | $0 < x < a$<br>$x^{\mu-2} (x^2 - a^2)^{-\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^\mu(x/a)$<br>$a < x < \infty$<br>$\operatorname{Re} \mu < 1$ | $(\frac{1}{2}\pi)^{\frac{1}{2}} a^{-1} e^{-\frac{1}{2}ay} W_{\mu-1, \nu}(ay)$<br>$\operatorname{Re} y > 0$                               |
| (5) | $0 < x < a$<br>$x^{-\mu} (x^2 - a^2)^{-\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^\mu(x/a)$<br>$a < x < \infty$<br>$\operatorname{Re} \mu < 1$  | $(2\pi)^{-\frac{1}{2}} a^{1-\mu} y^\mu K_\nu(\frac{1}{2}ay)$<br>$\times K_{\mu-\frac{1}{2}}(\frac{1}{2}ay)$<br>$\operatorname{Re} y > 0$ |
| (6) | $0 < x < a$<br>$x^{\mu-1} (x^2 - a^2)^{-\frac{1}{2}\mu} P_{\nu-3/2}^\mu(x/a)$<br>$a < x < \infty$<br>$\operatorname{Re} \mu < 1$         | $\left(\frac{\pi}{2ay}\right)^{\frac{1}{2}} e^{-\frac{1}{2}ay} W_{\mu-\frac{1}{2}, \nu-\frac{1}{2}}(y)$                                  |
| (7) | $x^{\frac{1}{2}} (x^2 + a^2)^{\frac{1}{2}\nu} P_\mu^\nu(1+2x^2 a^{-2})$<br>$\operatorname{Re} a > 0, \operatorname{Re} \nu < 1$          | $2^{-\nu} ay^{-\nu-\frac{1}{2}} S_{2\nu, 2\mu+1}(ay)$<br>$\operatorname{Re} y > 0$                                                       |

## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                 | $\int_0^\infty f(x) K_\nu(xy) (xy)^{\frac{1}{2}} dx$                                                                                                                                                               |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (8)  | $x^{\frac{1}{2}} (x^2 + a^2)^{\frac{1}{2}\nu}$<br>$\times [(\mu - \nu) P_\mu^\nu (1 + 2x^2 a^{-2})$<br>$+ (\mu + \nu) P_{-\mu}^\nu (1 + 2x^2 a^{-2})]$<br>$\text{Re } a > 0, \quad \text{Re } \nu < 1$ | $2^{1-\nu} \mu y^{-\nu-3/2} S_{2\nu+1, 2\mu}(ay)$<br>$\text{Re } y > 0$                                                                                                                                            |
| (9)  | $x^{\frac{1}{2}} (x^2 + a^2)^{\frac{1}{2}\nu-1}$<br>$\times [P_\mu^\nu (1 + 2x^2 a^{-2})$<br>$+ P_{-\mu}^\nu (1 + 2x^2 a^{-2})]$<br>$\text{Re } a > 0, \quad \text{Re } \nu < 1$                       | $2^{1-\nu} y^{\frac{1}{2}-\nu} S_{2\nu-1, 2\mu}(ay)$<br>$\text{Re } y > 0$                                                                                                                                         |
| (10) | $0 \qquad 0 < x < a$<br>$x^{\frac{1}{2}} (x^2 - a^2)^{-\frac{1}{2}\nu} P_\mu^\nu (2x^2 a^{-2} - 1)$<br>$a < x < \infty$<br>$\text{Re } \nu < 1$                                                        | $2^{-\nu} a y^{\nu-\frac{1}{2}} K_{\mu+1}(ay) \quad \text{Re } y > 0$                                                                                                                                              |
| (11) | $0 \qquad 0 < x < a$<br>$(x^2 - a^2)^{\frac{1}{2}\nu-\frac{1}{2}} P_\mu^{\frac{1}{2}-\nu} (2x^2 a^{-2} - 1)$<br>$a < x < \infty$<br>$\text{Re } \nu > -\frac{1}{2}$                                    | $\pi^{-\frac{1}{2}} 2^{\nu-1} a y^{\frac{1}{2}-\nu} [K_{\mu+\frac{1}{2}}(\frac{1}{2}ay)]^2$<br>$\text{Re } y > 0$                                                                                                  |
| (12) | $x^{-\nu-\frac{1}{2}} (x^2 + a^2)^{\frac{1}{2}-\frac{1}{2}\nu}$<br>$\times Q_{-\frac{1}{2}}^{\frac{1}{2}-\nu} (1 + 2a^2 x^{-2})$<br>$\text{Re } a > 0, \quad \text{Re } \nu < 1$                       | $ie^{-i\pi\nu} \pi^{3/2} 2^{-\nu-3}$<br>$\times a^{1/2-\nu} y^{\nu-1/2} [\Gamma(1-\nu)]^2$<br>$\times \{ [J_{\nu-\frac{1}{2}}(\frac{1}{2}ay)]^2 + [Y_{\nu-\frac{1}{2}}(\frac{1}{2}ay)]^2 \}$<br>$\text{Re } y > 0$ |

## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                      | $\int_0^\infty f(x) K_\nu(xy) (xy)^{\frac{1}{2}} dx$                                                                                                                                                                                                                                                                                                  |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (13) | $x^{-\nu-\frac{1}{2}} (x^2 + a^2)^{\frac{1}{4}-\frac{1}{2}\nu}$<br>$\times Q_\mu^{\frac{1}{2}-\nu} (1 + 2a^2 x^{-2})$<br>$\text{Re } a > 0, \quad \text{Re } \mu > -3/2$<br>$\text{Re } (\mu - \nu) > -3/2$ | $ie^{-i\pi\nu} \pi^{1/2} 2^{-\nu-1} a^{-\nu-1/2}$<br>$\times y^{\nu-3/2} [\Gamma(3/2 + \mu - \nu)]^2$<br>$\times W_{-\mu-\frac{1}{2}, \nu-\frac{1}{2}}(iay) W_{-\mu-\frac{1}{2}, \nu-\frac{1}{2}}(-iay)$<br>$\text{Re } y > 0$                                                                                                                        |
| (14) | $x^{\frac{1}{2}} P_\mu^\nu [(1+x^2)^{\frac{1}{2}}]$ $\text{Re } \nu < 1$                                                                                                                                    | $y^{-1} S_{\nu+\frac{1}{2}, \mu+\frac{1}{2}}(y)$ $\text{Re } y > 0$                                                                                                                                                                                                                                                                                   |
| (15) | $x^{\frac{1}{2}} (1+x^2)^{-\frac{1}{2}} P_\mu^\nu [(1+x^2)^{\frac{1}{2}}]$<br>$\text{Re } \nu < 1$                                                                                                          | $S_{\nu-\frac{1}{2}, \mu+\frac{1}{2}}(y)$ $\text{Re } y > 0$                                                                                                                                                                                                                                                                                          |
| (16) | $x^{\mu+\nu+\frac{1}{2}} J_\mu(ax)$<br>$\text{Re } \mu >  \text{Re } \nu  - 1$                                                                                                                              | $2^{\mu+\nu} a^\mu y^{\nu+\frac{1}{2}} \Gamma(\mu + \nu + 1)$<br>$\times (y^2 + a^2)^{-\mu-\nu-1} \quad \text{Re } y >  \text{Im } a $                                                                                                                                                                                                                |
| (17) | $x^{\sigma+\frac{1}{2}} J_\mu(ax)$<br>$\text{Re}(\mu + \sigma) >  \text{Re } \nu  - 2$                                                                                                                      | $\frac{2^\sigma a^\mu}{\Gamma(\mu + 1)} y^{-\sigma-\mu-3/2}$<br>$\times \Gamma\left(\frac{\mu+\nu+\sigma}{2} + 1\right) \Gamma\left(\frac{\mu-\nu+\sigma}{2} + 1\right)$<br>$\times {}_2F_1\left(\frac{\mu+\nu+\sigma}{2} + 1, \frac{\mu-\nu+\sigma}{2} + 1; \right.$<br>$\left. \mu + 1; -\frac{a^2}{y^2}\right)$<br>$\text{Re } y >  \text{Im } a $ |

## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                               | $\int_0^\infty f(x) K_\nu(xy) (xy)^{\frac{1}{2}} dx$                                                                                                                                                                                                                                                                                                         |
|------|------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (18) | $x^{-\frac{1}{2}} [J_\mu(ax)]^2$<br>$2 \operatorname{Re} \mu >  \operatorname{Re} \nu  - 1$          | $\frac{1}{2} \Gamma(\mu + \frac{1}{2} \nu + \frac{1}{2}) \Gamma(\mu - \frac{1}{2} \nu + \frac{1}{2}) y^{-\frac{1}{2}}$<br>$\times \{P_{\frac{1}{2} \nu - \frac{1}{2}}^{-\mu} [(1 + 4a^2 y^{-2})^{\frac{1}{2}}]\}^2$<br>$\operatorname{Re} y > 2  \operatorname{Im} \alpha $                                                                                  |
| (19) | $x^{\frac{1}{2}} [J_\mu(ax)]^2$<br>$2 \operatorname{Re} \mu >  \operatorname{Re} \nu  - 2$           | $\Gamma(\mu + \frac{1}{2} \nu + 1) \Gamma(\mu - \frac{1}{2} \nu + 1)$<br>$\times y^{-3/2} (1 + 4a^2 y^{-2})^{-1/2}$<br>$\times P_{\frac{1}{2} \nu}^{-\mu} [(1 + 4a^2 y^{-2})^{1/2}]$<br>$\times P_{\frac{1}{2} \nu - 1}^{-\mu} [(1 + 4a^2 y^{-2})^{1/2}]$<br>$\operatorname{Re} y > 2  \operatorname{Im} \alpha $                                            |
| (20) | $x^{\frac{1}{2}} J_\mu(ax) J_{\mu+1}(ax)$<br>$2 \operatorname{Re} \mu >  \operatorname{Re} \nu  - 3$ | $\Gamma\left(\mu + \frac{3 + \nu}{2}\right) \Gamma\left(\mu + \frac{3 - \nu}{2}\right)$<br>$\times y^{-3/2} (1 + 4a^2 y^{-2})^{1/2}$<br>$\times P_{\frac{1}{2} \nu - \frac{1}{2}}^{-\mu} [(1 + 4a^2 y^{-2})^{1/2}]$<br>$\times P_{\frac{1}{2} \nu - \frac{1}{2}}^{-\mu-1} [(1 + 4a^2 y^{-2})^{1/2}]$<br>$\operatorname{Re} y > 2  \operatorname{Im} \alpha $ |
| (21) | $x^{-\frac{1}{2}} J_\mu(ax) J_{-\mu}(ax)$<br>$-1 < \operatorname{Re} \nu < 1$                        | $\frac{1}{2} \pi y^{-\frac{1}{2}} \sec(\frac{1}{2} \nu \pi)$<br>$\times P_{\frac{1}{2} \nu - \frac{1}{2}}^{\mu} [(1 + 4a^2 y^{-2})^{\frac{1}{2}}]$<br>$\times P_{\frac{1}{2} \nu - \frac{1}{2}}^{-\mu} [(1 + 4a^2 y^{-2})^{\frac{1}{2}}]$<br>$\operatorname{Re} y > 2  \operatorname{Im} \alpha $                                                            |



## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                         | $\int_0^\infty f(x) K_\nu(xy) (xy)^{\frac{1}{2}} dx$                                                                                                                                                                                                                                                                                                                                                                                                                                                                               |
|------|--------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (22) | $x^{\frac{1}{2}} J_\mu(ax) J_{-\mu}(ax)$<br>$-2 < \operatorname{Re} \nu < 2$   | $-\frac{1}{2} \pi y^{-3/2} z^{-1} \csc(\frac{1}{2} \nu \pi)$<br>$\times [(\mu - \frac{1}{2} \nu) P_{\frac{1}{2} \nu}^\mu(z) P_{\frac{1}{2} \nu - 1}^{-\mu}(z)$<br>$- (\frac{1}{2} \nu + \mu) P_{\frac{1}{2} \nu - 1}^\mu(z) P_{\frac{1}{2} \nu}^{-\mu}(z)]$<br>$z = (1 + 4a^2 y^{-2})^{\frac{1}{2}}, \quad \operatorname{Re} y > 2  \operatorname{Im} a $                                                                                                                                                                          |
| (23) | $x^{\frac{1}{2}} J_\mu(ax) J_{1-\mu}(ax)$<br>$-3 < \operatorname{Re} \nu < 3$  | $\frac{a \Gamma(\frac{3+\nu}{2}) \Gamma(\frac{3-\nu}{2})}{y^{5/2} \Gamma(2-\mu) \Gamma(1+\mu)}$<br>$\times {}_4F_3\left(\frac{3+\nu}{2}, \frac{3-\nu}{2}, 1, \frac{3}{2};\right.$<br>$\left.2-\mu, 1+\mu, 2; -\frac{4a^2}{y^2}\right)$<br>$\operatorname{Re} y > 2  \operatorname{Im} a $                                                                                                                                                                                                                                          |
| (24) | $x^{\frac{1}{2}} J_\mu(ax) J_{-\mu-1}(ax)$<br>$-1 < \operatorname{Re} \nu < 1$ | $\frac{1}{2} \pi y^{-3/2} z^{-1} \sec(\frac{1}{2} \nu \pi)$<br>$\times [P_{\frac{1}{2} \nu - \frac{1}{2}}^{-\mu}(z) P_{\frac{1}{2} \nu - \frac{1}{2}}^{\mu+1}(z)$<br>$+ (\frac{1}{2} \nu - \frac{1}{2} - \mu) (\frac{1}{2} \nu + \frac{1}{2} + \mu)$<br>$\times P_{\frac{1}{2} \nu - \frac{1}{2}}^{-\mu-1}(z) P_{\frac{1}{2} \nu - \frac{1}{2}}^\mu(z)]$<br>$- \frac{1}{4} \pi^{-1} y^{1/2} \sin(\mu \pi) \sec(\frac{1}{2} \nu \pi)$<br>$z = (1 + 4a^2 y^{-2})^{\frac{1}{2}}, \quad \operatorname{Re} y > 2  \operatorname{Im} a $ |

## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                                    | $\int_0^\infty f(x) K_\nu(xy) (xy)^{\frac{1}{2}} dx$                                                                                                                                                                                                                                                                                                                                                                                                                                         |
|------|---------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (25) | $x^{\sigma+\frac{1}{2}} J_\mu(ax) J_\lambda(ax)$<br>$\operatorname{Re}(\sigma+\mu+\lambda) >  \operatorname{Re} \nu  - 2$ | $\frac{2^\sigma a^{\mu+\lambda} y^{-\mu-\lambda-\sigma-3/2}}{\Gamma(1+\mu) \Gamma(1+\lambda)}$ $\times \Gamma\left(\frac{\mu+\lambda+\nu+\sigma}{2} + 1\right)$ $\times \Gamma\left(\frac{\mu+\lambda+\sigma-\nu}{2} + 1\right)$ $\times {}_4F_3\left(\frac{\mu+\lambda+1}{2}, \frac{\mu+\lambda}{2}+1, \frac{\mu+\lambda+\nu+\sigma}{2}+1, \frac{\mu+\lambda-\nu+\sigma}{2}+1; 1+\mu, 1+\lambda, 1+\mu+\lambda; -\frac{4a^2}{y^2}\right) \quad \operatorname{Re} y >  \operatorname{Im} a $ |
| (26) | $x^{\sigma+\frac{1}{2}} J_\mu(ax) J_\lambda(\beta x)$                                                                     | see Bailey, W.N., 1936: <i>Proc. London Math. Soc.</i> 40, 37-48;<br><i>J. London Math. Soc.</i> 11, 16-20.                                                                                                                                                                                                                                                                                                                                                                                  |
| (27) | $x^{\frac{1}{2}} J_{\frac{1}{2}\nu}(ax^2)$<br>$a > 0, \quad \operatorname{Re} \nu > -1$                                   | $\frac{\pi y^{\frac{1}{2}}}{8a \cos(\frac{1}{2}\nu\pi)} \left[ \mathbf{H}_{-\frac{1}{2}\nu}\left(\frac{y^2}{4a}\right) - Y_{-\frac{1}{2}\nu}\left(\frac{y^2}{4a}\right) \right] \quad \operatorname{Re} y > 0$                                                                                                                                                                                                                                                                               |

## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                                                             | $\int_0^\infty f(x) K_\nu(xy) (xy)^{\frac{1}{2}} dx$                                                                                                                                                                                                                                                                    |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (28) | $x^{\frac{1}{2}} Y_{\frac{1}{2}\nu}(ax^2)$<br>$a > 0, \quad -1 < \operatorname{Re} \nu < 1$                                                        | $\frac{\pi y^{\frac{1}{2}}}{4a \sin(\frac{1}{2}\nu\pi)} \left[ \cos(\frac{1}{2}\nu\pi) \mathbf{H}_{-\frac{1}{2}\nu}\left(\frac{y^2}{4a}\right) - \sin(\frac{1}{2}\nu\pi) J_{-\frac{1}{2}\nu}\left(\frac{y^2}{4a}\right) - \mathbf{H}_{\frac{1}{2}\nu}\left(\frac{y^2}{4a}\right) \right] \quad \operatorname{Re} y > 0$ |
| (29) | $x^{\frac{1}{2}} J_{\frac{1}{4}\nu}(ax^2) J_{-\frac{1}{4}\nu}(ax^2)$<br>$a > 0, \quad -2 < \operatorname{Re} \nu < 2$                              | $\frac{\pi y^{\frac{1}{2}}}{32a \cos(\frac{1}{4}\nu\pi)} \left\{ \left[ J_{\frac{1}{4}\nu}\left(\frac{y^2}{16a}\right) \right]^2 + \left[ Y_{\frac{1}{4}\nu}\left(\frac{y^2}{16a}\right) \right]^2 \right\} \quad \operatorname{Re} y > 0$                                                                              |
| (30) | $x^{\frac{1}{2}} J_{\mu+\frac{1}{4}\nu}(ax^2) J_{\mu-\frac{1}{4}\nu}(ax^2)$<br>$a > 0, \quad 4\operatorname{Re} \mu >  \operatorname{Re} \nu  - 2$ | $\pi^{-1} y^{-3/2} \Gamma(\mu+\frac{1}{4}\nu+\frac{1}{2}) \Gamma(\mu-\frac{1}{4}\nu+\frac{1}{2}) \times W_{-\mu, \frac{1}{4}\nu}\left(\frac{y^2}{8a} e^{\frac{1}{2}i\pi}\right) \times W_{-\mu, \frac{1}{4}\nu}\left(\frac{y^2}{8a} e^{-\frac{1}{2}i\pi}\right) \quad \operatorname{Re} y > 0$                          |
| (31) | $x^{-\frac{1}{2}} J_\nu(a/x)$<br>$a > 0$<br>$-5/2 < \operatorname{Re} \nu < 5/2$                                                                   | $y^{-\frac{1}{2}} e^{\frac{1}{2}i(\nu+1)\pi} K_{2\nu}[2(ay)^{\frac{1}{2}} e^{\frac{1}{4}i\pi}] + y^{-\frac{1}{2}} e^{-\frac{1}{2}i(\nu+1)\pi} K_{2\nu}[2(ay)^{\frac{1}{2}} e^{-\frac{1}{4}i\pi}] \quad \operatorname{Re} y > 0$                                                                                         |

**Higher transcendental functions (cont'd)**

|      | $f(x)$                                                                                             | $\int_0^\infty f(x) K_\nu(xy) (xy)^{\frac{1}{2}} dx$                                                                                                                                                                                                               |
|------|----------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (32) | $x^{-5/2} J_\nu(a/x)$<br>$a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$         | $a^{-1} y^{\frac{1}{2}} e^{\frac{1}{2} i \nu \pi} K_{2\nu}[2(ay)^{\frac{1}{2}} e^{\frac{1}{2} i \pi}]$<br>$+ a^{-1} y^{\frac{1}{2}} e^{-\frac{1}{2} i \nu \pi} K_{2\nu}[2(ay)^{\frac{1}{2}} e^{-\frac{1}{2} i \pi}]$<br>$\operatorname{Re} y > 0$                  |
| (33) | $x^{2\nu-2} J_{\nu+\frac{1}{2}}(a/x)$<br>$a > 0, \quad \operatorname{Re} \nu > -1/3$               | $(2\pi)^{\frac{1}{2}} (y/a)^{-\nu+\frac{1}{2}} J_{2\nu}[(2ay)^{\frac{1}{2}}]$<br>$\times K_{2\nu}[(2ay)^{\frac{1}{2}}] \quad \operatorname{Re} y > 0$                                                                                                              |
| (34) | $x^{-2\nu} J_{\nu-\frac{1}{2}}(a/x)$<br>$a > 0, \quad \operatorname{Re} \nu < 1$                   | $(2\pi)^{\frac{1}{2}} (y/a)^{\nu-\frac{1}{2}} K_{2\nu-1}[(2ay)^{\frac{1}{2}}]$<br>$\times \{\sin(\nu\pi) J_{2\nu-1}[(2ay)^{\frac{1}{2}}]$<br>$+ \cos(\nu\pi) Y_{2\nu-1}[(2ay)^{\frac{1}{2}}]\}$<br>$\operatorname{Re} y > 0$                                       |
| (35) | $x^{2\nu} J_{\frac{1}{2}+\nu}(a/x)$<br>$a > 0, \quad \operatorname{Re} \nu > -1$                   | $(2\pi)^{\frac{1}{2}} (y/a)^{-\nu-\frac{1}{2}} J_{1+2\nu}[(2ay)^{\frac{1}{2}}]$<br>$\times K_{1+2\nu}[(2ay)^{\frac{1}{2}}] \quad \operatorname{Re} y > 0$                                                                                                          |
| (36) | $x^{\sigma-1} J_\mu(a/x)$<br>$a > 0, \quad \operatorname{Re} \sigma >  \operatorname{Re} \nu  - 2$ | $2^{-\sigma-3/2} a^\sigma$<br>$\times G_{04}^{\infty} \left( \frac{a^2 y^2}{4} \left  \begin{matrix} \mu - \sigma, \frac{1}{4} + \frac{\nu}{2}, \\ \frac{1}{4} - \frac{\nu}{2}, -\frac{\mu + \sigma}{2} \end{matrix} \right. \right)$<br>$\operatorname{Re} y > 0$ |

## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                     | $\int_0^\infty f(x) K_\nu(xy) (xy)^{\frac{1}{2}} dx$                                                                                                                                                                                                                     |
|------|--------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (37) | $x^{-\frac{1}{2}} Y_\nu(a/x)$<br>$a > 0, \quad -5/2 < \operatorname{Re} \nu < 5/2$         | $-y^{-\frac{1}{2}} e^{\frac{1}{2}i\nu\pi} K_{2\nu}[2(ay)^{\frac{1}{2}} e^{\frac{1}{4}i\pi}]$<br>$-y^{-\frac{1}{2}} e^{-\frac{1}{2}i\nu\pi} K_{2\nu}[2(ay)^{\frac{1}{2}} e^{-\frac{1}{4}i\pi}]$<br>$\operatorname{Re} y > 0$                                              |
| (38) | $x^{-5/2} Y_\nu(a/x)$<br>$a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$ | $a^{-1} y^{\frac{1}{2}} e^{\frac{1}{2}i(\nu+1)\pi}$<br>$\times K_{2\nu}[2(ay)^{\frac{1}{2}} e^{\frac{1}{4}i\pi}]$<br>$+ a^{-1} y^{\frac{1}{2}} e^{-\frac{1}{2}i(\nu+1)\pi}$<br>$\times K_{2\nu}[2(ay)^{\frac{1}{2}} e^{-\frac{1}{4}i\pi}] \quad \operatorname{Re} y > 0$ |
| (39) | $x^{2\nu-2} Y_{\nu+\frac{1}{2}}(a/x)$<br>$a > 0, \quad \operatorname{Re} \nu > -1/3$       | $(2\pi)^{\frac{1}{2}} (y/a)^{\frac{1}{2}-\nu} Y_{2\nu}[(2ay)^{\frac{1}{2}}]$<br>$\times K_{2\nu}[(2ay)^{\frac{1}{2}}] \quad \operatorname{Re} y > 0$                                                                                                                     |
| (40) | $x^{-2\nu} Y_{\nu-\frac{1}{2}}(a/x)$<br>$a > 0, \quad \operatorname{Re} \nu < 1$           | $-(\frac{1}{2}\pi)^{\frac{1}{2}} (y/a)^{\nu-\frac{1}{2}} \sec(\nu\pi)$<br>$\times K_{2\nu-1}[(2ay)^{\frac{1}{2}}] \{ J_{2\nu-1}[(2ay)^{\frac{1}{2}}]$<br>$- J_{1-2\nu}[(2ay)^{\frac{1}{2}}] \}$                                                                          |
| (41) | $x^{2\nu} Y_{\nu+\frac{1}{2}}(a/x)$<br>$a > 0, \quad \operatorname{Re} \nu < -1$           | $(2\pi)^{\frac{1}{2}} (y/a)^{-\nu-\frac{1}{2}} Y_{2\nu+1}[(2ay)^{\frac{1}{2}}]$<br>$\times K_{2\nu+1}[(2ay)^{\frac{1}{2}}] \quad \operatorname{Re} y > 0$                                                                                                                |
| (42) | $x^{-\frac{1}{2}} J_\mu(a/x) Y_\mu(a/x)$<br>$a > 0, \quad \nu = 0$                         | $-2y^{-\frac{1}{2}} J_{2\mu}(2a^{\frac{1}{2}} y^{\frac{1}{2}})$<br>$\times K_{2\mu}(2a^{\frac{1}{2}} y^{\frac{1}{2}}) \quad \operatorname{Re} y > 0$                                                                                                                     |

**Higher transcendental functions (cont'd)**

|      | $f(x)$                                                                                                                                                            | $\int_0^\infty f(x) K_\nu(xy)(xy)^{\frac{1}{2}} dx$                                                                                                                                                                                                                                                                                                   |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (43) | $x^{-\frac{1}{2}} \{ [J_\mu(a/x)]^2 - [Y_\mu(a/x)]^2 \}$<br>$a > 0, \quad \nu = 0$                                                                                | $4y^{-\frac{1}{2}} Y_{2\mu}(2a^{\frac{1}{2}} y^{\frac{1}{2}}) K_{2\mu}(2a^{\frac{1}{2}} y^{\frac{1}{2}})$<br>$\text{Re } y > 0$                                                                                                                                                                                                                       |
| (44) | $J_{2\nu-1}(ax^{\frac{1}{2}})$ $\text{Re } \nu > -\frac{1}{2}$                                                                                                    | $\frac{\pi a}{4y^{3/2}} \left[ I_{\nu-1}\left(\frac{a^2}{4y}\right) - \mathbf{L}_{\nu-1}\left(\frac{a^2}{4y}\right) \right]$<br>$\text{Re } y > 0$                                                                                                                                                                                                    |
| (45) | $x^{-\frac{1}{2}} J_{2\nu}(ax^{\frac{1}{2}})$ $\text{Re } \nu > -\frac{1}{2}$                                                                                     | $\frac{\pi}{2y^{\frac{1}{2}}} \left[ I_\nu\left(\frac{a^2}{4y}\right) - \mathbf{L}_\nu\left(\frac{a^2}{4y}\right) \right]$<br>$\text{Re } y > 0$                                                                                                                                                                                                      |
| (46) | $x^{-\frac{1}{2}} Y_{2\nu}(ax^{\frac{1}{2}})$<br>$-\frac{1}{2} < \text{Re } \nu < \frac{1}{2}$                                                                    | $\frac{\pi}{2y^{\frac{1}{2}}} \left[ \csc(2\nu\pi) \mathbf{L}_{-\nu}\left(\frac{a^2}{4y}\right) \right.$<br>$\quad - \cotn(2\nu\pi) \mathbf{L}_\nu\left(\frac{a^2}{4y}\right)$<br>$\quad - \tan(\nu\pi) I_\nu\left(\frac{a^2}{4y}\right)$<br>$\quad \left. - \frac{\sec(\nu\pi)}{\pi} K_\nu\left(\frac{a^2}{4y}\right) \right]$<br>$\text{Re } y > 0$ |
| (47) | 0 $0 < x < a$<br>$x^{\frac{1}{2}-\nu}(x^2 - a^2)^{\frac{1}{2}\mu}$<br>$\times J_\mu[\beta(x^2 - a^2)^{\frac{1}{2}}]$<br>$a < x < \infty$<br>$\text{Re } \mu > -1$ | $a^{\mu-\nu+1} \beta^\mu y^{\frac{1}{2}-\nu} (y^2 + \beta^2)^{\frac{1}{2}(\nu-\mu-1)}$<br>$\times K_{\nu-\mu-1}[a(y^2 + \beta^2)^{\frac{1}{2}}]$<br>$\text{Re } y > \text{Im } \beta$                                                                                                                                                                 |

**Higher transcendental functions (cont'd)**

|      | $f(x)$                                                                                                             | $\int_0^\infty f(x) K_\nu(xy) (xy)^{\frac{1}{2}} dx$                                                                                                                                                                                                                                                                                                                                                           |
|------|--------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (48) | $x^{\frac{1}{2}} K_\nu(ax) \quad -1 < \operatorname{Re} \nu < 1$                                                   | $\frac{\pi a^{-\nu} y^{\frac{1}{2} - \nu}}{2 \sin(\nu\pi)} \frac{a^{2\nu} - y^{2\nu}}{a^2 - y^2}$ $\operatorname{Re}(y + a) > 0$                                                                                                                                                                                                                                                                               |
| (49) | $x^{\sigma - 3/2} K_\mu(ax)$<br>$\operatorname{Re} \sigma >  \operatorname{Re} \mu  +  \operatorname{Re} \nu $     | $\frac{2^{\sigma-3} a^{-\nu-\sigma}}{\Gamma(\sigma)} \Gamma\left(\frac{\sigma+\mu+\nu}{2}\right) \Gamma\left(\frac{\sigma-\mu+\nu}{2}\right)$ $\times \Gamma\left(\frac{\sigma+\mu-\nu}{2}\right) \Gamma\left(\frac{\sigma-\mu-\nu}{2}\right) y^{\nu+\frac{1}{2}}$ $\times {}_2F_1\left(\frac{\sigma+\mu+\nu}{2}, \frac{\sigma-\mu+\nu}{2}; \sigma; 1 - \frac{y^2}{a^2}\right)$ $\operatorname{Re}(y + a) > 0$ |
| (50) | $x^{\frac{1}{2}} [2\pi^{-1} K_0(ax) - Y_0(ax)]$<br>$\nu = 0$                                                       | $2\pi^{-1} y^{\frac{1}{2}} [(y^2 + a^2)^{-1} + (y^2 - a^2)^{-1}]$ $\times \log(y/a)$ $\operatorname{Re} y >  \operatorname{Im} a , \quad \operatorname{Re}(y + a) > 0$                                                                                                                                                                                                                                         |
| (51) | $x^{\sigma + \frac{1}{2}} J_\mu(ax) K_\lambda(\beta x)$<br>$x^{\sigma + \frac{1}{2}} K_\mu(ax) K_\lambda(\beta x)$ | see Bailey, W.N., 1936: <i>J. London Math. Soc.</i> 11, 16-20; <i>Proc. London Math. Soc.</i> 40, 37-48.                                                                                                                                                                                                                                                                                                       |

**Higher transcendental functions (cont'd)**

|      | $f(x)$                                                                                                                                                                                               | $\int_0^\infty f(x) K_\nu(xy)(xy)^{\frac{1}{2}} dx$                                                                                                                                                                                                                                                          |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (52) | $x^{\frac{1}{2}} K_{\frac{1}{2}\nu}(ax^2)$<br>$\operatorname{Re} a > 0, \quad -1 < \operatorname{Re} \nu < 1$                                                                                        | $\frac{\pi y^{\frac{1}{2}}}{8a} \left\{ \sec(\frac{1}{2}\nu\pi) K_{\frac{1}{2}\nu}\left(\frac{y^2}{4a}\right) \right.$ $+ \pi \csc(\nu\pi) \left[ \mathbf{L}_{-\frac{1}{2}\nu}\left(\frac{y^2}{4a}\right) \right.$ $\left. \left. - \mathbf{L}_{\frac{1}{2}\nu}\left(\frac{y^2}{4a}\right) \right] \right\}$ |
| (53) | $x^{2\mu+\nu+\frac{1}{2}} \exp(-\frac{1}{2}ax^2)$<br>$\times I_\mu(\frac{1}{2}ax^2) \quad \operatorname{Re} a > 0$<br>$\operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re}(2\mu+\nu) > -1$ | $\pi^{-\frac{1}{2}} 2^{\mu-\frac{1}{2}} a^{-\frac{1}{2}\mu-\frac{1}{2}\nu-\frac{1}{4}} y^{-\mu-1}$ $\times \Gamma(2\mu+\nu+1) \Gamma(\mu+\frac{1}{2})$ $\times \exp\left(\frac{y^2}{8a}\right) W_{k,m}\left(\frac{y^2}{4a}\right)$ $2k = -3\mu - \nu - \frac{1}{2}$ $2m = \mu + \nu + \frac{1}{2}$           |
| (54) | $x^{-\frac{1}{2}} K_\nu(a/x) \quad \operatorname{Re} a > 0$                                                                                                                                          | $\pi y^{-\frac{1}{2}} K_{2\nu}(2a^{\frac{1}{2}} y^{\frac{1}{2}}) \quad \operatorname{Re} y > 0$                                                                                                                                                                                                              |
| (55) | $x^{-5/2} K_\nu(a/x) \quad \operatorname{Re} a > 0$                                                                                                                                                  | $\pi a^{-1} y^{\frac{1}{2}} K_{2\nu}(2a^{\frac{1}{2}} y^{\frac{1}{2}}) \quad \operatorname{Re} y > 0$                                                                                                                                                                                                        |
| (56) | $x^{2\nu} K_{\nu+\frac{1}{2}}(a/x) \quad \operatorname{Re} a > 0$                                                                                                                                    | $(2\pi)^{\frac{1}{2}} (y/a)^{-\nu-\frac{1}{2}} K_{2\nu+1}[(2ay)^{\frac{1}{2}} e^{\frac{1}{4}i\pi}]$ $\times K_{2\nu+1}[(2ay)^{\frac{1}{2}} e^{-\frac{1}{4}i\pi}] \quad \operatorname{Re} y > 0$                                                                                                              |
| (57) | $x^{2\nu-2} K_{\nu+\frac{1}{2}}(a/x) \quad \operatorname{Re} a > 0$                                                                                                                                  | $(2\pi)^{\frac{1}{2}} (y/a)^{\frac{1}{2}-\nu} K_{2\nu}[(2ay)^{\frac{1}{2}} e^{\frac{1}{4}i\pi}]$ $\times K_{2\nu}[(2ay)^{\frac{1}{2}} e^{-\frac{1}{4}i\pi}] \quad \operatorname{Re} y > 0$                                                                                                                   |



## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                          | $\int_0^\infty f(x) K_\nu(xy) (xy)^{\frac{1}{2}} dx$                                                                                                                                                                                                                                                               |
|------|-----------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (58) | $x^{\sigma-1} K_\mu(a/x) \quad \text{Re } a > 0$                                                                | $2^{-\sigma-5/2} a^\sigma$<br>$\times G_{04}^{40} \left( \frac{a^2 y^2}{4} \middle  \frac{\mu-\sigma}{2}, \frac{1}{4}, \frac{\nu}{2}, \frac{1}{4}, -\frac{\nu}{2}, -\frac{\mu+\sigma}{2} \right)$                                                                                                                  |
| (59) | $x^{-\frac{1}{2}} [K_\mu(a/x)]^2$<br>$\text{Re } a > 0, \quad \nu = 0$                                          | $2\pi y^{-\frac{1}{2}} K_{2\mu}(2a^{\frac{1}{2}} y^{\frac{1}{2}} e^{\frac{1}{4}i\pi})$<br>$\times K_{2\mu}(2a^{\frac{1}{2}} y^{\frac{1}{2}} e^{-\frac{1}{4}i\pi}) \quad \text{Re } y > 0$                                                                                                                          |
| (60) | $x^{-\frac{1}{2}} I_{2\nu}(ax^{\frac{1}{2}}) \quad \text{Re } \nu > -\frac{1}{2}$                               | $\frac{\pi}{2y^{\frac{1}{2}}} \left[ I_\nu\left(\frac{a^2}{4y}\right) + \mathbf{L}_\nu\left(\frac{a^2}{4y}\right) \right]$<br>$\text{Re } y > 0$                                                                                                                                                                   |
| (61) | $x^{-\frac{1}{2}} [J_{2\nu}(ax^{\frac{1}{2}}) + I_{2\nu}(ax^{\frac{1}{2}})]$<br>$\text{Re } \nu > -\frac{1}{2}$ | $\frac{\pi}{y^{\frac{1}{2}}} I_\nu\left(\frac{a^2}{4y}\right) \quad \text{Re } y > 0$                                                                                                                                                                                                                              |
| (62) | $x^{-\frac{1}{2}} [I_{2\nu}(ax^{\frac{1}{2}}) - J_{2\nu}(ax^{\frac{1}{2}})]$<br>$\text{Re } \nu > -\frac{1}{2}$ | $\frac{\pi}{y^{\frac{1}{2}}} \mathbf{L}_\nu\left(\frac{a^2}{4y}\right) \quad \text{Re } y > 0$                                                                                                                                                                                                                     |
| (63) | $x^{-\frac{1}{2}} K_{2\nu}(ax^{\frac{1}{2}})$<br>$-\frac{1}{2} < \text{Re } \nu < \frac{1}{2}$                  | $\frac{\pi y^{-\frac{1}{2}}}{4 \cos(\nu\pi)} \left\{ K_\nu\left(\frac{a^2}{4y}\right) \right.$<br>$\left. + \frac{\pi}{2 \sin(\nu\pi)} \left[ \mathbf{L}_{-\nu}\left(\frac{a^2}{4y}\right) \right. \right.$<br>$\left. \left. - \mathbf{L}_\nu\left(\frac{a^2}{4y}\right) \right] \right\} \quad \text{Re } y > 0$ |

## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                                | $\int_0^\infty f(x) K_\nu(xy) (xy)^{\frac{1}{2}} dx$                                                                                                                                                                                                                                                                                                  |
|------|-----------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (64) | $x^{\nu+\frac{1}{2}} I_{2\nu}(ax^{\frac{1}{2}}) J_{2\nu}(ax^{\frac{1}{2}})$<br>$\operatorname{Re} \nu > -\frac{1}{2}$ | $\pi^{\frac{1}{2}} 2^{-\nu-1} a^{2\nu+1} y^{-2\nu-2}$<br>$\times J_{\nu-\frac{1}{2}}\left(\frac{a^2}{2y}\right) \quad \operatorname{Re} y > 0$                                                                                                                                                                                                        |
| (65) | $x^{\nu-\frac{1}{2}} I_{2\nu-1}(ax^{\frac{1}{2}}) J_{2\nu-1}(ax^{\frac{1}{2}})$<br>$\operatorname{Re} \nu > 0$        | $\pi^{\frac{1}{2}} 2^{-\nu} a^{2\nu-1} y^{-2\nu}$<br>$\times J_{\nu-\frac{1}{2}}\left(\frac{a^2}{2y}\right) \quad \operatorname{Re} y > 0$                                                                                                                                                                                                            |
| (66) | $x^{\nu-\frac{1}{2}} I_{2\nu-1}(ax^{\frac{1}{2}}) Y_{2\nu-1}(ax^{\frac{1}{2}})$<br>$\operatorname{Re} \nu > 0$        | $\pi^{\frac{1}{2}} 2^{-\nu-1} a^{2\nu-1} y^{-2\nu} \csc(\nu\pi)$<br>$\times \left[ \mathbf{H}_{\frac{1}{2}-\nu}\left(\frac{a^2}{2y}\right) \right.$<br>$\left. + \cos(\nu\pi) J_{\nu-\frac{1}{2}}\left(\frac{a^2}{2y}\right) \right.$<br>$\left. + \sin(\nu\pi) Y_{\nu-\frac{1}{2}}\left(\frac{a^2}{2y}\right) \right] \quad \operatorname{Re} y > 0$ |
| (67) | $x^{\nu-\frac{1}{2}} J_{2\nu-1}(ax^{\frac{1}{2}}) K_{2\nu-1}(ax^{\frac{1}{2}})$<br>$\operatorname{Re} \nu > 0$        | $\pi^{3/2} 2^{-\nu-2} a^{2\nu-1} y^{-2\nu} \csc(\nu\pi)$<br>$\times \left[ \mathbf{H}_{\frac{1}{2}-\nu}\left(\frac{a^2}{2y}\right) - Y_{\frac{1}{2}-\nu}\left(\frac{a^2}{2y}\right) \right]$<br>$\operatorname{Re} y > 0$                                                                                                                             |

## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                      | $\int_0^\infty f(x) K_\nu(xy) (xy)^{\frac{1}{2}} dx$                                                                                                                                                                                                     |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (68) | $x^{-\nu-\frac{1}{2}} I_{2\nu+1}(ax^{\frac{1}{2}})$<br>$\times J_{-2\nu-1}(ax^{\frac{1}{2}}) \quad \text{Re } \nu < \frac{1}{2}$                                                                            | $-\pi^{\frac{1}{2}} 2^\nu a^{-2\nu-1} y^{2\nu}$<br>$\times \left[ \cos(\nu\pi) \mathbf{H}_{\nu+\frac{1}{2}}\left(\frac{a^2}{2y}\right) \right.$<br>$\left. + \sin(\nu\pi) J_{\nu+\frac{1}{2}}\left(\frac{a^2}{2y}\right) \right] \quad \text{Re } y > 0$ |
| (69) | $x^{-\nu-\frac{1}{2}} I_{-2\nu-1}(ax^{\frac{1}{2}})$<br>$\times J_{2\nu+1}(ax^{\frac{1}{2}}) \quad \text{Re } \nu < \frac{1}{2}$                                                                            | $\pi^{\frac{1}{2}} 2^\nu a^{-2\nu-1} y^{2\nu}$<br>$\times \left[ \cos(\nu\pi) \mathbf{H}_{\nu+\frac{1}{2}}\left(\frac{a^2}{2y}\right) \right.$<br>$\left. - \sin(\nu\pi) J_{\nu+\frac{1}{2}}\left(\frac{a^2}{2y}\right) \right] \quad \text{Re } y > 0$  |
| (70) | $x^{\frac{1}{2}-\nu} [I_{2\nu}(ax^{\frac{1}{2}}) J_{-2\nu}(ax^{\frac{1}{2}})$<br>$- J_{2\nu}(ax^{\frac{1}{2}}) I_{-2\nu}(ax^{\frac{1}{2}})]$<br>$\text{Re } \nu < 3/2$                                      | $-\pi^{\frac{1}{2}} 2^\nu a^{1-2\nu} \sin(\nu\pi) y^{2\nu-2}$<br>$\times J_{\nu+\frac{1}{2}}\left(\frac{a^2}{2y}\right) \quad \text{Re } y > 0$                                                                                                          |
| (71) | $x^{-\frac{1}{2}} K_\mu(ax^{\frac{1}{2}})$<br>$\times [\sin(\frac{1}{2}\mu\pi) J_\mu(ax^{\frac{1}{2}})$<br>$+ \cos(\frac{1}{2}\mu\pi) Y_\mu(ax^{\frac{1}{2}})]$<br>$-1 < \text{Re } \mu < 1, \quad \nu = 0$ | $-\frac{\pi^2 y^{-\frac{1}{2}}}{16 \cos(\frac{1}{2}\mu\pi)} H_{\frac{1}{2}\mu}^{(1)}\left(\frac{a^2}{4y}\right)$<br>$\times H_{\frac{1}{2}\mu}^{(2)}\left(\frac{a^2}{4y}\right) \quad \text{Re } y > 0$                                                  |

**Higher transcendental functions (cont'd)**

|      | $f(x)$                                                                                                                                                                                                                                     | $\int_0^\infty f(x) K_\nu(xy) (xy)^{\frac{1}{2}} dx$                                                                                                                                                                                                                                                                                          |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (72) | $x^{-\frac{1}{2}} K_\mu(ax^{\frac{1}{2}})$<br>$\times \{ \sin[\frac{1}{2}(\mu-\nu)\pi] J_\mu(ax^{\frac{1}{2}})$<br>$+ \cos[\frac{1}{2}(\mu-\nu)\pi] Y_\mu(ax^{\frac{1}{2}}) \}$<br>$ \operatorname{Re} \mu  +  \operatorname{Re} \nu  < 1$ | $-\frac{1}{2} a^{-2} y^{\frac{1}{2}} \Gamma\left(\frac{1+\mu-\nu}{2}\right) \Gamma\left(\frac{1-\mu-\nu}{2}\right)$<br>$\times W_{\frac{1}{2}\nu, \frac{1}{2}\mu}\left(\frac{a^2}{2y} e^{\frac{1}{2}i\pi}\right)$<br>$\times W_{\frac{1}{2}\nu, \frac{1}{2}\mu}\left(\frac{a^2}{2y} e^{-\frac{1}{2}i\pi}\right)$<br>$\operatorname{Re} y > 0$ |
| (73) | $x^{\frac{1}{2}} \mathbf{H}_\nu(ax) \quad \operatorname{Re} \nu > -3/2$                                                                                                                                                                    | $a^{\nu+1} y^{-\nu-\frac{1}{2}} (y^2 + a^2)^{-1}$<br>$\operatorname{Re} y >  \operatorname{Im} a $                                                                                                                                                                                                                                            |
| (74) | $x^{\mu+\nu+\frac{1}{2}} \mathbf{H}_\mu(ax) \quad \operatorname{Re} \mu > -3/2$<br>$\operatorname{Re}(\mu + \nu) > -3/2$                                                                                                                   | $\pi^{-\frac{1}{2}} 2^{\mu+\nu+1} a^{\mu+1}$<br>$\times y^{-2\mu-\nu-5/2} \Gamma(\mu + \nu + 3/2)$<br>$\times {}_2F_1\left(\mu + \nu + \frac{3}{2}, 1; \frac{3}{2}; -\frac{a^2}{y^2}\right)$<br>$\operatorname{Re} y >  \operatorname{Im} a $                                                                                                 |
| (75) | $x^{\frac{1}{2}} \mathbf{H}_{\frac{1}{2}\nu}(ax^2)$<br>$a > 0, \quad \operatorname{Re} \nu > -2$                                                                                                                                           | $\frac{y^{\frac{1}{2}} \Gamma(1 + \frac{1}{2}\nu)}{2^{1-\frac{1}{2}\nu} a \pi} S_{-\frac{1}{2}\nu-1, \frac{1}{2}\nu}\left(\frac{y^2}{4a}\right)$<br>$\operatorname{Re} y > 0$                                                                                                                                                                 |
| (76) | $x^{3/2} \mathbf{H}_{\frac{1}{2}\nu+\frac{1}{2}}(ax^2)$<br>$a > 0, \quad \operatorname{Re} \nu > -3$                                                                                                                                       | $\frac{2^{\nu/2+1/2} y^{3/2}}{a^2 \pi} \Gamma\left(\frac{3+\nu}{2}\right)$<br>$\times S_{-\frac{\nu+5}{2}, \frac{\nu-1}{2}}\left(\frac{y^2}{4a}\right) \quad \operatorname{Re} y > 0$                                                                                                                                                         |

## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                                           | $\int_0^\infty f(x) K_\nu(xy) (xy)^{\frac{1}{2}} dx$                                                                                                                                                                                                                       |
|------|----------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (77) | $x^{5/2} \mathbf{H}_{\frac{1}{2}\nu}(ax^2)$<br>$a > 0, \quad \operatorname{Re} \nu > -3$                                         | $\frac{y^{5/2} \Gamma(2 + \frac{1}{2}\nu)}{2^{1-\frac{1}{2}\nu} a^3 \pi} S_{-3-\frac{1}{2}\nu, \frac{1}{2}\nu} \left( \frac{y^2}{4a} \right)$<br>$\operatorname{Re} y > 0$                                                                                                 |
| (78) | $x^{\frac{1}{2}} s_{\mu, \frac{1}{2}\nu}(ax^2) \quad a > 0$<br>$\operatorname{Re} \mu > \frac{1}{2}  \operatorname{Re} \nu  - 2$ | $\frac{y^{\frac{1}{2}}}{4a} \Gamma(\mu + \frac{1}{2}\nu + 1) \Gamma(\mu - \frac{1}{2}\nu + 1)$<br>$\times S_{-\mu-1, \frac{1}{2}\nu} \left( \frac{y^2}{4a} \right) \quad \operatorname{Re} y > 0$                                                                          |
| (79) | $x^{3/2} s_{\mu, \frac{1}{2}\nu + \frac{1}{2}}(ax^2) \quad a > 0$<br>$2 \operatorname{Re} \mu >  \operatorname{Re} \nu  - 5$     | $\frac{y^{3/2}}{8a^2} \left( \mu + \frac{3-\nu}{2} \right)$<br>$\times \Gamma\left(\mu + \frac{1-\nu}{2}\right) \Gamma\left(\mu + \frac{3+\nu}{2}\right)$<br>$\times S_{-\mu-2, \frac{1}{2}\nu - \frac{1}{2}} \left( \frac{y^2}{4a} \right) \quad \operatorname{Re} y > 0$ |
| (80) | $x^{5/2} s_{\mu, \frac{1}{2}\nu}(ax^2) \quad a > 0$<br>$\operatorname{Re} \mu > \frac{1}{2}  \operatorname{Re} \nu  - 3$         | $\frac{y^{5/2}}{16a^3} (2 + \mu + \frac{1}{2}\nu) (2 + \mu - \frac{1}{2}\nu)$<br>$\times \Gamma(1 + \mu + \frac{1}{2}\nu) \Gamma(1 + \mu - \frac{1}{2}\nu)$<br>$\times S_{-\mu-3, \frac{1}{2}\nu} \left( \frac{y^2}{4a} \right) \quad \operatorname{Re} y > 0$             |
| (81) | $D_{\nu-\frac{1}{2}}(ax^{-\frac{1}{2}}) D_{-\nu-\frac{1}{2}}(ax^{-\frac{1}{2}})$<br>$ \arg a  < \frac{1}{4}\pi$                  | $\frac{\pi}{2y} \exp[-a(2y)^{\frac{1}{2}}] \quad \operatorname{Re} y > 0$                                                                                                                                                                                                  |

**Higher transcendental functions (cont'd)**

|      | $f(x)$                                                                                                                                                                    | $\int_0^\infty f(x) K_\nu(xy) (xy)^{\frac{1}{2}} dx$                                                                                                                                                                                                                                                                                      |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (82) | $x^{2\mu+\nu-\frac{1}{2}} \exp(-\frac{1}{2}ax^2)$<br>$\times M_{\kappa,\mu}(ax^2) \quad \text{Re } a > 0$<br>$\text{Re } \mu > -\frac{1}{2}, \text{ Re } (2\mu+\nu) > -1$ | $2^{\mu-\kappa-\frac{1}{2}} a^{\frac{1}{4}-\frac{1}{2}(\mu+\nu+\kappa)} y^{\kappa-\mu-1}$<br>$\times \Gamma(2\mu+1) \Gamma(2\mu+\nu+1)$<br>$\times \exp\left(\frac{y^2}{8a}\right) W_{\kappa,m}\left(\frac{y^2}{4a}\right)$<br>$2k = -3\mu - \nu - \kappa - \frac{1}{2}$<br>$2m = \mu + \nu - \kappa + \frac{1}{2}$<br>$\text{Re } y > 0$ |
| (83) | $x^{-3/2} M_{\kappa,0}(iax^2) M_{\kappa,0}(-iax^2)$<br>$a > 0, \quad \nu = 0$                                                                                             | $\frac{\pi y^{\frac{1}{2}}}{16} \left\{ \left[ J_\kappa\left(\frac{y^2}{8a}\right) \right]^2 \right.$<br>$\left. + \left[ Y_\kappa\left(\frac{y^2}{8a}\right) \right]^2 \right\}$                                                                                                                                                         |
| (84) | $x^{-3/2} M_{\kappa,\mu}(iax^2) M_{\kappa,\mu}(-iax^2)$<br>$a > 0, \quad \text{Re } \mu > -\frac{1}{2}, \quad \nu = 0$                                                    | $ay^{-3/2} [\Gamma(2\mu+1)]^2 W_{-\mu,\kappa}\left(\frac{iy^2}{4a}\right)$<br>$\times W_{-\mu,\kappa}\left(-\frac{iy^2}{4a}\right) \quad \text{Re } y > 0$                                                                                                                                                                                |
| (85) | $x^{\frac{1}{2}} W_{\frac{1}{2}\nu,\mu}(a/x) W_{-\frac{1}{2}\nu,\mu}(a/x)$<br>$\text{Re } a > 0$                                                                          | $2ay^{-\frac{1}{2}} K_{2\mu}[(2ay)^{\frac{1}{2}} e^{\frac{1}{4}i\pi}]$<br>$\times K_{2\mu}[(2ay)^{\frac{1}{2}} e^{-\frac{1}{4}i\pi}] \quad \text{Re } y > 0$                                                                                                                                                                              |
| (86) | $x^{\nu+\frac{1}{2}} {}_2F_1(a, \beta; \nu+1; -\lambda^2 x^2)$<br>$\text{Re } \lambda > 0, \quad \text{Re } \nu > -1$                                                     | $2^{\nu+1} \lambda^{-\alpha-\beta} y^{\alpha+\beta-\nu-3/2} \Gamma(\nu+1)$<br>$\times S_{1,-\alpha-\beta,\alpha-\beta}(y/\lambda) \quad \text{Re } y > 0$                                                                                                                                                                                 |

## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                                                                                       | $\int_0^\infty f(x) K_\nu(xy)(xy)^{\frac{1}{2}} dx$                                                                                                                                                                                                                                                                                                               |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (87) | $x^{\nu+2\gamma-3/2}$<br>$\times {}_3F_2(1, \alpha, \beta; \gamma, \gamma+\nu; -\lambda^2 x^2)$<br>$\text{Re } \lambda > 0, \quad \text{Re } \gamma > 0$<br>$\text{Re } (\gamma + \nu) > 0$                                                                                                                  | $2^{\nu+2\gamma-2} \lambda^{-\alpha-\beta} \gamma^{\alpha+\beta-2\gamma-\nu+\frac{1}{2}}$<br>$\times \Gamma(\gamma) \Gamma(\gamma+\nu) S_{1-\alpha-\beta, \alpha-\beta}(\gamma/\lambda)$<br>$\text{Re } \gamma > 0$                                                                                                                                               |
| (88) | $x^{\mu-3/2}$<br>$\times {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; -\lambda x^2)$<br>$p \leq q-1, \quad \text{Re } \mu >  \text{Re } \nu $                                                                                                                                                 | $2^{\mu-2} \gamma^{\frac{1}{2}-\mu} \Gamma\left(\frac{\mu+\nu}{2}\right) \Gamma\left(\frac{\mu-\nu}{2}\right)$<br>$\times {}_{p+2}F_q\left(\alpha_1, \dots, \alpha_p, \frac{\mu+\nu}{2}, \frac{\mu-\nu}{2}; \right.$<br>$\left. \beta_1, \dots, \beta_q; \frac{4\lambda}{\gamma^2}\right)$<br>$\text{Re } \gamma > 0$                                             |
| (89) | $x^{\mu-3/2}$<br>$\times E(\alpha_1, \dots, \alpha_p; \rho_1, \dots, \rho_q; ax^{-2})$<br>$a > 0, \quad \text{Re } \mu >  \text{Re } \nu $                                                                                                                                                                   | $2^{\mu-2} a^{-\mu} \gamma^{\frac{1}{2}}$<br>$\times E(\alpha_1, \dots, \alpha_{p+2}; \rho_1, \dots, \rho_q; \frac{1}{4} a \gamma^2)$<br>$\alpha_{p+1} = \frac{\mu+\nu}{2}, \quad \alpha_{p+2} = \frac{\mu-\nu}{2}$<br>$\text{Re } \gamma > 0$                                                                                                                    |
| (90) | $G_{pq}^{mn} \left( \lambda x^2 \left  \begin{matrix} \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q \end{matrix} \right. \right)$<br>$p+q < 2(m+n)$<br>$ \arg \lambda  < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$<br>$\text{Re } \beta_j > \frac{1}{2} \text{Re } \nu  - \frac{3}{4}$<br>$j = 1, \dots, m$ | $2^{-3/2} \lambda^{-1/2}$<br>$\times G_{q, p+2}^{n+2, m} \left( \frac{\gamma^2}{4\lambda} \left  \begin{matrix} \frac{1}{2}-\beta_1, \dots, \frac{1}{2}-\beta_q \\ h, k, \frac{1}{2}-\alpha_1, \dots, \frac{1}{2}-\alpha_p \end{matrix} \right. \right)$<br>$h = \frac{1}{4} + \frac{1}{2}\nu, \quad k = \frac{1}{4} - \frac{1}{2}\nu$<br>$\text{Re } \gamma > 0$ |





## CHAPTER XI

### H-TRANSFORMS

We call

$$\int_0^\infty f(x) \mathbf{H}_\nu(xy)(xy)^{\frac{1}{2}} dx$$

the **H**-transform of order  $\nu$  of  $f(x)$  and regard  $y$  as a positive real variable. The inversion formula 11.1(1) was given by Titchmarsh (1937, p. 215). The **H**-transform is the reciprocal of the **Y**-transform (see chapter IX).

From the transform pairs given in this chapter further transform pairs may be derived by means of the methods indicated in the introduction to vol. I, and also by means of the general formulas of sec. 11.1. Moreover, **H**-transforms being reciprocal to **Y**-transforms when  $-\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$ , many further formulas may be obtained from the tables in chapter IX: the extension of such formulas by means of analytic continuation to a wider range of  $\operatorname{Re} \nu$  (the range of absolute convergence of the integral) is immediate.

### REFERENCE

Titchmarsh, E.C., 1937: *Introduction to the theory of Fourier integrals*. Oxford.



## H-TRANSFORMS

### 11.1. General formulas

|     |                                                                                                                                                                                |                                                                                                                                                              |
|-----|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------|
|     | $f(x)$                                                                                                                                                                         | $\int_0^\infty f(x) \mathbf{H}_\nu(xy)(xy)^{\frac{1}{2}} dx$<br>$= g(y; \nu)$<br>$y > 0$                                                                     |
| (1) | $\int_0^\infty g(y; \nu) Y_\nu(xy)(xy)^{\frac{1}{2}} dy$<br>$-\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                                                               | $g(y; \nu)$                                                                                                                                                  |
| (2) | $f(ax)$<br>$a > 0$                                                                                                                                                             | $a^{-1} g(a^{-1} y; \nu)$                                                                                                                                    |
| (3) | $x^m f(x)$<br>$m = 0, 1, 2, \dots$                                                                                                                                             | $y^{\frac{1}{2}-\nu} \left( \frac{d}{y dy} \right)^m [y^{\nu-\frac{1}{2}+m} g(y; \nu+m)]$                                                                    |
| (4) | $x^{\frac{1}{2}+\nu} \left( \frac{d}{x dx} \right)^m [x^{m-\nu-\frac{1}{2}} f(x)]$<br>$m = 0, 1, 2, \dots$                                                                     | $(-y)^m g(y; \nu-m)$                                                                                                                                         |
| (5) | $x^{\frac{1}{2}+\nu} \int_x^\infty \xi^{\frac{1}{2}-\nu-\mu} (\xi^2 - x^2)^{\mu-1}$<br>$\times f(\xi) d\xi$<br>$\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > -3/2$ | $2^{\mu-1} \Gamma(\mu) y^{-\mu} g(y; \nu+\mu)$                                                                                                               |
| (6) | $x^{-\mu} f(x)$<br>$\operatorname{Re} \nu + 3/2 > \operatorname{Re} \mu > 0$                                                                                                   | $2^{1-\mu} [\Gamma(\mu)]^{-1} y^{\frac{1}{2}-\nu}$<br>$\times \int_0^y \eta^{\frac{1}{2}-\mu+\nu} (y^2 - \eta^2)^{\mu-1}$<br>$\times g(\eta; \nu-\mu) d\eta$ |

## 11.2. Elementary functions

|     | $f(x)$                                                                                                               | $\int_0^\infty f(x) \mathbf{H}_\nu(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                             |
|-----|----------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $x^{-\frac{1}{2}} \quad -2 < \operatorname{Re} \nu < 0$                                                              | $-\operatorname{ctn}(\frac{1}{2}\nu\pi) y^{-\frac{1}{2}}$                                                                                                                                                                             |
| (2) | $x^{\nu+\frac{1}{2}} \quad 0 < x < a$<br>$0 \quad a < x < \infty$<br>$\operatorname{Re} \nu > -3/2$                  | $a^{\nu+1} y^{-\frac{1}{2}} \mathbf{H}_{\nu+1}(ay)$                                                                                                                                                                                   |
| (3) | $x^{\frac{1}{2}-\nu} \quad 0 < x < a$<br>$0 \quad a < x < \infty$                                                    | $\frac{ay^{\nu-\frac{1}{2}}}{2^{\nu-1} \pi^{\frac{1}{2}} \Gamma(\nu+\frac{1}{2})}$<br>$-a^{1-\nu} y^{-\frac{1}{2}} \mathbf{H}_{\nu-1}(ay)$                                                                                            |
| (4) | $x^{\lambda-\frac{1}{2}} \quad \operatorname{Re} \lambda < \frac{1}{2}$<br>$-2 < \operatorname{Re}(\lambda+\nu) < 0$ | $2^\lambda y^{-\lambda-\frac{1}{2}} \tan[\frac{1}{2}(\lambda+\nu+1)\pi]$<br>$\times \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\lambda+\frac{1}{2}\nu)}{\Gamma(\frac{1}{2}-\frac{1}{2}\lambda+\frac{1}{2}\nu)}$                              |
| (5) | $x^{\lambda-\frac{1}{2}} \quad 0 < x < a$<br>$0 \quad a < x < \infty$<br>$\operatorname{Re}(\lambda+\nu) > -2$       | $\frac{a^{\lambda+\nu+2} y^{\nu+3/2}}{2^\nu \pi^{\frac{1}{2}} \Gamma(\nu+3/2) (\lambda+\nu+2)}$<br>$\times {}_2F_3\left(1, \frac{\lambda+\nu}{2}+1; \frac{3}{2}, \nu+\frac{3}{2}, \frac{\lambda+\nu}{2}+2; -\frac{a^2 y^2}{4}\right)$ |
| (6) | $x^{-\frac{1}{2}} (x^2 + a^2)^{-1} \quad \operatorname{Re} a > 0, \quad \nu = 1$                                     | $\frac{\pi y^{\frac{1}{2}}}{2a} [I_1(ay) - \mathbf{L}_1(ay)]$                                                                                                                                                                         |

## Elementary functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                                | $\int_0^\infty f(x) \mathbf{H}_\nu(xy) (xy)^{\frac{1}{2}} dx \quad \gamma > 0$                                                                                                                                                                                                                                                       |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (7)  | $x^{-\frac{1}{2}} (x^2 + a^2)^{-1}$<br>$\operatorname{Re} a > 0, \quad -2 < \operatorname{Re} \nu < 2$                                                                                                                | $-\frac{\pi \gamma^{1/2}}{2 a \sin(\frac{1}{2} \nu \pi)} \mathbf{L}_\nu(a\gamma)$ $+ \frac{\gamma^{3/2} \operatorname{ctn}(\frac{1}{2} \nu \pi)}{1 - \nu^2}$ $\times {}_1F_2\left(1; \frac{3-\nu}{2}, \frac{3+\nu}{2}; \frac{a^2 \gamma^2}{4}\right)$                                                                                |
| (8)  | $x^{\nu+\frac{1}{2}} (x^2 + a^2)^{\mu-1}$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -3/2$<br>$\operatorname{Re}(\mu + \nu) < 1/2$<br>$\operatorname{Re}(2\mu + \nu) < 3/2$                           | $\frac{2^{\mu-1} \pi a^{\mu+\nu} \gamma^{\frac{1}{2}-\mu}}{\Gamma(1-\mu) \cos[(\mu + \nu) \pi]}$ $\times [I_{-\mu-\nu}(a\gamma) - \mathbf{L}_{\mu+\nu}(a\gamma)]$                                                                                                                                                                    |
| (9)  | $x^{\frac{1}{2}-\nu} (x^2 + a^2)^{\mu-1}$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \mu < 1/2$<br>$\operatorname{Re}(2\mu - \nu) < 3/2$                                                                    | $\frac{2^{\mu-1} \pi a^{\mu-\nu} \gamma^{\frac{1}{2}-\mu}}{\Gamma(1-\mu) \cos(\mu\pi)} I_{\nu-\mu}(a\gamma)$ $+ \frac{a^{2\mu+1} \gamma^{\nu+3/2} \Gamma(-1/2-\mu)}{2^{\nu+2} \Gamma(1-\mu) \Gamma(\nu+3/2)}$ $\times {}_1F_2\left(1; \mu + \frac{3}{2}, \nu + \frac{3}{2}; \frac{a^2 \gamma^2}{4}\right)$                           |
| (10) | $x^{\lambda-\frac{1}{2}} (x^2 + a^2)^{\mu-1}$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re}(\lambda + \nu) > -2$<br>$\operatorname{Re}(\lambda + 2\mu) < 5/2$<br>$\operatorname{Re}(\lambda + 2\mu + \nu) < 2$ | $2^{-1/2} [\Gamma(1-\mu)]^{-1} a^{\lambda+2\mu-3/2}$ $\times G_{24}^{22}\left(\frac{a^2 \gamma^2}{4} \left  \begin{matrix} l, m \\ l, m-\mu, h, k \end{matrix} \right.\right)$ $h = \frac{1}{4} + \frac{\nu}{2}, \quad k = \frac{1}{4} - \frac{\nu}{2}$ $l = \frac{3}{4} + \frac{\nu}{2}, \quad m = \frac{3}{4} - \frac{\lambda}{2}$ |

## Elementary functions (cont'd)

|      | $f(x)$                                                                                                                                                   | $\int_0^\infty f(x) \mathbf{H}_\nu(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                         |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (11) | $(x^2 + a^2)^{-\frac{1}{2}}$<br>$\times [x + (x^2 + a^2)^{\frac{1}{2}}]^{\nu+1}$<br>$\text{Re } a > 0, \quad -2 < \text{Re } \nu < 0$                    | $\frac{\pi^{\frac{1}{2}} a^{\nu+\frac{1}{2}}}{y^{\frac{1}{2}} \sin(\nu\pi)} [\sinh(\frac{1}{2}ay) I_{\nu+\frac{1}{2}}(\frac{1}{2}ay)$<br>$- \cosh(\frac{1}{2}ay) I_{-\nu-\frac{1}{2}}(\frac{1}{2}ay)]$                                                                                                                                            |
| (12) | $x^{\nu+\frac{1}{2}} (a^2 - x^2)^{\mu-1} \quad 0 < x < a$<br>$0 \quad a < x < \infty$<br>$\text{Re } \mu > 0, \quad \text{Re } \nu > -3/2$               | $2^{\mu-1} a^{\mu+\nu} y^{\frac{1}{2}-\mu} \Gamma(\mu) \mathbf{H}_{\mu+\nu}(ay)$                                                                                                                                                                                                                                                                  |
| (13) | $x^{\lambda-\frac{1}{2}} (a^2 - x^2)^{\mu-1} \quad 0 < x < a$<br>$0 \quad a < x < \infty$<br>$\text{Re } \mu > 0, \quad \text{Re } (\lambda + \nu) > -2$ | $\frac{a^{2\mu+\nu+\lambda} y^{\nu+3/2} \Gamma(\mu) \Gamma(\frac{\lambda+\nu}{2} + 1)}{2^{\nu+1} \pi^{1/2} \Gamma(\nu + 3/2) \Gamma(\frac{\lambda+\nu}{2} + \mu + 1)}$<br>$\times {}_2F_3\left(1, \frac{\lambda + \nu}{2} + 1; \frac{3}{2}, \nu + \frac{3}{2}, \right.$<br>$\left. \frac{\lambda + \nu}{2} + \mu + 1; -\frac{a^2 y^2}{4} \right)$ |
| (14) | $0 \quad 0 < x < a$<br>$x^{-\nu-\frac{1}{2}} (x^2 - a^2)^{-\nu-\frac{1}{2}} \quad a < x < \infty$<br>$-\frac{1}{2} < \text{Re } \nu < 1$                 | $2^{-\nu-1} \pi^{\frac{1}{2}} a^{-2\nu} y^{\nu+\frac{1}{2}}$<br>$\times \Gamma(\frac{1}{2} - \nu) [J_\nu(\frac{1}{2}ay)]^2$                                                                                                                                                                                                                       |
| (15) | $0 \quad 0 < x < a$<br>$x^{\nu+\frac{1}{2}} (x^2 - a^2)^m \quad a < x < \infty$<br>$m = 0, 1, 2, \dots,$<br>$\text{Re } \nu < -2m - \frac{1}{2}$         | $(-1)^{m+1} 2^m a^{m+\nu+1} y^{-m-\frac{1}{2}} m!$<br>$\times \mathbf{H}_{\nu+m+1}(ay)$                                                                                                                                                                                                                                                           |

## Elementary functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                                   | $\int_0^\infty f(x) \mathbf{H}_\nu(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                    |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (16) | $0 \quad 0 < x < a$<br>$x^{\nu+\frac{1}{2}} (x^2 - a^2)^{\mu-1} \quad a < x < \infty$<br>$\operatorname{Re} \mu > 0, \quad \operatorname{Re}(\mu + \nu) < \frac{1}{2}$<br>$\operatorname{Re}(2\mu + \nu) < 3/2$          | $2^{\mu-1} a^{\mu+\nu} y^{\frac{1}{2}-\mu} \Gamma(\mu) \sec[(\mu+\nu)\pi]$<br>$\times [\sin(\mu\pi) J_{-\mu-\nu}(ay)$<br>$+ \cos(\nu\pi) \mathbf{H}_{\mu+\nu}(ay)]$                                                                                                                                                          |
| (17) | $0 \quad 0 < x < a$<br>$x^{-\nu-\frac{1}{2}} (x^2 - a^2)^\mu \quad a < x < \infty$<br>$-1 < \operatorname{Re} \mu < 0$<br>$\operatorname{Re} \nu > 2 \operatorname{Re} \mu - \frac{1}{2}$                                | $-\pi 2^{2\mu-\nu} y^{\nu-2\mu-\frac{1}{2}}$<br>$\times [\Gamma(\frac{1}{2}-\mu) \Gamma(\frac{1}{2}+\nu-\mu) \sin(\mu\pi)]^{-1}$<br>$\times {}_1F_2(-\mu; \frac{1}{2}-\mu, \frac{1}{2}+\nu-\mu; -\frac{1}{4}a^2y^2)$                                                                                                         |
| (18) | $0 \quad 0 < x < a$<br>$x^{\lambda-\frac{1}{2}} (x^2 - a^2)^{\mu-1} \quad a < x < \infty$<br>$\operatorname{Re} \mu > 0, \quad \operatorname{Re}(\lambda + 2\mu) < 5/2$<br>$\operatorname{Re}(\lambda + 2\mu + \nu) < 2$ | $2^{-1/2} \Gamma(\mu) a^{2\mu+\lambda-3/2}$<br>$\times G_{24}^{21} \left( \frac{a^2 y^2}{4} \middle  \begin{matrix} l, m \\ l, m-\mu, h, k \end{matrix} \right)$<br>$h = \frac{1}{4} + \frac{\nu}{2}, \quad k = \frac{1}{4} - \frac{\nu}{2}$<br>$l = \frac{3}{4} + \frac{\nu}{2}, \quad m = \frac{3}{4} - \frac{\lambda}{2}$ |
| (19) | $x^{\lambda-\frac{1}{2}} e^{-ax}$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re}(\lambda + \nu) > -2$                                                                                                              | $\frac{y^{\nu+3/2} \Gamma(\lambda + \nu + 2)}{2^\nu a^{\lambda+\nu+2} \pi^{1/2} \Gamma(\nu + 3/2)}$<br>$\times {}_3F_2 \left( 1, \frac{\lambda + \nu}{2} + 1, \frac{\lambda + \nu + 3}{2}; \right.$<br>$\left. \frac{3}{2}, \nu + \frac{3}{2}; -\frac{y^2}{a^2} \right)$                                                     |

## Elementary functions (cont'd)

|      | $f(x)$                                                                                                                                                                               | $\int_0^\infty f(x) \mathbf{H}_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                 |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (20) | $x^{\lambda+\frac{1}{2}} \exp(-ax^2)$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re}(\lambda + \nu) > -3$                                                                      | $2^{-\nu-1} \pi^{-1/2} a^{-(\lambda+\nu+3)/2} y^{\nu+3/2}$<br>$\times \frac{\Gamma(\frac{\lambda+\nu+3}{2})}{\Gamma(\nu+3/2)}$<br>$\times {}_2F_2\left(1, \frac{\lambda+\nu+3}{2}; \frac{3}{2}, \nu+\frac{3}{2}; -\frac{y^2}{4a}\right)$ |
| (21) | $x^{-\nu-\frac{1}{2}} \sin(ax)$<br>$a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$                                                                                               | $0 \quad 0 < y < a$<br>$\pi^{\frac{1}{2}} 2^{-\nu} y^{\frac{1}{2}-\nu} [\Gamma(\nu+\frac{1}{2})]^{-1}$<br>$\times (y^2 - a^2)^{\nu-\frac{1}{2}} \quad a < y < \infty$                                                                    |
| (22) | $x^{\frac{1}{2}} \cos[(\nu+1)\theta]/\sin\theta$<br>$0 < x < a$<br>$0 \quad a < x < \infty$<br>$0 < \theta < \frac{1}{2}\pi, \quad x = a \cos\theta$<br>$\operatorname{Re} \nu > -2$ | $\pi^{\frac{1}{2}} a^{\frac{1}{2}} \sin(\frac{1}{2}ay) J_{\nu+\frac{1}{2}}(\frac{1}{2}ay)$                                                                                                                                               |

## 11.3. Higher transcendental functions

|     |                                                                              |                                                                                                                                                             |
|-----|------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $J_{\nu+\frac{1}{2}}(ax)$<br>$a > 0, \quad -3/2 < \operatorname{Re} \nu < 1$ | $0 \quad 0 < y < a$<br>$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\frac{y}{a}\right)^{\nu+\frac{1}{2}} (y^2 - a^2)^{-\frac{1}{2}}$<br>$a < y < \infty$ |
|-----|------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------|



## Higher transcendental functions (cont'd)

|     | $f(x)$                                                                                                                                                                                                                 | $\int_0^\infty f(x) \mathbf{H}_\nu(xy) (xy)^{\frac{1}{2}} dx$                                                                                                            | $y > 0$                           |
|-----|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------|
| (2) | $x^{-\frac{1}{2}} Y_{\nu+1}(ax)$<br>$a > 0, \quad -3/2 < \operatorname{Re} \nu < 3/2$                                                                                                                                  | 0<br>$-a^{-\nu-1} y^{\frac{1}{2}+\nu}$                                                                                                                                   | $0 < y < a$<br>$a < y < \infty$   |
| (3) | $x^{\mu-\nu+\frac{1}{2}} Y_\mu(ax)$<br>$a > 0, \quad \operatorname{Re}(\nu - \mu) > 0$<br>$-3/2 < \operatorname{Re} \mu < 1/2$                                                                                         | 0<br>$\frac{2^{1+\mu-\nu} a^\mu}{\Gamma(\nu - \mu)} y^{\frac{1}{2}-\nu} (y^2 - a^2)^{\nu-\mu-1}$                                                                         | $0 < y < a$<br>$a < y < \infty$   |
| (4) | $x^{\frac{1}{2}-\mu} [\sin(\mu\pi) J_{\mu+\nu}(ax)$<br>$+ \cos(\mu\pi) Y_{\mu+\nu}(ax)]$<br>$a > 0, \quad 1 < \operatorname{Re} \mu < 3/2$<br>$\operatorname{Re} \nu > -3/2, \quad \operatorname{Re}(\nu - \mu) < 1/2$ | 0<br>$\frac{y^{\frac{1}{2}+\nu} (y^2 - a^2)^{\mu-1}}{2^{\mu-1} a^{\mu+\nu} \Gamma(\mu)}$                                                                                 | $0 < y < a$<br>$a < y < \infty$   |
| (5) | $x^{\nu+\frac{1}{2}} J_\nu(ax) Y_\nu(ax)$<br>$a > 0, \quad -\frac{3}{4} < \operatorname{Re} \nu < 0$                                                                                                                   | $\frac{\Gamma(2\nu + 3/2) y^{\nu+3/2}}{\pi^{3/2} 2^{\nu+2} a^{2\nu+3} \Gamma(\nu + 2)}$<br>$\times {}_2F_1\left(1, 2\nu + \frac{3}{2}; \nu + 2; \frac{y^2}{4a^2}\right)$ | $0 < y < 2a$                      |
| (6) | $x^{\nu+\frac{1}{2}} \{[J_\nu(ax)]^2 - [Y_\nu(ax)]^2\}$<br>$a > 0, \quad -\frac{3}{4} < \operatorname{Re} \nu < 0$                                                                                                     | 0<br>$\frac{2^{3\nu+2} a^{2\nu} y^{-\nu-\frac{1}{2}}}{\pi^{\frac{1}{2}} \Gamma(\frac{1}{2} - \nu)} (y^2 - 4a^2)^{-\nu-\frac{1}{2}}$                                      | $0 < y < 2a$<br>$2a < y < \infty$ |

## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                                         | $\int_0^\infty f(x) \mathbf{H}_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                              |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (7)  | $x^{\frac{1}{2}} \{ [J_{\frac{1}{2}\nu}(ax)]^2 - [Y_{\frac{1}{2}\nu}(ax)]^2 \}$<br>$a > 0, \quad -3/2 < \text{Re } \nu < 0$                                                                                                    | $0 \quad 0 < y < 2a$<br>$4\pi^{-1} y^{-\frac{1}{2}} (y^2 - 4a^2)^{-\frac{1}{2}} \quad 2a < y < \infty$                                                                                                                                |
| (8)  | $x^{\frac{1}{2}} [J_{\frac{1}{2}\nu+\frac{1}{2}\mu}(ax) J_{\frac{1}{2}\nu-\frac{1}{2}\mu}(ax)$<br>$- Y_{\frac{1}{2}\nu+\frac{1}{2}\mu}(ax) Y_{\frac{1}{2}\nu-\frac{1}{2}\mu}(ax)]$<br>$a > 0, \quad -3/2 < \text{Re } \nu < 0$ | $0 \quad 0 < y < 2a$<br>$4\pi^{-1} y^{-\frac{1}{2}} (y^2 - 4a^2)^{-\frac{1}{2}} \cosh(\mu u)$<br>$y = 2a \cosh u, \quad u > 0$                                                                                                        |
| (9)  | $J_{2\nu+1}(ax^{\frac{1}{2}})$<br>$a > 0, \quad -3/2 < \text{Re } \nu < 1/4$                                                                                                                                                   | $-\frac{a}{2y^{3/2}} Y_{\nu+1}\left(\frac{a^2}{4y}\right)$                                                                                                                                                                            |
| (10) | $x^{-\frac{1}{2}} J_{2\nu}(ax^{\frac{1}{2}})$<br>$a > 0, \quad -1 < \text{Re } \nu < 5/4$                                                                                                                                      | $-y^{-\frac{1}{2}} Y_\nu\left(\frac{a^2}{4y}\right)$                                                                                                                                                                                  |
| (11) | $x^{\frac{1}{2}} \{ [J_{\frac{1}{2}\nu}[b(z-a)]]^2$<br>$- [Y_{\frac{1}{2}\nu}[b(z+a)]]^2 \}$<br>$\text{Re } a > 0, \quad b > 0$<br>$-3/2 < \text{Re } \nu < 1$<br>$z = (x^2 + a^2)^{\frac{1}{2}}$                              | $-\frac{4 \sin[a(4b^2 - y^2)^{\frac{1}{2}}]}{\pi y^{\frac{1}{2}} (4b^2 - y^2)^{\frac{1}{2}}} \quad 0 < y < 2b$<br>$\frac{4 \exp[-a(y^2 - 4b^2)^{\frac{1}{2}}]}{\pi y^{\frac{1}{2}} (y^2 - 4b^2)^{\frac{1}{2}}} \quad 2b < y < \infty$ |
| (12) | $x^{\frac{1}{2}} K_\nu(ax)$<br>$\text{Re } a > 0, \quad \text{Re } \nu > -3/2$                                                                                                                                                 | $a^{-\nu-1} y^{\nu+3/2} (y^2 + a^2)^{-1}$                                                                                                                                                                                             |

## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                                                | $\int_0^\infty f(x) \mathbf{H}_\nu(xy) (xy)^{1/2} dx \quad y > 0$                                                                                                                                                                                                                                                   |
|------|---------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (13) | $x^{\mu+\nu+1/2} K_\mu(ax)$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -3/2$<br>$\operatorname{Re}(\mu + \nu) > -3/2$ | $2^{\mu+\nu+1} \pi^{-1/2} a^{-\mu-2\nu-3}$<br>$\times y^{\nu+3/2} \Gamma(\mu + \nu + 3/2)$<br>$\times {}_2F_1\left(1, \mu + \nu + \frac{3}{2}; \frac{3}{2}; -\frac{y^2}{a^2}\right)$                                                                                                                                |
| (14) | $x^{\mu-\nu+1/2} K_\mu(ax)$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \mu > -3/2$                                          | $\frac{2^{\mu-\nu} y^{\nu+3/2} \Gamma(\mu + 3/2)}{a^{\mu+3} \Gamma(\nu + 3/2)}$<br>$\times {}_2F_1\left(1, \mu + \frac{3}{2}; \nu + \frac{3}{2}; -\frac{y^2}{a^2}\right)$                                                                                                                                           |
| (15) | $x^{\sigma-1/2} K_\mu(ax) \quad \operatorname{Re} a > 0$<br>$\operatorname{Re}(\sigma + \nu) >  \operatorname{Re} \mu  - 2$           | $2^\sigma \pi^{-1/2} a^{-\nu-\sigma-2} y^{\nu+3/2}$<br>$\times \frac{\Gamma(1 + \frac{\nu+\sigma+\mu}{2}) \Gamma(1 + \frac{\nu+\sigma-\mu}{2})}{\Gamma(\nu + 3/2)}$<br>$\times {}_3F_2\left(1, 1 + \frac{\nu+\sigma+\mu}{2}, 1 + \frac{\nu+\sigma-\mu}{2}; \frac{3}{2}, \nu + \frac{3}{2}; -\frac{y^2}{a^2}\right)$ |
| (16) | $x^{\sigma-1/2} e^{-ax} K_\nu(ax)$                                                                                                    | see Mohan, Brij, 1942: <i>Bull. Calcutta Math. Soc.</i> 34, 55-59.                                                                                                                                                                                                                                                  |
| (17) | $x^{\nu+1/2} [K_\nu(ax)]^2$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -3/4$                                          | $\pi^{1/2} 2^{-\nu-3} a^{-2\nu-3} y^{\nu+3/2}$<br>$\times \Gamma(2\nu + 3/2) [\Gamma(\nu + 2)]^{-1}$<br>$\times {}_2F_1\left(1, 2\nu + \frac{3}{2}; \nu + 2; -\frac{y^2}{4a^2}\right)$                                                                                                                              |

**Higher transcendental functions (cont'd)**

|      | $f(x)$                                                                                                                                                             | $\int_0^\infty f(x) \mathbf{H}_\nu(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (18) | $x^{\frac{1}{2}} [K_\mu(ax)]^2 \quad \nu = 0$<br>$\operatorname{Re} a > 0, \quad -3/2 < \operatorname{Re} \mu < 3/2$                                               | $-\pi 2^{-\mu-1} a^{-2\mu} y^{-\frac{1}{2}} z^{-1} \sec(\mu\pi)$<br>$\times [(z+y)^{2\mu} + (z-y)^{2\mu}]$<br>$z = (y^2 + 4a^2)^{\frac{1}{2}}$                                                                                                                                                                                                                                                                                                                                                                                                                                                         |
| (19) | $x^{-\nu-\frac{1}{2}} K_0(ax) K_1(ax)$<br>$\operatorname{Re} a > 0$                                                                                                | $\frac{\pi^{3/2} y^{\nu+3/2}}{2^{\nu+3} a^2 \Gamma(\nu+3/2)}$<br>$\times {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \nu + \frac{3}{2}; -\frac{y^2}{4a^2}\right)$                                                                                                                                                                                                                                                                                                                                                                                                                                           |
| (20) | $x^{-\nu-\frac{1}{2}} K_\nu(ax) K_{\nu+1}(ax)$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu < \frac{1}{2}$                                             | $\pi^{1/2} 2^{-\nu-2} a^{-2} y^{\nu+3/2} \Gamma(\frac{1}{2} - \nu)$<br>$\times {}_2F_1\left(\frac{1}{2}, \frac{1}{2} - \nu; \frac{3}{2}; -\frac{y^2}{4a^2}\right)$                                                                                                                                                                                                                                                                                                                                                                                                                                     |
| (21) | $x^{\sigma-5/2} K_\lambda(ax) K_\mu(ax)$<br>$\operatorname{Re} a > 0$<br>$\operatorname{Re}(\sigma + \nu) >  \operatorname{Re} \lambda  +  \operatorname{Re} \mu $ | $\frac{2^{\sigma-3} y^{\nu+3/2} \Gamma(\frac{\sigma+\nu+\lambda+\mu}{2})}{\pi^{1/2} a^{\sigma+\nu} \Gamma(\nu+3/2) \Gamma(\sigma+\nu)}$<br>$\times \Gamma\left(\frac{\sigma+\nu+\lambda-\mu}{2}\right) \Gamma\left(\frac{\sigma+\nu-\lambda+\mu}{2}\right)$<br>$\times \Gamma\left(\frac{\sigma+\nu-\lambda-\mu}{2}\right)$<br>$\times {}_5F_4\left(1, \frac{\sigma+\nu+\lambda+\mu}{2}, \frac{\sigma+\nu+\lambda-\mu}{2}, \frac{\sigma+\nu-\lambda+\mu}{2}, \frac{\sigma+\nu-\lambda-\mu}{2}; \frac{3}{2}, \nu + \frac{3}{2}, \frac{\sigma+\nu}{2}, \frac{\sigma+\nu+1}{2}; -\frac{y^2}{4a^2}\right)$ |

## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                                                                                        | $\int_0^\infty f(x) \mathbf{H}_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                                           |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (22) | $x^{\nu+3/2} K_\nu(ax) K_{\nu+1}(ax)$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -5/4$                                                                        | $\pi^{1/2} 2^{-\nu-3} a^{-2\nu-4} y^{\nu+3/2}$<br>$\times \Gamma(2\nu+5/2) [\Gamma(\nu+2)]^{-1}$<br>$\times {}_2F_1\left(1, 2\nu+\frac{5}{2}; \nu+2; -\frac{y^2}{4a^2}\right)$                                                                                                                                                                                                                     |
| (23) | $x^{\sigma-5/2} \exp(-\frac{1}{2}a^2 x^2)$<br>$\times K_\mu(\frac{1}{2}a^2 x^2)$<br>$ \arg a  < \frac{1}{4}\pi$<br>$\operatorname{Re}(\sigma+\nu) > 2 \operatorname{Re} \mu $ | $\pi^{1/2} 2^{-\nu-2} a^{-\nu-\sigma} y^{\nu+3/2}$<br>$\times \frac{\Gamma(\frac{\nu+\sigma}{2}+\mu) \Gamma(\frac{\nu+\sigma}{2}-\mu)}{\Gamma(\frac{3}{2}) \Gamma(\nu+\frac{3}{2}) \Gamma(\frac{\nu+\sigma}{2})}$<br>$\times {}_3F_3\left(1, \frac{\nu+\sigma}{2}+\mu, \frac{\nu+\sigma}{2}-\mu; \frac{3}{2}, \right.$<br>$\left. \nu+\frac{3}{2}, \frac{\nu+\sigma}{2}; -\frac{y^2}{4a^2}\right)$ |
| (24) | $x^{\frac{1}{2}} \exp\left(\frac{a^2 x^2}{8}\right) K_{\frac{1}{2}\nu}\left(\frac{a^2 x^2}{8}\right)$<br>$ \arg a  < \frac{3}{4}\pi$<br>$-3/2 < \operatorname{Re} \nu < 0$    | $2\pi^{-\frac{1}{2}} a^{-\frac{1}{2}\nu-1} y^{\frac{1}{2}\nu-\frac{1}{2}}$<br>$\times \cos(\frac{1}{2}\nu\pi) \Gamma(-\frac{1}{2}\nu)$<br>$\times \exp\left(\frac{y^2}{2a^2}\right) W_{k,m}\left(\frac{y^2}{a^2}\right)$<br>$k = \frac{1}{4}\nu, \quad m = \frac{1}{2} + \frac{1}{4}\nu$                                                                                                           |

## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                                                                                  | $\int_0^\infty f(x) \mathbf{H}_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                                                                                                                                    |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (25) | $x^\sigma \exp(ax^2) K_\mu(ax^2)$<br>$ \arg a  < 3\pi/2, \quad \operatorname{Re} \sigma < 1$<br>$ \operatorname{Re} \mu  - 5/2 < \operatorname{Re}(\sigma + \nu) < 1/2$ | $\pi^{-\frac{1}{2}} 2^{-1-\frac{1}{2}\sigma} a^{-\frac{1}{2}-\frac{1}{2}\sigma} \cos(\mu\pi)$<br>$\times G_{34}^{23} \left( \frac{y^2}{8a} \left  \begin{matrix} l, \frac{1-\sigma}{2} + \mu, \frac{1-\sigma}{2} - \mu \\ l, -\frac{\sigma}{2}, h, k \end{matrix} \right. \right)$<br>$h = \frac{1}{4} + \frac{\nu}{2}, \quad k = \frac{1}{4} - \frac{\nu}{2}, \quad l = \frac{3}{4} + \frac{\nu}{2}$                                                                                       |
| (26) | $K_{2\nu-1}(2ax^{\frac{1}{2}})$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1$                                                                          | $2^{\nu+1} \pi^{-1} a y^{-3/2} \Gamma(\nu+1)$<br>$\times S_{-\nu-2, \nu-1}(a^2/y)$                                                                                                                                                                                                                                                                                                                                                                                                          |
| (27) | $x^{-\frac{1}{2}} K_{2\nu}(2ax^{\frac{1}{2}})$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1$                                                           | $2^\nu \pi^{-1} y^{-\frac{1}{2}} \Gamma(\nu+1) S_{-\nu-1, \nu}(a^2/y)$                                                                                                                                                                                                                                                                                                                                                                                                                      |
| (28) | $x^{\frac{1}{2}} K_{2\nu}(2ax^{\frac{1}{2}})$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -2$                                                            | $2^\nu \pi^{-1} a^2 y^{-5/2} \Gamma(\nu+2)$<br>$\times S_{-\nu-3, \nu}(a^2/y)$                                                                                                                                                                                                                                                                                                                                                                                                              |
| (29) | $x^\sigma K_\mu(2ax^{\frac{1}{2}})$<br>$\operatorname{Re} a > 0$<br>$2\operatorname{Re}(\sigma + \nu) >  \operatorname{Re} \mu  - 5$                                    | $2^{2\sigma-\frac{1}{2}} a^{-2\sigma-2} \pi^{-1}$<br>$\times G_{53}^{15} \left( \frac{4y^2}{a^4} \left  \begin{matrix} l, \beta_1, \dots, \beta_4 \\ l, h, k \end{matrix} \right. \right)$<br>$h = \frac{1}{4} + \frac{\nu}{2}, \quad k = \frac{1}{4} - \frac{\nu}{2}, \quad l = \frac{3}{4} + \frac{\nu}{2}$<br>$2\beta_1 = 1 - \sigma + \frac{1}{2}\mu, \quad 2\beta_2 = 1 - \sigma - \frac{1}{2}\mu$<br>$2\beta_3 = -\sigma + \frac{1}{2}\mu, \quad 2\beta_4 = -\sigma - \frac{1}{2}\mu$ |

## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                                                 | $\int_0^\infty f(x) \mathbf{H}_\nu(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                         |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (30) | $x^{-\frac{1}{2}} [2\pi^{-1} K_{2\nu}(2ax^{\frac{1}{2}})$<br>$+ Y_{2\nu}(2ax^{\frac{1}{2}})]$<br>$a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                                                                                                     | $y^{-\frac{1}{2}} J_\nu(a^2/y)$                                                                                                                                   |
| (31) | $x^{\frac{1}{2}-\nu} [J_{2\nu}(ax^{\frac{1}{2}})$<br>$- J_{-2\nu}(ax^{\frac{1}{2}})] K_{2\nu}(ax^{\frac{1}{2}})$<br>$ \arg a  < \frac{1}{4}\pi$<br>$-3/2 < \operatorname{Re} \nu < 3/2$                                                                                | $2^\nu \pi^{-\frac{1}{2}} a^{1-2\nu} y^{2\nu-2}$<br>$\times \sin(\nu\pi) K_{\nu+\frac{1}{2}}\left(\frac{a^2}{2y}\right)$                                          |
| (32) | $x^{\frac{1}{2}} Y_\nu(ax^{\frac{1}{2}}) K_\nu(ax^{\frac{1}{2}})$<br>$ \arg a  < \frac{1}{4}\pi, \quad \operatorname{Re} \nu > -3/2$                                                                                                                                   | $\frac{1}{2y^{3/2}} \exp\left(-\frac{a^2}{2y}\right)$                                                                                                             |
| (33) | $x^{\nu-\frac{1}{2}} Y_{2\nu-1}(ax^{\frac{1}{2}}) K_{2\nu-1}(ax^{\frac{1}{2}})$<br>$ \arg a  < \frac{1}{4}\pi, \quad \operatorname{Re} \nu > -\frac{1}{4}$                                                                                                             | $\frac{a^{2\nu-1}}{\pi^{\frac{1}{2}} 2^\nu y^{2\nu}} K_{\nu-\frac{1}{2}}\left(\frac{a^2}{2y}\right)$                                                              |
| (34) | $x^{\nu+\frac{1}{2}} Y_{2\nu}(ax^{\frac{1}{2}}) K_{2\nu}(ax^{\frac{1}{2}})$<br>$ \arg a  < \frac{1}{4}\pi, \quad \operatorname{Re} \nu > -\frac{3}{4}$                                                                                                                 | $\frac{a^{2\nu+1}}{\pi^{\frac{1}{2}} 2^{\nu+1} y^{2\nu+2}} K_{\nu-\frac{1}{2}}\left(\frac{a^2}{2y}\right)$                                                        |
| (35) | $x^{-\frac{1}{2}} \{\cos[\frac{1}{2}(\mu-\nu)\pi] J_\mu(ax^{\frac{1}{2}})$<br>$- \sin[\frac{1}{2}(\mu-\nu)\pi] Y_\mu(ax^{\frac{1}{2}})\}$<br>$\times K_\mu(ax^{\frac{1}{2}}) \quad  \arg a  < \frac{1}{4}\pi$<br>$\operatorname{Re} \nu >  \operatorname{Re} \mu  - 2$ | $a^{-2} y^{\frac{1}{2}} W_{\frac{1}{2}\nu, \frac{1}{2}\mu}\left(\frac{a^2}{2y}\right)$<br>$\times W_{-\frac{1}{2}\nu, \frac{1}{2}\mu}\left(\frac{a^2}{2y}\right)$ |

**Higher transcendental functions (cont'd)**

|      | $f(x)$                                                                                                                                                                                               | $\int_0^\infty f(x) \mathbf{H}_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                                                                                                                                           |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (36) | $x^{\nu-\frac{1}{2}} K_{2\nu-1}(ax^{\frac{1}{2}} e^{\frac{1}{4}i\pi})$<br>$\times K_{2\nu-1}(ax^{\frac{1}{2}} e^{-\frac{1}{4}i\pi})$<br>$\text{Re } a > 0, \quad \text{Re } \nu > -\frac{1}{4}$      | $\pi^{-\frac{1}{2}} 2^{3\nu-1} y^{-\nu-\frac{1}{2}}$<br>$\times \Gamma(\nu+1) \Gamma(2\nu+\frac{1}{2})$<br>$\times S_{-3\nu-\frac{1}{2}, \nu-\frac{1}{2}}\left(\frac{a}{2^{\frac{1}{2}} y^{\frac{1}{2}}}\right)$                                                                                                                                                                                                                                                                                   |
| (37) | $x^{-\frac{1}{2}} \mathbf{H}_\nu(a^2/x)$<br>$a > 0, \quad \text{Re } \nu > -3/2$                                                                                                                     | $-y^{-\frac{1}{2}} J_{2\nu}(2ay^{\frac{1}{2}})$                                                                                                                                                                                                                                                                                                                                                                                                                                                    |
| (38) | $x^{-3/2} \mathbf{H}_{\nu-1}(a^2/x)$<br>$a > 0, \quad \text{Re } \nu > -\frac{1}{2}$                                                                                                                 | $-a^{-1} J_{2\nu-1}(2ay^{\frac{1}{2}})$                                                                                                                                                                                                                                                                                                                                                                                                                                                            |
| (39) | $x^{-\frac{1}{2}} [J_{-\nu}(a^2/x)$<br>$+ \sin(\nu\pi) \mathbf{H}_\nu(a^2/x)]$<br>$a > 0, \quad -3/2 < \text{Re } \nu < 0$                                                                           | $y^{-\frac{1}{2}} [2\pi^{-1} K_{2\nu}(2ay^{\frac{1}{2}})$<br>$- Y_{2\nu}(2ay^{\frac{1}{2}})]$                                                                                                                                                                                                                                                                                                                                                                                                      |
| (40) | $x^\sigma S_{\lambda, \mu}(ax)$<br>$ \arg a  < \pi, \quad \text{Re}(\lambda + \sigma) < 1$<br>$\text{Re}(\sigma + \nu) >  \text{Re } \mu  - 5/2$<br>$-3/2 < \text{Re}(\lambda + \nu + \sigma) < 1/2$ | $\frac{2^{\lambda+\sigma-\frac{1}{2}} a^{-\sigma-1}}{\Gamma(\frac{1}{2}-\frac{1}{2}\lambda-\frac{1}{2}\mu) \Gamma(\frac{1}{2}-\frac{1}{2}\lambda+\frac{1}{2}\mu)}$ $\times G_{44}^{24}\left(\frac{y^2}{a^2} \left  \begin{matrix} l, -\frac{\lambda+\sigma}{2}, \frac{1-\sigma+\mu}{2}, \frac{1-\sigma-\mu}{2} \\ l, -\frac{\lambda+\sigma}{2}, h, k \end{matrix} \right. \right)$ $h = \frac{1}{4} + \frac{\nu}{2}, \quad k = \frac{1}{4} - \frac{\nu}{2}, \quad l = \frac{3}{4} + \frac{\nu}{2}$ |



## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                        | $\int_0^\infty f(x) \mathbf{H}_\nu(xy) (xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                                                                                                                                                      |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (41) | $x^{-\nu-\frac{1}{2}} \exp(-\frac{1}{4}x^2)$<br>$\times [D_\mu(x) - D_\mu(-x)]$<br>$\text{Re}(\mu + \nu) > -3/2$<br>$\text{Re } \mu > -1$                                                                     | $\frac{2^{3/2} \Gamma(\frac{1}{2}\mu + \frac{1}{2})}{\Gamma(\frac{1}{2}\mu + \nu + 1)} y^{\mu+\nu+\frac{1}{2}} \sin(\frac{1}{2}\mu\pi)$<br>$\times {}_1F_1(\frac{1}{2}\mu + \frac{1}{2}; \frac{1}{2}\mu + \nu + 1; -\frac{1}{2}y^2)$                                                                                                                                                                                                                                                                           |
| (42) | $x^{2\lambda} \exp(-\frac{1}{4}x^2) M_{\kappa,\mu}(\frac{1}{2}x^2)$<br>$\text{Re}(2\lambda + 2\mu + \nu) > -7/2$<br>$\text{Re}(\kappa - \lambda) > 0$<br>$\text{Re}(2\lambda - 2\kappa + \nu) < -\frac{1}{2}$ | $\frac{2^{-\lambda} \Gamma(2\mu + 1)}{\Gamma(\frac{1}{2} + \kappa + \mu)}$<br>$\times G_{34}^{22} \left( \frac{y^2}{2} \left  \begin{matrix} l, -\mu - \lambda, \mu - \lambda \\ l, \kappa - \lambda - \frac{1}{2}, h, k \end{matrix} \right. \right)$<br>$h = \frac{1}{4} + \frac{\nu}{2}, \quad k = \frac{1}{4} - \frac{\nu}{2}, \quad l = \frac{3}{4} + \frac{\nu}{2}$                                                                                                                                      |
| (43) | $x^{2\lambda} \exp(-\frac{1}{4}x^2) W_{\kappa,\mu}(\frac{1}{2}x^2)$<br>$\text{Re}(2\lambda + \nu) > 2 \text{Re } \mu  - 7/2$                                                                                  | $2^{1/4-\lambda-\nu/2} \pi^{-1/2} y^{\nu+3/2}$<br>$\times \frac{\Gamma(\frac{7}{4} + \frac{\nu}{2} + \lambda + \mu) \Gamma(\frac{7}{4} + \frac{\nu}{2} + \lambda - \mu)}{\Gamma(\nu + \frac{3}{2}) \Gamma(\frac{9}{4} + \lambda - \kappa - \frac{\nu}{2})}$<br>$\times {}_3F_3 \left( 1, \frac{7}{4} + \frac{\nu}{2} + \lambda + \mu, \frac{7}{4} + \frac{\nu}{2} + \lambda - \mu; \right.$<br>$\left. \frac{3}{2}, \nu + \frac{3}{2}, \frac{9}{4} + \lambda - \kappa + \frac{\nu}{2}; -\frac{y^2}{2} \right)$ |
| (44) | $x^{-\frac{1}{2}} \exp(\frac{1}{2}x^2)$<br>$\times W_{-\frac{1}{2}\nu-\frac{1}{2}, \frac{1}{2}\nu}(x^2)$<br>$\text{Re } \nu > -1$                                                                             | $2^{-\nu-1} y^{\nu+\frac{1}{2}} \pi \exp(\frac{1}{4}y^2) \text{Erfc}(\frac{1}{2}y)$                                                                                                                                                                                                                                                                                                                                                                                                                            |

## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                                                                                                                                                                     | $\int_0^\infty f(x) \mathbf{H}_\nu(xy)(xy)^{\frac{1}{2}} dx \quad y > 0$                                                                                                                                                                                                                                                                                                             |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (45) | $x^{-\frac{1}{2}} \exp(\frac{1}{4}x^2) W_{\kappa, \frac{1}{2}\nu}(\frac{1}{2}x^2)$<br>$-3/2 < \operatorname{Re} \nu < -2\operatorname{Re} \kappa$<br>$\operatorname{Re} \kappa < \frac{1}{4}$                                                                                                                                                                                              | $\frac{2^{\frac{1}{2}\kappa - \frac{1}{4}\nu} y^{\frac{1}{2}\nu - \kappa - \frac{1}{2}} \Gamma(-\kappa - \frac{1}{2}\nu)}{\Gamma(\frac{1}{2} - \kappa + \frac{1}{2}\nu) \Gamma(\frac{1}{2} - \kappa - \frac{1}{2}\nu)}$ $\times \exp(\frac{1}{4}y^2) W_{k, m}(\frac{1}{2}y^2)$ $2k = \kappa + \frac{1}{2}\nu, \quad 2m = \kappa + \frac{1}{2}\nu + 1$                                |
| (46) | $x^{2\lambda} \exp(\frac{1}{4}x^2) W_{\kappa, \lambda}(\frac{1}{2}x^2)$<br>$\operatorname{Re}(2\lambda + \nu) > 2 \operatorname{Re} \mu  - 7/2$<br>$\operatorname{Re}(2\kappa + 2\lambda + \nu) < -1/2$<br>$\operatorname{Re}(\kappa + \lambda) < 0$                                                                                                                                       | $[2^\lambda \Gamma(\frac{1}{2} - \kappa + \mu) \Gamma(\frac{1}{2} - \kappa - \mu)]^{-1}$ $\times G_{34}^{23} \left( \frac{y^2}{2} \left  \begin{matrix} l, -\mu - \lambda, \mu - \lambda \\ l, -\kappa - \lambda - \frac{1}{2}, h, k \end{matrix} \right. \right)$ $h = \frac{1}{4} + \frac{\nu}{2}, \quad k = \frac{1}{4} - \frac{\nu}{2}, \quad l = \frac{3}{4} + \frac{\nu}{2}$   |
| (47) | $G_{pq}^m \left( \lambda x^2 \left  \begin{matrix} \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q \end{matrix} \right. \right)$<br>$p + q < 2(m + n)$<br>$ \arg \lambda  < (m + n - \frac{1}{2}p - \frac{1}{2}q)\pi$<br>$\operatorname{Re} \alpha_j < \min(1, \frac{3}{4} - \frac{1}{2}\nu)$<br>$j = 1, \dots, n$<br>$\operatorname{Re}(2\beta_j + \nu) > -5/2$<br>$j = 1, \dots, m$ | $(2\lambda)^{-\frac{1}{2}}$ $\times G_{q+1, p+3}^{n+1, m+1} \left( \frac{y^2}{4\lambda} \left  \begin{matrix} l, \frac{1}{2} - \beta_1, \dots, \frac{1}{2} - \beta_q \\ l, \frac{1}{2} - \alpha_1, \dots, \frac{1}{2} - \alpha_p, h, k \end{matrix} \right. \right)$ $h = \frac{1}{4} + \frac{\nu}{2}, \quad k = \frac{1}{4} - \frac{\nu}{2}, \quad l = \frac{3}{4} + \frac{\nu}{2}$ |

## CHAPTER XII

### KONTOROVICH-LEBEDEV TRANSFORMS

The pair of reciprocal formulas

$$g(y) = \int_0^{\infty} f(x) K_{ix}(y) dx$$

$$f(x) = 2\pi^{-2} x \sinh(\pi x) \int_0^{\infty} g(y) K_{ix}(y) y^{-1} dy$$

was given by Kontorovich and Lebedev (1938, 1939) who used these formulas in the solution of certain boundary value problems. Further applications to boundary value problems were given by Lebedev and Kontorovich, and the mathematical theory was developed by Lebedev (1946, 1949). It should be noted that  $K_{ix}(y)$  is real when  $x$  is real and  $y$  is positive. Alternative forms of this inversion were stated in the papers referred to above. See also Erdélyi, et al. (1953, p. 75).

In this chapter we give a short list of integrals corresponding to the first of the above formulas; integrals corresponding to the second formula may be evaluated by means of the tables given in chapter X. We take  $y$  to be a positive real variable, although some of the integrals given below are valid for complex  $y$ .

## REFERENCES

- Erdélyi, Arthur, Wilhelm Magnus, Fritz Oberhettinger, and F.G. Tricomi, 1953: *Higher transcendental functions*, vol. II, McGraw-Hill.
- Kontorovich, M.J. and N.N. Lebedev, 1938: *J. Exper. Theor. Phys. USSR*, 8, 1192-1206.
- Kontorovich, M.J. and N.N. Lebedev, 1939: *Acad. Sci. USSR, J. Phys.* 1, 229-241.
- Lebedev, N.N. and M.J. Kontorovich, 1939: *J. Exper. Theor. Phys. USSR*, 9, 729-741.
- Lebedev, N.N., 1946: *Acad. Sci. USSR, Doklady* 52, 655-658.
- Lebedev, N.N., 1949: *Acad. Sci. USSR, Doklady* 65, 621-624.

# KONTOROVICH-LEBEDEV TRANSFORMS

## 12.1. Formulas

|     | $f(x)$                                                                          | $\int_0^\infty f(x) K_{ix}(y) dx \quad y > 0$                              |
|-----|---------------------------------------------------------------------------------|----------------------------------------------------------------------------|
| (1) | $x \sin(ax) \quad  \operatorname{Im} a  < \frac{1}{2}\pi$                       | $\frac{1}{2}\pi y \sinh a \exp(-y \cosh a)$                                |
| (2) | $\cos ax \quad  \operatorname{Im} a  < \frac{1}{2}\pi$                          | $\frac{1}{2}\pi \exp(-y \cosh a)$                                          |
| (3) | $x \tanh(\pi x) P_{-\frac{1}{2}+ix}(z)$                                         | $(\frac{1}{2}\pi y)^{\frac{1}{2}} e^{-zy}$                                 |
| (4) | $x \tanh(\pi x) K_{ix}(\beta) \quad  \arg \beta  < \pi$                         | $\frac{1}{2}\pi(\beta y)^{\frac{1}{2}} (\beta + y)^{-1} \exp(-\beta - y)$  |
| (5) | $x \sinh(\pi x) K_{2ix}(a) \quad  \arg a  < \frac{1}{4}\pi$                     | $\frac{\pi^{3/2} a}{2^{5/2} y^{1/2}} \exp\left(-y - \frac{a^2}{8y}\right)$ |
| (6) | $x \sin(\frac{1}{2}\pi x) K_{\frac{1}{2}ix}(a) \quad  \arg a  < \frac{1}{2}\pi$ | $\frac{\pi^{3/2} y}{2^{1/2} a^{1/2}} \exp\left(-a - \frac{y^2}{8a}\right)$ |
| (7) | $\cosh(ax) K_{ix}(\beta) \quad  \operatorname{Re} a  +  \arg \beta  < \pi$      | $\frac{1}{2}\pi K_0[(y^2 + \beta^2 + 2\beta y \cos a)^{\frac{1}{2}}]$      |

## Formulas (cont'd)

|      | $f(x)$                                                                                                                                          | $\int_0^\infty f(x) K_{ix}(y) dx$                                                                                                                     | $y > 0$                         |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------|
| (8)  | $x(x^2 + n^2)^{-1} \sinh(\pi x)$<br>$\times K_{ix}(a)$<br>$a > 0, \quad n = 0, 1, 2, 3, \dots$                                                  | $\frac{1}{2} \pi^2 I_n(y) K_n(a)$<br>$\frac{1}{2} \pi^2 I_n(a) K_n(y)$                                                                                | $0 < y < a$<br>$a < y < \infty$ |
| (9)  | $x \sinh(\pi x) K_{ix}(a) K_{ix}(\beta)$<br>$ \arg a  +  \arg \beta  < \frac{1}{2} \pi$                                                         | $\frac{\pi^2}{4} \exp \left[ -\frac{y}{2} \left( \frac{a}{\beta} + \frac{\beta}{a} + \frac{a\beta}{y^2} \right) \right]$                              |                                 |
| (10) | $x \sinh(\frac{1}{2} \pi x) K_{\frac{1}{2}ix}(a) K_{\frac{1}{2}ix}(\beta)$<br>$ \arg a  +  \arg \beta  < \pi$                                   | $\frac{\pi^2 y}{2z} \exp \left[ -\frac{(a+\beta)z}{2(a\beta)^{\frac{1}{2}}} \right]$<br>$z = (y^2 + 4a\beta)^{\frac{1}{2}}$                           |                                 |
| (11) | $x \sinh(\pi x) K_{\frac{1}{2}ix+\lambda}(a)$<br>$\times K_{\frac{1}{2}ix-\lambda}(a)$<br>$a > 0$                                               | $0$<br>$\frac{\pi^2 y}{2^{2\lambda+1} a^{2\lambda} z} [(y+z)^{2\lambda} + (y-z)^{2\lambda}]$<br>$2a < y < \infty$<br>$z = (y^2 - 4a^2)^{\frac{1}{2}}$ | $0 < y < 2a$                    |
| (12) | $x \sinh(\pi x) \Gamma(\lambda + ix)$<br>$\times \Gamma(\lambda - ix) K_{ix}(a)$<br>$ \arg a  < \pi, \quad \operatorname{Re} \lambda > 0$       | $2^{\nu-1} \pi^{3/2} (ay)^\lambda (y+a)^{-\lambda}$<br>$\times \Gamma(\lambda + \frac{1}{2}) K_\lambda(y+a)$                                          |                                 |
| (13) | $x \sinh(2\pi x) \Gamma(\lambda + ix)$<br>$\times \Gamma(\lambda - ix) K_{ix}(a)$<br>$a > 0, \quad 0 < \operatorname{Re} \lambda < \frac{1}{2}$ | $\frac{2^\lambda \pi^{5/2}}{\Gamma(\frac{1}{2} - \lambda)} \left( \frac{ay}{ y-a } \right)^\lambda K_\lambda( y-a )$                                  |                                 |

## Formulas (cont'd)

|      | $f(x)$                                                                                                                                                                                                                           | $\int_0^\infty f(x) K_{ix}(y) dx$ <span style="float: right;"><math>y &gt; 0</math></span>                                                                   |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (14) | $x \sinh(\pi x) \Gamma(\lambda + \frac{1}{2}ix)$<br>$\times \Gamma(\lambda - \frac{1}{2}ix) K_{ix}(a)$<br>$ \arg a  < \frac{1}{2}\pi, \quad \operatorname{Re} \lambda > 0$                                                       | $2\pi^2 \left(\frac{ay}{2z}\right)^{2\lambda} K_{2\lambda}(z)$<br>$z = (y^2 + a^2)^{\frac{1}{2}}$                                                            |
| (15) | $\frac{x \tanh(\pi x) K_{ix}(a)}{\Gamma(\frac{3}{4} + \frac{1}{2}ix) \Gamma(\frac{3}{4} - \frac{1}{2}ix)}$<br>$ \arg a  < \frac{1}{2}\pi$                                                                                        | $\frac{1}{2} \left(\frac{\pi ay}{y^2 + a^2}\right)^{\frac{1}{2}} \exp[-(y^2 + a^2)^{\frac{1}{2}}]$                                                           |
| (16) | $x \sinh(\pi x) \Gamma(\lambda + ix) \Gamma(\lambda - ix)$<br>$\times P_{-\frac{1}{2}+ix}^{\frac{1}{2}-\lambda}(\beta) K_{ix}(a)$<br>$ \arg a  < \frac{1}{2}\pi$<br>$ \arg(\beta - 1)  < \pi$<br>$\operatorname{Re} \lambda > 0$ | $2^{-1/2} \pi^{3/2} (ay/z)^\lambda (\beta^2 - 1)^{\frac{1}{2}\lambda - \frac{1}{4}}$<br>$\times K_\lambda(z)$<br>$z = (y^2 + a^2 + 2a\beta y)^{\frac{1}{2}}$ |





## MISCELLANEOUS TRANSFORMS



## CHAPTER XIII

### FRACTIONAL INTEGRALS

We call

$$g(y; \mu) = \Re_{\mu} \{f(x); y\} = \frac{1}{\Gamma(\mu)} \int_0^y f(x) (y-x)^{\mu-1} dx$$

the *Riemann-Liouville* (fractional) *integral of order  $\mu$* , and

$$h(y; \mu) = \Im_{\mu} \{f(x); y\} = \frac{1}{\Gamma(\mu)} \int_y^{\infty} f(x) (x-y)^{\mu-1} dx$$

the *Weyl* (fractional) *integral of order  $\mu$* , of  $f(x)$ . In general,  $\mu$  and  $y$  are envisaged as complex numbers, the path of integration being the segment  $x = yt$ ,  $0 < t < 1$  in  $g$ , and one of the rays  $x = yt$ ,  $t > 1$  or  $x = y + t$ ,  $t > 0$  in  $h$ .

Many authors denote  $g(y; \mu)$  by  $I^{\mu}f$  or  $I_{+}^{\mu}f$ , and  $h(y; \mu)$  by  $K^{\mu}f$  or  $K_{-}^{\mu}f$ . The integral

$$\frac{1}{\Gamma(\mu)} \int_y^a f(x) (x-y)^{\mu-1} dx,$$

sometimes denoted by  $I_{-}^{\mu}f$ , may be expressed as  $\Re_{\mu} \{f(a-x); a-y\}$  by a change of variables; alternatively it may be written as  $\Im_{\mu} \{f(x); y\}$  by adopting the convention that  $f(x) = 0$  when  $x > a$ .

Fractional derivatives of order  $\alpha$  may be defined by the formulas

$$D_0^{\alpha} f(x) = \frac{d^n}{dx^n} \Re_{n-\alpha} \{f(t); x\} \quad n-1 < \operatorname{Re} \alpha < n$$

$$D_{\infty}^{\alpha} f(x) = \frac{d^n}{dx^n} \Im_{n-\alpha} \{f(t); x\} \quad n-1 < \operatorname{Re} \alpha < n$$

so that tables of fractional integrals may be used to evaluate fractional derivatives.

On p. 184 we give a brief selection of books and papers which contain information on the theory of fractional integrals and derivatives. Hardy and Littlewood (1928) give further references. As far as we know, there is no extensive table of fractional integrals although numerous integrals of this kind occur in almost any table of integrals.

An extension of the operators  $\mathfrak{H}_\mu$  and  $\mathfrak{B}_\mu$  was introduced by Kober (1940) and Erdélyi (1940). Kober (1941b) also discussed fractional integrals of imaginary order. Fractional integration by parts over a finite interval is expressed by the formula

$$\int_0^a g_1(x; \mu) f_2(a-x) dx = \int_0^a f_1(a-x) g_2(x; \mu) dx,$$

and was discussed by Young and Love (1938). For the infinite interval the formula is

$$\int_0^\infty f_1(x) g_2(x; \mu) dx = \int_0^\infty h_1(x; \mu) f_2(x) dx,$$

and was discussed by Kober (1940). In these formulas  $g_{1,2} = \mathfrak{H}f_{1,2}$ , and  $h_1 = \mathfrak{B}f_1$ .

The operators  $\mathfrak{H}_\mu$ ,  $\mathfrak{B}_\mu$  are connected with differentiation and integration, and with each other, by a number of relations. We list a few here, others being given in the list of general formulas in sections 13.1 and 13.2.

$$g(x; 1) = \int_0^x f(t) dt, \quad h(x; 1) = \int_x^\infty f(t) dt$$

$$\frac{d}{dx} g(x; \mu) = g(x; \mu - 1), \quad -\frac{d}{dx} h(x; \mu) = h(x; \mu - 1)$$

$$\mathfrak{H}_\mu \mathfrak{H}_\nu = \mathfrak{H}_{\mu+\nu}, \quad \mathfrak{B}_\mu \mathfrak{B}_\nu = \mathfrak{B}_{\mu+\nu}.$$

The functions  $g(x; \mu)$  and  $h(x; \mu)$  may be regarded as  $\mu$  times repeated indefinite integrals of  $f(x)$ , the fixed limit being 0 in the case of  $g$ , and  $\infty$  in the case of  $h$ .

The connection of fractional integrals with other integral transforms may be seen from the following formulas.

$$\mathfrak{L}\{g(t; \mu); p\} = p^{-\mu} \mathfrak{L}\{f(t); p\}$$

$$\mathfrak{F}_e\{h(x; \mu); y\} = e^{\frac{1}{2}\mu\pi i} y^{-\mu} \mathfrak{F}_e\{f(x); y\}$$

$$\mathfrak{M}\{g(x; \mu); s\} = \frac{\Gamma(1-s-\mu)}{\Gamma(1-s)} \mathfrak{M}\{f(x); s+\mu\}$$

$$\mathfrak{M}\{h(x; \mu); s\} = \frac{\Gamma(s)}{\Gamma(s+\mu)} \mathfrak{M}\{f(x); s+\mu\}$$

which may be used in conjunction with the tables of vol. I to evaluate fractional integrals by means of tables of Fourier, Laplace, Mellin transforms and their inversions. These formulas may also be used to derive from a known pair of, say, Fourier transforms a new pair by means of integration of fractional order.

The connection of fractional integrals and Laplace transforms is discussed in Doetsch (1937, p. 293-305) and Widder (1941, p. 70-75). Doetsch also discusses Abel's integral equation,  $g = \mathfrak{R}_{\mu} f$ . For the connection of fractional integrals and Fourier transforms see Kober (1941a, Lemma 3). For the connection of fractional integrals and Mellin transforms see Kober (1940). For the connection of fractional integrals and Hankel transforms see Erdélyi and Kober (1940) and Erdélyi (1940); see also 8.1(13) to 8.1(16). For fractional integrals in the theory of Fourier series see Zygmund (1935, p. 222 ff.).

Fractional integrals occur in the solution by definite integrals of linear differential equations. In this context fractional integrals are often called *Euler transforms* (see, for instance, Ince, 1927, p. 191 ff.),  $\mathfrak{R}_{\mu} f$  being the Euler transform of the first kind, and  $\mathfrak{B}_{\mu} f$  the Euler transform of the second kind, of  $f$ .

M. Riesz (1949) has developed a theory of fractional integrals of functions of several variables: this theory has been applied by Riesz and others to the solution of partial differential equations. (See, for instance, Baker and Copson, 1950, Chap. I, § 7.)

From the fractional integrals given in the tables, further fractional integrals may be derived by the general methods enumerated in the introduction to vol. I, by means of the general formulas stated above and in sections 13.1 and 13.2, and by means of the connection, also stated above, between fractional integrals and other integral transforms.

## REFERENCES

- Baker, B.B. and E.T. Copson, 1950: *The mathematical theory of Huygens' principle*. Oxford, Clarendon Press.
- Doetsch, Gustav, 1937: *Theorie und Anwendung der Laplace-Transformation*. Berlin, Springer.
- Erdélyi, Arthur, 1940: *Quart. J. Math., Oxford Ser.* 11, 293-303.
- Erdélyi, Arthur and Hermann Kober, 1940: *Quart J. Math., Oxford Ser.* 11, 212-221.
- Hardy, G.H., 1918: *Messenger of Math.* 47, 145-150.
- Hardy, G.H. and J.E. Littlewood, 1925: *Proc. London Math. Soc.* (2) 24, xxxvii-xli.
- Hardy, G.H. and J.E. Littlewood, 1928: *Math. Z.* 27, 565-606.
- Hardy, G.H. and J.E. Littlewood, 1932: *Math. Z.* 34, 403-439.
- Hardy, G.H., Littlewood, J.E. and G. Polya, 1934: *Inequalities*, Cambridge.
- Ince, E.L., 1927: *Ordinary differential equations*. Longmans, Green & Co., London.
- Kober, Hermann, 1940: *Quart. J. Math., Oxford Ser.* 11, 193-211.
- Kober, Hermann, 1941a: *Quart. J. Math., Oxford Ser.*, 12, 78-85.
- Kober, Hermann, 1941b: *Trans. Amer. Math. Soc.* 50, 160-174.
- Kuttner, Brian, 1953: *Proc. London Math. Soc.* (3) 3, 480-497.
- Love, E.R., 1938: *Proc. London Math. Soc.* (2) 44, 363-397.
- Riesz, Marcel, 1949: *Acta Math.* 81, 1-223.
- Weyl, Hermann, 1917: *Vierteljschr. Naturforsch. Ges. Zürich*, 62, 296-302.
- Widder, D.V., 1941: *The Laplace transform*. Princeton University Press, Princeton, New Jersey.
- Young, L.C. and E.R. Love, 1938: *Proc. London Math. Soc.* (2) 44, 1-28.
- Zygmund, Antoni, 1935: *Trigonometrical series*. Warszawa-Lwow.

## FRACTIONAL INTEGRALS

### 13.1. Riemann-Liouville fractional integrals

|     |                                                                                                                        |                                                                                              |
|-----|------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------|
|     | $f(x)$                                                                                                                 | $[\Gamma(\mu)]^{-1} \int_0^y f(x) (y-x)^{\mu-1} dx$<br>$= g(y; \mu)$                         |
| (1) | $f(ax)$                                                                                                                | $a^{-\mu} g(ay; \mu)$                                                                        |
| (2) | $f(a/x)$                                                                                                               | $ay^{\mu-1} \mathfrak{B}_{\mu}\{t^{-\mu-1} f(t); a/y\}$<br>For tables see sec 13.2.          |
| (3) | $f'(x)$                                                                                                                | $g(y; \mu-1) - f(0) y^{\mu-1}/\Gamma(\mu)$                                                   |
| (4) | $\int_0^x f(t) dt$                                                                                                     | $g(y; \mu+1)$                                                                                |
| (5) | $g(x; \nu)$                                                                                                            | $g(y; \mu+\nu)$                                                                              |
| (6) | 1 <span style="float: right;"><math>\operatorname{Re} \mu &gt; 0</math></span>                                         | $\frac{y^{\mu}}{\Gamma(\mu+1)}$                                                              |
| (7) | $x^{\nu-1}$ <span style="float: right;"><math>\operatorname{Re} \mu &gt; 0, \operatorname{Re} \nu &gt; 0</math></span> | $\frac{\Gamma(\nu)}{\Gamma(\mu+\nu)} y^{\mu+\nu-1}$                                          |
| (8) | $(x+a)^{\nu}$ <span style="float: right;"><math>\operatorname{Re} \mu &gt; 0</math></span>                             | $\frac{a^{\nu} y^{\mu}}{\Gamma(\mu+1)} {}_2F_1(1, -\nu; 1+\mu; -y/a)$<br>$ \arg(y/a)  < \pi$ |

**Riemann-Liouville integrals (cont'd)**

|      | $f(x)$                                                                                                                                                                     | $[\Gamma(\mu)]^{-1} \int_0^y f(x) (y-x)^{\mu-1} dx$                                                                                                                                                                                                                                    |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (9)  | $x^{\nu-1} (x+a)^\lambda$<br>$\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0$                                                                                  | $\frac{a^\lambda y^{\mu+\nu-1} \Gamma(\nu)}{\Gamma(\mu+\nu)}$ $\times {}_2F_1(-\lambda, \nu; \mu+\nu; -y/a)$ $ \arg y/a  < \pi$                                                                                                                                                        |
| (10) | $x^{\nu-1} (x^2+a^2)^\lambda$<br>$\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0$                                                                              | $\frac{a^{2\lambda} y^{\mu+\nu-1} \Gamma(\nu)}{\Gamma(\mu+\nu)}$ $\times {}_3F_2\left(-\lambda, \frac{\nu}{2}, \frac{\nu+1}{2}; \frac{\mu+\nu}{2}, \frac{\mu+\nu+1}{2}; -\frac{y^2}{a^2}\right) \quad \operatorname{Re}(y/a) > 0$                                                      |
| (11) | $x^{\nu-1} (x^k+a^k)^\lambda$<br>$k = 1, 2, \dots,$<br>$\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0$                                                        | $\frac{a^{k\lambda} y^{\mu+\nu+1} \Gamma(\nu)}{\Gamma(\mu+\nu)}$ $\times {}_{k+1}F_k\left(-\lambda, \frac{\nu}{k}, \frac{\nu+1}{k}, \dots, \frac{\nu+k-1}{k}; \frac{\mu+\nu}{k}, \frac{\mu+\nu+1}{k}, \dots, \frac{\mu+\nu+k-1}{k}; -\frac{y^k}{a^k}\right) \quad  \arg(y/a)  < \pi/k$ |
| (12) | $x^{-\frac{1}{2}} (x+2)^{-\frac{1}{2}} \{[(x+2)^{\frac{1}{2}} + x^{\frac{1}{2}}]^{2\nu} + [(x+2)^{\frac{1}{2}} - x^{\frac{1}{2}}]^{2\nu}\}$<br>$\operatorname{Re} \mu > 0$ | $2^{\mu+\frac{1}{2}} \pi^{\frac{1}{2}} [y(y+2)]^{\frac{1}{2}\mu-\frac{1}{4}}$ $\times P_{\nu-\frac{1}{2}}^{\frac{1}{2}-\mu}(y+1) \quad  \arg y  < \pi$                                                                                                                                 |



**Riemann-Liouville integrals (cont'd)**

|      | $f(x)$                                                                                                          | $[\Gamma(\mu)]^{-1} \int_0^y f(x) (y-x)^{\mu-1} dx$                                                                                                                                                                 |
|------|-----------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (13) | $x^{\mu-1} e^{ax} \quad \operatorname{Re} \mu > 0$                                                              | $\pi^{1/2} (y/a)^{\mu-1/2} \exp(\frac{1}{2} ay) \times I_{\mu-1/2}(\frac{1}{2} ay)$                                                                                                                                 |
| (14) | $x^{\nu-1} e^{ax} \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0$                             | $\frac{\Gamma(\nu)}{\Gamma(\mu+\nu)} y^{\mu+\nu-1} {}_1F_1(\nu; \mu+\nu; ay)$                                                                                                                                       |
| (15) | $x^{\nu-1} \exp(ax^k) \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0$<br>$k = 2, 3, 4, \dots$ | $\frac{\Gamma(\nu)}{\Gamma(\mu+\nu)} y^{\mu+\nu-1} \times {}_kF_k\left(\frac{\nu}{k}, \frac{\nu+1}{k}, \dots, \frac{\nu+k-1}{k}; \frac{\mu+\nu}{k}, \frac{\mu+\nu+1}{k}, \dots, \frac{\mu+\nu+k-1}{k}; ay^k\right)$ |
| (16) | $x^{-\mu-1} \exp(-a/x) \quad \operatorname{Re} \mu > 0$                                                         | $a^{-\mu} y^{\mu-1} \exp(-a/y) \quad  \arg y  < \pi$                                                                                                                                                                |
| (17) | $x^{-2\mu} \exp(-a/x) \quad \operatorname{Re} \mu > 0$                                                          | $(\pi y)^{-1/2} a^{1/2-\mu} \exp\left(-\frac{a}{2y}\right) \times K_{\mu-1/2}\left(\frac{a}{2y}\right) \quad \operatorname{Re}(a/y) > 0$                                                                            |
| (18) | $x^{\nu-1} \exp(-a/x) \quad \operatorname{Re} \mu > 0$                                                          | $a^{1/2\nu-1/2} y^{-\kappa} \exp\left(-\frac{a}{2y}\right) \times W_{\kappa, 1/2\nu}\left(\frac{a}{y}\right) \quad \kappa = 1/2 - \mu - 1/2\nu, \quad \operatorname{Re}(a/y) > 0$                                   |

**Riemann-Liouville integrals (cont'd)**

|      | $f(x)$                                                                       | $[\Gamma(\mu)]^{-1} \int_0^y f(x) (y-x)^{\mu-1} dx$                                                                                                                                                                                                                                          |
|------|------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (19) | $\exp(ax^{1/2}) \quad \text{Re } \mu > 0$                                    | $\frac{y^\mu}{\Gamma(\mu+1)} + \pi^{1/2} (\tfrac{1}{2}a)^{1/2-\mu} y^{1/2\mu+1/4}$ $\times [I_{\mu+1/2}(ay^{1/2}) + L_{\mu+1/2}(ay^{1/2})]$                                                                                                                                                  |
| (20) | $x^{-1/2} \exp(ax^{1/2})$                                                    | $\pi^{1/2} (\tfrac{1}{2}a)^{1/2-\mu} y^{1/2\mu-1/4}$ $\times [I_{\mu-1/2}(ay^{1/2}) + L_{\mu-1/2}(ay^{1/2})]$                                                                                                                                                                                |
| (21) | $x^{\nu-1} \exp(ax^{1/2})$<br>$\text{Re } \mu > 0, \quad \text{Re } \nu > 0$ | $\frac{y^{\mu+\nu-1} \Gamma(\nu)}{\Gamma(\mu+\nu)}$ $\times {}_1F_2\left(\nu; \frac{1}{2}, \mu+\nu; \frac{a^2 y}{4}\right)$ $+ \frac{ay^{\mu+\nu-1/2} \Gamma(\nu+1/2)}{\Gamma(\mu+\nu+1/2)}$ $\times {}_1F_2\left(\nu+\frac{1}{2}; \frac{3}{2}, \mu+\nu+\frac{1}{2}; \frac{a^2 y}{4}\right)$ |
| (22) | $x^{-3/2} \exp(-ax^{-1/2}) \quad \mu = 1$                                    | $2a^{-1} \exp(-ay^{-1/2})$                                                                                                                                                                                                                                                                   |
| (23) | $x^{-\mu-1/2} \exp(-ax^{-1/2})$<br>$\text{Re } \mu > 0$                      | $2^{\mu+1/2} \pi^{-1/2} a^{1/2-\mu} y^{1/2\mu-1/4}$ $\times K_{\mu-1/2}(ay^{-1/2}) \quad \text{Re}(ay^{-1/2}) > 0$                                                                                                                                                                           |
| (24) | $x^{\nu-1} \log x$<br>$\text{Re } \mu > 0, \quad \text{Re } \nu > 0$         | $\frac{y^{\mu+\nu-1} \Gamma(\nu)}{\Gamma(\mu+\nu)}$ $\times [\log y + \psi(\nu) - \psi(\mu+\nu)]$                                                                                                                                                                                            |

**Riemann-Liouville integrals (cont'd)**

|      | $f(x)$                                                                                                         | $[\Gamma(\mu)]^{-1} \int_0^y f(x)(y-x)^{\mu-1} dx$                                                                                                                                    |
|------|----------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (25) | $x^{\mu-1} \sin(ax) \quad \operatorname{Re} \mu > 0$                                                           | $\pi^{\frac{1}{2}} (y/a)^{\mu-\frac{1}{2}} \sin(\frac{1}{2}ay) J_{\mu-\frac{1}{2}}(\frac{1}{2}ay)$                                                                                    |
| (26) | $x^{\nu-1} \sin(ax) \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > -1$                         | $\frac{y^{\mu+\nu-1} \Gamma(\nu)}{2i \Gamma(\mu+\nu)} [{}_1F_1(\nu; \mu+\nu; iay) - {}_1F_1(\nu; \mu+\nu; -iay)]$                                                                     |
| (27) | $\sin(ax^{\frac{1}{2}}) \quad \operatorname{Re} \mu > 0$                                                       | $2^{\mu-\frac{1}{2}} \pi^{\frac{1}{2}} a^{\frac{1}{2}-\mu} y^{\frac{1}{2}\mu+\frac{1}{4}} J_{\mu+\frac{1}{2}}(ay^{\frac{1}{2}})$                                                      |
| (28) | $x^{-\frac{1}{2}} \sin(ax^{\frac{1}{2}}) \quad \operatorname{Re} \mu > 0$                                      | $\pi^{\frac{1}{2}} 2^{\mu-\frac{1}{2}} a^{\frac{1}{2}-\mu} y^{\frac{1}{2}\mu-\frac{1}{4}} \mathbf{H}_{\mu-\frac{1}{2}}(ay^{\frac{1}{2}})$                                             |
| (29) | $x^{\nu-1} \sin(ax^{\frac{1}{2}}) \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$ | $\frac{ay^{\mu+\nu-\frac{1}{2}} \Gamma(\nu+\frac{1}{2})}{\Gamma(\mu+\nu+\frac{1}{2})} \times {}_1F_2\left(\nu+\frac{1}{2}; \frac{3}{2}, \mu+\nu+\frac{1}{2}; -\frac{a^2 y}{4}\right)$ |
| (30) | $x^{-\mu-\frac{1}{2}} \sin(ax^{-\frac{1}{2}}) \quad 0 < \operatorname{Re} \mu < 1, \quad a > 0$                | $2^{\mu-\frac{1}{2}} \pi^{\frac{1}{2}} a^{\frac{1}{2}-\mu} y^{\frac{1}{2}\mu-\frac{1}{4}} J_{\frac{1}{2}-\mu}(ay^{-\frac{1}{2}}) \quad  \arg y  < \pi$                                |
| (31) | $x^{\mu-1} \cos(ax) \quad \operatorname{Re} \mu > 0$                                                           | $\pi^{\frac{1}{2}} (y/a)^{\mu-\frac{1}{2}} \cos(\frac{1}{2}ay) J_{\mu-\frac{1}{2}}(\frac{1}{2}ay)$                                                                                    |
| (32) | $x^{\nu-1} \cos(ax) \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0$                          | $\frac{y^{\mu+\nu-1} \Gamma(\nu)}{2\Gamma(\mu+\nu)} [{}_1F_1(\nu; \mu+\nu; iay) + {}_1F_1(\nu; \mu+\nu; -iay)]$                                                                       |

## Riemann-Liouville integrals (cont'd)

|      | $f(x)$                                                                                                                                       | $[\Gamma(\mu)]^{-1} \int_0^y f(x) (y-x)^{\mu-1} dx$                                                                                                                                                                                                     |
|------|----------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (33) | $\cos(ax^{\frac{1}{2}}) \quad \text{Re } \mu > 0$                                                                                            | $\frac{y^\mu}{\Gamma(\mu+1)} - 2^{\mu-\frac{1}{2}} a^{\frac{1}{2}-\mu} \pi^{\frac{1}{2}} \times y^{\frac{1}{2}\mu+\frac{1}{4}} \mathbf{H}_{\mu+\frac{1}{2}}(ay^{\frac{1}{2}})$                                                                          |
| (34) | $x^{-\frac{1}{2}} \cos(ay^{\frac{1}{2}}) \quad \text{Re } \mu > 0$                                                                           | $2^{\mu-\frac{1}{2}} \pi^{\frac{1}{2}} a^{\frac{1}{2}-\mu} y^{\frac{1}{2}\mu-\frac{1}{4}} J_{\mu-\frac{1}{2}}(ay^{\frac{1}{2}})$                                                                                                                        |
| (35) | $x^{\nu-1} \cos(ax^{\frac{1}{2}}) \quad \text{Re } \mu > 0, \quad \text{Re } \nu > 0$                                                        | $\frac{y^{\mu+\nu-1} \Gamma(\nu)}{\Gamma(\mu+\nu)} \times {}_1F_2\left(\nu; \frac{1}{2}, \mu+\nu; -\frac{a^2 y}{4}\right)$                                                                                                                              |
| (36) | $x^{-\mu-\frac{1}{2}} \cos(ax^{-\frac{1}{2}}) \quad 0 < \text{Re } \mu < 1, \quad a > 0$                                                     | $-2^{\mu-\frac{1}{2}} \pi^{\frac{1}{2}} a^{\frac{1}{2}-\mu} y^{\frac{1}{2}\mu-\frac{1}{4}} Y_{\frac{1}{2}-\mu}(ay^{-\frac{1}{2}}) \quad  \arg y  < \pi$                                                                                                 |
| (37) | $P_n(1-\gamma x) \quad \text{Re } \mu > 0$                                                                                                   | $\frac{n! y^\mu}{\Gamma(\mu+n+1)} P_n^{(\mu, -\mu)}(1-\gamma y)$                                                                                                                                                                                        |
| (38) | $x^{\nu-1} P_n(1-\gamma x) \quad \text{Re } \mu > 0, \quad \text{Re } \nu > 0$                                                               | $\frac{\Gamma(\nu) y^{\mu+\nu-1}}{\Gamma(\mu+\nu)} \times {}_3F_2(-n, n+1, \nu; 1, \mu+\nu; \frac{1}{2}\gamma y)$                                                                                                                                       |
| (39) | $x^{\lambda-\frac{1}{2}} C_n^\lambda(1-\gamma x) \quad \text{Re } \lambda > -1, \quad \lambda \neq 0, -\frac{1}{2} \quad \text{Re } \mu > 0$ | $\frac{(2\lambda)_n \Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda + \mu + n + \frac{1}{2})} \times y^{\lambda+\mu-\frac{1}{2}} P_n^{(\alpha, \beta)}(1-\gamma y) \quad \alpha = \lambda + \mu - \frac{1}{2}, \quad \beta = \lambda - \mu - \frac{1}{2}$ |

**Riemann-Liouville integrals (cont'd)**

|      | $f(x)$                                                                                                                                   | $[\Gamma(\mu)]^{-1} \int_0^y f(x) (y-x)^{\mu-1} dx$                                                                                                                                                                                          |
|------|------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (40) | $x^{\nu-1} C_n^\lambda (1-\gamma x)$<br>$2\lambda \neq 0, -1, -2, \dots$<br>$\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0$ | $\frac{(2\lambda)_n \Gamma(\nu)}{n! \Gamma(\mu+\nu)} y^{\mu+\nu-1}$ $\times {}_3F_2(-n, n+2\lambda, \nu; \lambda+\frac{1}{2}; \mu+\nu; \frac{1}{2}\gamma y)$                                                                                 |
| (41) | $x^{\nu-1} C_{2n}^\lambda (\gamma x^{\frac{1}{2}})$<br>$\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0$                      | $(-1)^n \frac{(\lambda)_n \Gamma(\nu)}{n! \Gamma(\mu+\nu)} y^{\mu+\nu-1}$ $\times {}_3F_2(-n, n+\lambda, \nu; \frac{1}{2}, \mu+\nu; \gamma^2 y)$                                                                                             |
| (42) | $x^{\nu-1} C_{2n+1}^\lambda (\gamma x^{\frac{1}{2}})$<br>$\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$         | $2(-1)^n \gamma y^{\mu+\nu-\frac{1}{2}}$ $\times \frac{(\lambda)_{n+1} \Gamma(\nu+\frac{1}{2})}{n! \Gamma(\mu+\nu+\frac{1}{2})}$ $\times {}_3F_2\left(-n, n+\lambda+1, \nu+\frac{1}{2}; \frac{3}{2}, \mu+\nu+\frac{1}{2}; \gamma^2 y\right)$ |
| (43) | $x^\alpha P_n^{(\alpha, \beta)} (1-\gamma x)$<br>$\operatorname{Re} \alpha > -1, \quad \operatorname{Re} \mu > 0$                        | $\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+\mu+n+1)} y^{\alpha+\mu}$ $\times P_n^{(\alpha+\mu, \beta-\mu)} (1-\gamma y)$                                                                                                                       |
| (44) | $x^\beta P_n^{(\alpha, \beta)} (\gamma x-1)$<br>$\operatorname{Re} \beta > -1, \quad \operatorname{Re} \mu > 0$                          | $\frac{\Gamma(\beta+n+1)}{\Gamma(\beta+\mu+n+1)} y^{\beta+\mu}$ $\times P_n^{(\alpha-\mu, \beta+\mu)} (\gamma y-1)$                                                                                                                          |

**Riemann-Liouville integrals (cont'd)**

|      | $f(x)$                                                                                                                                                             | $[\Gamma(\mu)]^{-1} \int_0^y f(x) (y-x)^{\mu-1} dx$                                                                                                                                                             |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (45) | $x^{-\beta-\mu-n-1} (1 - \frac{1}{2}\gamma x)^\beta$<br>$\times P_n^{(\alpha, \beta)}(1 - \gamma x)$<br>$0 < \operatorname{Re} \mu < -\operatorname{Re} \beta - n$ | $\frac{\Gamma(-\beta - \mu - n)}{\Gamma(-\beta - \mu)} y^{-\beta-n-1} (1 - \frac{1}{2}\gamma y)^{\beta+\mu}$<br>$\times P_n^{(\alpha, \beta+\mu)}(1 - \gamma y)$                                                |
| (46) | $x^{\lambda-1} P_n^{(\alpha, \beta)}(1 - \gamma x)$<br>$\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \mu > 0$                                            | $\frac{\Gamma(a+n+1) \Gamma(\lambda)}{n! \Gamma(a+1) \Gamma(\lambda+\mu)} y^{\lambda+\mu-1}$<br>$\times {}_3F_2(-n, n+a+\beta+1, \lambda;$<br>$\quad a+1, \lambda+\mu; \frac{1}{2}\gamma y)$                    |
| (47) | $x^{\lambda-1} P_n^{(\alpha, \beta)}(\gamma x - 1)$<br>$\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \mu > 0$                                            | $(-1)^n \frac{\Gamma(\beta+n+1) \Gamma(\lambda)}{n! \Gamma(\beta+1) \Gamma(\lambda+\mu)} y^{\lambda+\mu-1}$<br>$\times {}_3F_2(-n, n+a+\beta+1, \lambda;$<br>$\quad \beta+1, \lambda+\mu; \frac{1}{2}\gamma y)$ |
| (48) | $x^{\lambda-1} (1 - \frac{1}{2}\gamma x)^\beta P_n^{(\alpha, \beta)}(1 - \gamma x)$<br>$\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \mu > 0$            | $\frac{\Gamma(n+a+1) \Gamma(\lambda)}{n! \Gamma(a+1) \Gamma(\lambda+\mu)} y^{\lambda+\mu-1}$<br>$\times {}_3F_2(a+n+1, -\beta-n, \lambda;$<br>$\quad a+1, \lambda+\mu; \frac{1}{2}\gamma y)$                    |
| (49) | $x^\alpha L_n^\alpha(\beta x)$<br>$\operatorname{Re} \alpha > -1, \operatorname{Re} \mu > 0$                                                                       | $\frac{\Gamma(a+n+1)}{\Gamma(a+\mu+n+1)} y^{a+\mu} L_n^{a+\mu}(\beta y)$                                                                                                                                        |
| (50) | $x^{\lambda-1} L_n^\alpha(\beta x)$<br>$\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \mu > 0$                                                            | $\frac{\Gamma(a+n+1) \Gamma(\lambda)}{n! \Gamma(a+1) \Gamma(\lambda+\mu)} y^{\lambda+\mu-1}$<br>$\times {}_2F_2(-n, \lambda; a+1, \lambda+\mu; \beta y)$                                                        |

**Riemann-Liouville integrals (cont'd)**

|      | $f(x)$                                                                                                                                                                                   | $[\Gamma(\mu)]^{-1} \int_0^y f(x) (y-x)^{\mu-1} dx$                                                                                                                                                                                                                                                |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (51) | $x^{\lambda-1} e^{-\beta x} L_n^\alpha(\beta x)$<br>$\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \mu > 0$                                                                     | $\frac{\Gamma(a+n+1) \Gamma(\lambda)}{n! \Gamma(a+1) \Gamma(\lambda+\mu)} y^{\lambda+\mu-1}$<br>$\times {}_2F_2(a+n+1, \lambda; a+1, \lambda+\mu; -\beta y)$                                                                                                                                       |
| (52) | $[x(1+\frac{1}{2}\gamma x)]^{-\frac{1}{2}\lambda} P_\nu^\lambda(1+\gamma x)$<br>$\operatorname{Re} \lambda < 1, \quad \operatorname{Re} \mu > 0$                                         | $(2/\gamma)^{\frac{1}{2}\mu} [y(1+\frac{1}{2}\gamma y)]^{\frac{1}{2}\mu-\frac{1}{2}\lambda}$<br>$\times P_\nu^{\lambda-\mu}(1+\gamma y) \quad  \arg \gamma y  < \pi$                                                                                                                               |
| (53) | $x^{\kappa+\frac{1}{2}\lambda-1} (1+\frac{1}{2}\gamma x)^{-\frac{1}{2}\lambda}$<br>$\times P_\nu^\lambda(1+\gamma x)$<br>$\operatorname{Re} \kappa > 0, \quad \operatorname{Re} \mu > 0$ | $\frac{(\frac{1}{2}\gamma)^{-\frac{1}{2}\lambda} \Gamma(\kappa)}{\Gamma(1-\lambda) \Gamma(\kappa+\mu)} y^{\kappa+\mu-1}$<br>$\times {}_3F_2(-\nu, 1+\nu, \kappa; 1-\lambda, \kappa+\mu; -\frac{1}{2}\gamma y)$<br>$ \gamma y  < 1$                                                                 |
| (54) | $[x(1-x)]^{-\frac{1}{2}\lambda} P_\nu^\lambda(1-2x)$<br>$\operatorname{Re} \lambda < 1, \quad \operatorname{Re} \mu > 0$                                                                 | $[y(1-y)]^{\frac{1}{2}\mu-\frac{1}{2}\lambda} P_\nu^{\lambda-\mu}(1-2y)$<br>$0 < y < 1$                                                                                                                                                                                                            |
| (55) | $x^{\kappa+\frac{1}{2}\lambda-1} (1-x)^{-\frac{1}{2}\lambda} P_\nu^\lambda(1-2x)$<br>$\operatorname{Re} \kappa > 0, \quad \operatorname{Re} \mu > 0$                                     | $\frac{\Gamma(\kappa) y^{\kappa+\mu-1}}{\Gamma(\kappa+\mu) \Gamma(1-\lambda)}$<br>$\times {}_3F_2(-\nu, 1+\nu, \kappa; 1-\lambda, \kappa+\mu; y)$<br>$0 < y < 1$                                                                                                                                   |
| (56) | $x^{\lambda-1} J_\nu(ax)$<br>$\operatorname{Re} \mu > 0, \quad \operatorname{Re}(\lambda+\nu) > 0$                                                                                       | $\frac{\Gamma(\lambda+\nu)}{\Gamma(\nu+1) \Gamma(\lambda+\mu+\nu)} (\frac{1}{2}a)^\nu y^{\lambda+\mu+\nu-1}$<br>$\times {}_2F_3\left(\frac{\lambda+\nu}{2}, \frac{\lambda+\nu+1}{2}; \nu+1, \right.$<br>$\left. \frac{\lambda+\mu+\nu}{2}, \frac{\lambda+\mu+\nu+1}{2}; -\frac{a^2 y^2}{4}\right)$ |

## Riemann-Liouville integrals (cont'd)

|      | $f(x)$                                                                                                                            | $[\Gamma(\mu)]^{-1} \int_0^y f(x) (y-x)^{\mu-1} dx$                                                                                                                                 |
|------|-----------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (57) | $x^\nu e^{\pm iax} J_\nu(ax)$<br>$\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$                          | $\frac{(2a)^\nu y^{\mu+2\nu} \Gamma(\nu + \frac{1}{2})}{\pi^{\frac{1}{2}} \Gamma(\mu + 2\nu + 1)}$<br>$\times {}_1F_1(\nu + \frac{1}{2}; \mu + 2\nu + 1; \pm 2iay)$                 |
| (58) | $x^{\lambda-\nu-1} e^{\pm iax} J_\nu(ax)$<br>$\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \mu > 0$                     | $\frac{2^{-\nu} a^\nu y^{\lambda+\mu-1} \Gamma(\lambda)}{\Gamma(\lambda + \mu) \Gamma(\nu + 1)}$<br>$\times {}_2F_2(\lambda, \nu + \frac{1}{2}; \lambda + \mu, 2\nu + 1; \pm 2iay)$ |
| (59) | $x^{-\frac{1}{2}} J_{2\nu}(ax^{\frac{1}{2}})$<br>$\mu = \frac{1}{2}, \quad \operatorname{Re} \nu > -\frac{1}{2}$                  | $\pi^{\frac{1}{2}} [J_\nu(\frac{1}{2}ay^{\frac{1}{2}})]^2$                                                                                                                          |
| (60) | $x^{\frac{1}{2}\nu-\frac{1}{2}} J_\nu(ax^{\frac{1}{2}})$<br>$\mu = \nu + \frac{1}{2}, \quad \operatorname{Re} \nu > -\frac{1}{2}$ | $\pi^{\frac{1}{2}} \left(\frac{2y}{a}\right)^\nu [J_\nu(\frac{1}{2}ay^{\frac{1}{2}})]^2$                                                                                            |
| (61) | $x^{\frac{1}{2}\nu-\frac{1}{2}} J_\nu(ax^{\frac{1}{2}})$<br>$\mu = \nu - \frac{1}{2}, \quad \operatorname{Re} \nu > \frac{1}{2}$  | $(\frac{1}{2}a)^{1-\nu} \pi^{\frac{1}{2}} y^{\nu-\frac{1}{2}} J_\nu(\frac{1}{2}ay^{\frac{1}{2}})$<br>$\times J_{\nu-1}(\frac{1}{2}ay^{\frac{1}{2}})$                                |
| (62) | $x^{-\frac{1}{2}\nu-\frac{1}{2}} J_\nu(ax^{\frac{1}{2}})$<br>$\mu = \frac{1}{2} - \nu, \quad \operatorname{Re} \nu < \frac{1}{2}$ | $\pi^{\frac{1}{2}} \left(\frac{a}{2y}\right)^\nu J_\nu(\frac{1}{2}ay^{\frac{1}{2}}) J_{-\nu}(\frac{1}{2}ay^{\frac{1}{2}})$                                                          |
| (63) | $x^{\frac{1}{2}\nu} J_\nu(ax^{\frac{1}{2}})$<br>$\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > -1$                     | $2^\mu a^{-\mu} y^{\frac{1}{2}\mu+\frac{1}{2}\nu} J_{\mu+\nu}(ay^{\frac{1}{2}})$                                                                                                    |
| (64) | $x^{-\frac{1}{2}\nu} J_\nu(ax^{\frac{1}{2}})$<br>$\operatorname{Re} \mu > 0$                                                      | $\frac{2^{2-\nu} a^{-\mu} y^{\frac{1}{2}\mu-\frac{1}{2}\nu}}{\Gamma(\mu) \Gamma(\nu)} s_{\mu+\nu-1, \mu-\nu}(ay^{\frac{1}{2}})$                                                     |



**Riemann-Liouville integrals((cont'd)**

|      | $f(x)$                                                                                                                                                                                                                                | $[\Gamma(\mu)]^{-1} \int_0^y f(x) (y-x)^{\mu-1} dx$                                                                                                                                                            |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (65) | $x^{\lambda-\frac{1}{2}\nu-1} J_\nu(ax^{\frac{1}{2}})$<br>$\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \mu > 0$                                                                                                            | $\frac{a^\nu y^{\lambda+\mu-1} \Gamma(\lambda)}{2^\nu \Gamma(\lambda+\mu) \Gamma(\nu+1)}$ $\times {}_1F_2(\lambda; \nu+1, \lambda+\mu; -\frac{1}{4}a^2 y)$                                                     |
| (66) | $x^{\lambda-\nu-1} [J_\nu(ax^{\frac{1}{2}})]^2$<br>$\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \mu > 0$<br>For several particular cases<br>see Bailey, W.N., 1938:<br><i>Quart. J. Math. Oxford Ser.</i> ,<br>9, 141-147. | $\frac{(\frac{1}{2}a)^{2\nu} y^{\lambda+\mu-1} \Gamma(\lambda)}{[\Gamma(\nu+1)]^2 \Gamma(\lambda+\mu)}$ $\times {}_2F_3(\lambda, \nu+\frac{1}{2}; \lambda+\mu, \nu+1, 2\nu+1; -a^2 y)$                         |
| (67) | $x^{\lambda-1} J_\nu(ax^{\frac{1}{2}}) J_{-\nu}(ax^{\frac{1}{2}})$<br>$\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \mu > 0$                                                                                                | $\frac{\Gamma(\lambda) \sin(\nu\pi)}{\nu\pi \Gamma(\lambda+\mu)} y^{\lambda+\mu-1}$ $\times {}_2F_3(\frac{1}{2}, \lambda; 1+\nu, 1-\nu, \lambda+\mu; -a^2 y)$                                                  |
| (68) | $x^{-\frac{1}{2}} Y_\nu(ax^{\frac{1}{2}})$<br>$\mu = \frac{1}{2}, \quad -1 < \operatorname{Re} \nu < 1$                                                                                                                               | $\pi^{\frac{1}{2}} \operatorname{ctn}(\nu\pi) [J_{\frac{1}{2}\nu}(\frac{1}{2}ay^{\frac{1}{2}})]^2$ $- \pi^{\frac{1}{2}} \operatorname{csc}(\nu\pi) [J_{-\frac{1}{2}\nu}(\frac{1}{2}ay^{\frac{1}{2}})]^2$       |
| (69) | $x^{\frac{1}{2}\nu} Y_\nu(ax^{\frac{1}{2}})$<br>$\mu = \frac{1}{2}, \quad \operatorname{Re} \nu > -1$                                                                                                                                 | $2^{\frac{1}{2}} a^{-\frac{1}{2}} y^{\frac{1}{2}\nu+\frac{1}{4}} \operatorname{csc}(\nu\pi)$ $\times [\cos(\nu\pi) J_{\nu+\frac{1}{2}}(ay^{\frac{1}{2}})$ $- \mathbf{H}_{-\nu-\frac{1}{2}}(ay^{\frac{1}{2}})]$ |
| (70) | $x^{-\frac{1}{2}\nu} Y_\nu(ax^{\frac{1}{2}})$<br>$\mu = \frac{1}{2}, \quad \operatorname{Re} \nu < 1$                                                                                                                                 | $2^{\frac{1}{2}} a^{-\frac{1}{2}} y^{\frac{1}{4}-\frac{1}{2}\nu} \operatorname{csc}(\nu\pi)$ $\times [\cos(\nu\pi) \mathbf{H}_{\nu-\frac{1}{2}}(ay^{\frac{1}{2}})$ $- J_{\frac{1}{2}-\nu}(ay^{\frac{1}{2}})]$  |

## Riemann-Liouville integrals (cont'd)

|      | $f(x)$                                                                                                                     | $[\Gamma(\mu)]^{-1} \int_0^y f(x) (y-x)^{\mu-1} dx$                                                                                                                                                                                                                   |
|------|----------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (71) | $x^{\frac{1}{2}\nu-\frac{1}{2}} Y_\nu(ax^{\frac{1}{2}})$<br>$\mu = \nu + \frac{1}{2}, \quad \text{Re } \nu > -\frac{1}{2}$ | $\pi^{\frac{1}{2}} \left(\frac{2y}{a}\right)^\nu J_\nu(\frac{1}{2}ay^{\frac{1}{2}}) Y_\nu(\frac{1}{2}ay^{\frac{1}{2}})$                                                                                                                                               |
| (72) | $x^{-\frac{1}{2}\nu-\frac{1}{2}} Y_\nu(ax^{\frac{1}{2}})$<br>$\mu = \frac{1}{2} - \nu, \quad \text{Re } \nu < \frac{1}{2}$ | $\pi^{\frac{1}{2}} \left(\frac{a}{2y}\right)^\nu J_{-\nu}(\frac{1}{2}ay^{\frac{1}{2}}) Y_\nu(\frac{1}{2}ay^{\frac{1}{2}})$                                                                                                                                            |
| (73) | $x^{\frac{1}{2}\nu} Y_\nu(ax^{\frac{1}{2}})$<br>$\text{Re } \mu > 0, \quad \text{Re } \nu > -1$                            | $\left(\frac{2}{a}\right)^\mu y^{\frac{1}{2}\mu+\frac{1}{2}\nu} \text{ctn}(\nu\pi) J_{\mu+\nu}(ay^{\frac{1}{2}})$<br>$+ \frac{2^{\nu+2} y^{\frac{1}{2}\mu+\frac{1}{2}\nu} \Gamma(\nu+1)}{\pi a^\mu \Gamma(\mu)}$<br>$\times S_{\mu-\nu-1, \mu+\nu}(ay^{\frac{1}{2}})$ |
| (74) | $x^{\frac{1}{2}\nu} Y_\nu(ax^{\frac{1}{2}})$<br>$\text{Re } \mu > 0, \quad \text{Re } \nu > -1$                            | $\left(\frac{2}{a}\right)^\mu y^{\frac{1}{2}\mu+\frac{1}{2}\nu} Y_{\mu+\nu}(ay^{\frac{1}{2}})$<br>$+ \frac{2^{\nu+2} y^{\frac{1}{2}\mu+\frac{1}{2}\nu} \Gamma(\nu+1)}{\pi a^\mu \Gamma(\mu)}$<br>$\times S_{\mu-\nu-1, \mu+\nu}(ay^{\frac{1}{2}})$                    |
| (75) | $x^{-\frac{1}{2}\nu} Y_\nu(ax^{\frac{1}{2}})$<br>$\text{Re } \mu > 0, \quad \text{Re } \nu < 1$                            | $\frac{y^{\frac{1}{2}\mu-\frac{1}{2}\nu} \text{ctn}(\nu\pi)}{2^{\nu-2} a^\mu \Gamma(\mu) \Gamma(\nu)} S_{\mu+\nu-1, \mu-\nu}(ay^{\frac{1}{2}})$<br>$- (2/a)^\mu y^{\frac{1}{2}\mu-\frac{1}{2}\nu} \text{csc}(\nu\pi)$<br>$\times J_{\mu-\nu}(ay^{\frac{1}{2}})$       |

## Riemann-Liouville integrals (cont'd)

|      | $f(x)$                                                                                                                                        | $[\Gamma(\mu)]^{-1} \int_0^y f(x) (y-x)^{\mu-1} dx$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (76) | $x^{\lambda-1} Y_\nu(ax^{\frac{1}{2}})$<br>$\operatorname{Re} \lambda > \frac{1}{2}  \operatorname{Re} \nu , \quad \operatorname{Re} \mu > 0$ | $2^{-\nu} a^\nu y^{\lambda+\mu+\frac{1}{2}\nu-1} \operatorname{ctn}(\nu\pi)$<br>$\times \frac{\Gamma(\lambda + \frac{1}{2}\nu)}{\Gamma(1+\nu) \Gamma(\lambda + \mu + \frac{1}{2}\nu)}$<br>$\times {}_1F_2(\lambda + \frac{1}{2}\nu; 1+\nu, \lambda + \mu + \frac{1}{2}\nu;$<br>$\quad -\frac{1}{4}a^2y)$<br>$- 2^\nu a^{-\nu} y^{\lambda+\mu-\frac{1}{2}\nu-1} \operatorname{csc}(\nu\pi)$<br>$\times \frac{\Gamma(\lambda - \frac{1}{2}\nu)}{\Gamma(1-\nu) \Gamma(\lambda + \mu - \frac{1}{2}\nu)}$<br>$\times {}_1F_2(\lambda - \frac{1}{2}\nu; 1-\nu, \lambda + \mu - \frac{1}{2}\nu;$<br>$\quad -\frac{1}{4}a^2y)$ |
| (77) | $x^\nu e^{\pm ax} I_\nu(ax)$<br>$\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$                                       | $\frac{(2a)^\nu y^{\mu+2\nu} \Gamma(\nu + \frac{1}{2})}{\pi^{\frac{1}{2}} \Gamma(\mu + 2\nu + 1)}$<br>$\times {}_1F_1(\nu + \frac{1}{2}; \mu + 2\nu + 1; \pm 2ay)$                                                                                                                                                                                                                                                                                                                                                                                                                                                     |
| (78) | $x^{\lambda-1} e^{\pm ax} I_\nu(ax)$<br>$\operatorname{Re} \mu > 0, \quad \operatorname{Re}(\lambda + \nu) > 0$                               | $\frac{(\frac{1}{2}a)^\nu y^{\lambda+\mu+\nu-1} \Gamma(\lambda + \nu)}{\Gamma(\nu + 1) \Gamma(\lambda + \mu + \nu)}$<br>$\times {}_2F_2(\nu + \frac{1}{2}, \lambda + \nu; 2\nu + 1, \mu + \lambda + \nu;$<br>$\quad \pm 2ay)$                                                                                                                                                                                                                                                                                                                                                                                          |
| (79) | $x^{-\frac{1}{2}} I_{2\nu}(ax^{\frac{1}{2}})$<br>$\mu = \frac{1}{2}, \quad \operatorname{Re} \nu > -\frac{1}{2}$                              | $\pi^{\frac{1}{2}} [I_\nu(\frac{1}{2}ay^{\frac{1}{2}})]^2$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             |
| (80) | $x^{\frac{1}{2}\nu-\frac{1}{2}} I_\nu(ax^{\frac{1}{2}})$<br>$\mu = \nu + \frac{1}{2}, \quad \operatorname{Re} \nu > -\frac{1}{2}$             | $\pi^{\frac{1}{2}} \left(\frac{2y}{a}\right)^\nu [I_\nu(\frac{1}{2}ay^{\frac{1}{2}})]^2$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               |

## Riemann-Liouville integrals (cont'd)

|      | $f(x)$                                                                                                                     | $[\Gamma(\mu)]^{-1} \int_0^y f(x) (y-x)^{\mu-1} dx$                                                                                                                                                                                                                                                                                                                                                                                                                                                      |
|------|----------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (81) | $x^{\frac{1}{2}\nu-\frac{1}{2}} I_\nu(ax^{\frac{1}{2}})$<br>$\mu = \nu - \frac{1}{2}, \quad \text{Re } \nu > \frac{1}{2}$  | $(\frac{1}{2}a)^{1-\nu} \pi^{\frac{1}{2}} y^{\nu-\frac{1}{2}} I_\nu(\frac{1}{2}ay^{\frac{1}{2}})$<br>$\times I_{\nu-1}(\frac{1}{2}ay^{\frac{1}{2}})$                                                                                                                                                                                                                                                                                                                                                     |
| (82) | $x^{-\frac{1}{2}\nu-\frac{1}{2}} I_\nu(ax^{\frac{1}{2}})$<br>$\mu = \frac{1}{2} - \nu, \quad \text{Re } \nu < \frac{1}{2}$ | $\pi^{\frac{1}{2}} \left(\frac{a}{2y}\right)^\nu I_\nu(\frac{1}{2}ay^{\frac{1}{2}}) I_{-\nu}(\frac{1}{2}ay^{\frac{1}{2}})$                                                                                                                                                                                                                                                                                                                                                                               |
| (83) | $x^{\frac{1}{2}\nu} I_\nu(ax^{\frac{1}{2}})$<br>$\text{Re } \mu > 0, \quad \text{Re } \nu > -1$                            | $2^\mu a^{-\mu} y^{\frac{1}{2}\mu+\frac{1}{2}\nu} I_{\mu+\nu}(ay^{\frac{1}{2}})$                                                                                                                                                                                                                                                                                                                                                                                                                         |
| (84) | $x^{\lambda-\frac{1}{2}\nu-1} I_\nu(ax^{\frac{1}{2}})$<br>$\text{Re } \lambda > 0, \quad \text{Re } \mu > 0$               | $\frac{a^\nu y^{\lambda+\mu-1} \Gamma(\lambda)}{2^\nu \Gamma(\nu+1) \Gamma(\lambda+\mu)}$<br>$\times {}_1F_2(\lambda; \nu+1, \lambda+\mu; \frac{1}{4}a^2y)$                                                                                                                                                                                                                                                                                                                                              |
| (85) | $x^{-\frac{1}{2}} K_{2\nu}(ax^{\frac{1}{2}})$<br>$\mu = \frac{1}{2}, \quad -\frac{1}{2} < \text{Re } \nu < \frac{1}{2}$    | $\frac{1}{2} \sec(\nu\pi) [I_\nu(\frac{1}{2}ay^{\frac{1}{2}})$<br>$+ I_{-\nu}(\frac{1}{2}ay^{\frac{1}{2}})] K_\nu(\frac{1}{2}ay^{\frac{1}{2}})$                                                                                                                                                                                                                                                                                                                                                          |
| (86) | $x^{\frac{1}{2}\nu-\frac{1}{2}} K_\nu(ax^{\frac{1}{2}})$<br>$\mu = \nu + \frac{1}{2}, \quad \text{Re } \nu > -\frac{1}{2}$ | $\pi^{\frac{1}{2}} (2y/a)^\nu I_\nu(\frac{1}{2}ay^{\frac{1}{2}}) K_\nu(\frac{1}{2}ay^{\frac{1}{2}})$                                                                                                                                                                                                                                                                                                                                                                                                     |
| (87) | $x^{\lambda-1} K_\nu(ax^{\frac{1}{2}})$<br>$\text{Re } \lambda > \frac{1}{2} \text{Re } \nu , \quad \text{Re } \mu > 0$    | $2^{\nu-1} a^{-\nu} y^{\lambda+\mu-\frac{1}{2}\nu-1} \frac{\Gamma(\nu)\Gamma(\lambda-\frac{1}{2}\nu)}{\Gamma(\lambda+\mu-\frac{1}{2}\nu)}$<br>$\times {}_1F_2\left(\lambda-\frac{\nu}{2}; 1-\nu, \lambda+\mu-\frac{\nu}{2}; \frac{a^2y}{4}\right)$<br>$+ 2^{1-\nu} a^\nu y^{\lambda+\mu+\frac{1}{2}\nu-1} \frac{\Gamma(-\nu)\Gamma(\lambda+\frac{1}{2}\nu)}{\Gamma(\lambda+\mu+\frac{1}{2}\nu)}$<br>$\times {}_1F_2\left(\lambda+\frac{\nu}{2}; 1+\nu, \lambda+\mu+\frac{\nu}{2}; \frac{a^2y}{4}\right)$ |

**Riemann-Liouville integrals (cont'd)**

|      | $f(x)$                                                                                                                                            | $[\Gamma(\mu)]^{-1} \int_0^y f(x) (y-x)^{\mu-1} dx$                                                                                                                                                                                        |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (88) | $x^{\frac{1}{2}\nu} \mathbf{H}_\nu(\gamma x^{\frac{1}{2}})$<br>$\operatorname{Re} \nu > -3/2, \quad \operatorname{Re} \mu > 0$                    | $(\frac{1}{2}\gamma)^{-\mu} \gamma^{\frac{1}{2}\mu+\frac{1}{2}\nu} \mathbf{H}_{\mu+\nu}(\gamma y^{\frac{1}{2}})$                                                                                                                           |
| (89) | $x^{\lambda-\nu/2-3/2} \mathbf{H}_\nu(\gamma x^{\frac{1}{2}})$<br>$\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \mu > 0$                | $\frac{\Gamma(\lambda) \gamma^{\nu+1} \gamma^{\lambda+\mu-1}}{2^\nu \pi^{\frac{1}{2}} \Gamma(\nu+3/2) \Gamma(\lambda+\mu)}$<br>$\times {}_2F_3\left(1, \lambda; \frac{3}{2}, \nu+\frac{3}{2}, \lambda+\mu; -\frac{\gamma^2 y}{4}\right)$   |
| (90) | $x^{\frac{1}{2}\nu} \mathbf{L}_\nu(\gamma x^{\frac{1}{2}})$<br>$\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > -3/2$                    | $(\frac{1}{2}\gamma)^{-\mu} \gamma^{\frac{1}{2}\mu+\frac{1}{2}\nu} \mathbf{L}_{\mu+\nu}(\gamma y^{\frac{1}{2}})$                                                                                                                           |
| (91) | $x^{\lambda-\nu/2-3/2} \mathbf{L}_\nu(\gamma x^{\frac{1}{2}})$<br>$\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \mu > 0$                | $\frac{\gamma^{\nu+1} \gamma^{\lambda+\mu-1} \Gamma(\lambda)}{2^\nu \pi^{\frac{1}{2}} \Gamma(\nu+3/2) \Gamma(\lambda+\mu)}$<br>$\times {}_2F_3\left(1, \lambda; \frac{3}{2}, \nu+\frac{3}{2}, \lambda+\mu; \frac{\gamma^2 y}{4}\right)$    |
| (92) | $x^{\lambda-\frac{1}{2}\kappa-\frac{1}{2}} s_{\kappa,\nu}(ax^{\frac{1}{2}})$<br>$\operatorname{Re} \lambda > -1, \quad \operatorname{Re} \mu > 0$ | $\frac{a^\kappa \gamma^{\lambda+\mu} \Gamma(\lambda+1)}{(\kappa-\nu+1)(\kappa+\nu+1) \Gamma(\lambda+\mu+1)}$<br>$\times {}_2F_3\left(1, \lambda+1; \frac{\kappa-\nu+3}{2}, \frac{\kappa+\nu+3}{2}, \lambda+\mu+1; -\frac{a^2 y}{4}\right)$ |

## Riemann-Liouville integrals (cont'd)

|      | $f(x)$                                                                                                                                                                                                                     | $[\Gamma(\mu)]^{-1} \int_0^y f(x) (\gamma - x)^{\mu-1} dx$                                                                                                                                                                                                                                                    |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (93) | $x^{\kappa-\mu-1} e^{-\frac{1}{2}ax} W_{\kappa, \lambda}(ax)$<br>$0 < \operatorname{Re} \mu < \operatorname{Re} \kappa -  \operatorname{Re} \lambda  + \frac{1}{2}$                                                        | $y^{\kappa-1} e^{-\frac{1}{2}ay} \sec[(\kappa - \mu - \lambda)\pi]$<br>$\times \left\{ \sin(\mu\pi) \frac{\Gamma(\kappa - \mu + \lambda + \frac{1}{2})}{\Gamma(2\lambda + 1)} \right.$<br>$\times M_{\kappa-\mu, \lambda}(ay)$<br>$\left. + \cos[(\kappa - \lambda)\pi] W_{\kappa-\mu, \lambda}(ay) \right\}$ |
| (94) | $x^{\nu-1} {}_pF_q(a_1, \dots, a_p;$<br>$\nu, b_2, \dots, b_q; ax)$<br>$p \leq q+1$<br>$\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0$                                                                        | $\frac{y^{\mu+\nu-1} \Gamma(\nu)}{\Gamma(\mu + \nu)}$<br>$\times {}_pF_q(a_1, \dots, a_p; \mu + \nu, b_2, \dots, b_q; ay)$<br>$ ay  < 1 \text{ if } p = q+1$                                                                                                                                                  |
| (95) | $x^{\nu-1} {}_pF_q(a_1, \dots, a_p;$<br>$b_1, \dots, b_q; ax)$<br>$p \leq q+1$<br>$\operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0$                                                                             | $\frac{y^{\mu+\nu-1} \Gamma(\nu)}{\Gamma(\mu + \nu)}$<br>$\times {}_{p+1}F_{q+1}(\nu, a_1, \dots, a_p;$<br>$\mu + \nu, b_1, \dots, b_q; ay)$<br>$ ay  < 1 \text{ if } p = q+1$                                                                                                                                |
| (96) | $G_{pq}^{mn} \left( ax \left  \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right)$<br>$p \leq q, \quad \operatorname{Re} \mu > 0$<br>$\operatorname{Re} b_j > -1 \quad j = 1, \dots, m$         | $y^\mu G_{p+1, q+1}^{m, n+1} \left( ay \left  \begin{matrix} 0, a_1, \dots, a_p \\ b_1, \dots, b_q, -\mu \end{matrix} \right. \right)$<br>$ ay  < 1 \text{ if } p = q$                                                                                                                                        |
| (97) | $G_{pq}^{mn} \left( ax \left  \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right)$<br>$p + q < 2(m + n), \quad \operatorname{Re} \mu > 0$<br>$\operatorname{Re} b_j > -1 \quad j = 1, \dots, m$ | $y^\mu G_{p+1, q+1}^{m, n+1} \left( ay \left  \begin{matrix} 0, a_1, \dots, a_p \\ b_1, \dots, b_q, -\mu \end{matrix} \right. \right)$<br>$ \arg ay  < (m + n - \frac{1}{2}p - \frac{1}{2}q)\pi$                                                                                                              |

## 13.2. Weyl fractional integrals

|     |                                                                                           |                                                                                                                                                                                      |
|-----|-------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
|     | $f(x)$                                                                                    | $[\Gamma(\mu)]^{-1} \int_y^\infty f(x) (x-y)^{\mu-1} dx$<br>$= h(y; \mu)$                                                                                                            |
| (1) | $f(ax)$                                                                                   | $a^{-\mu} h(ay; \mu)$                                                                                                                                                                |
| (2) | $f(a/x)$                                                                                  | $a y^{\mu-1} \mathfrak{R}_\mu \{ t^{-\mu-1} f(t); a/y \}$<br>For tables see sec. 13.1.                                                                                               |
| (3) | $f'(x)$                                                                                   | $-h(y; \mu-1)$                                                                                                                                                                       |
| (4) | $\int_x^\infty f(t) dt$                                                                   | $h(y; \mu+1)$                                                                                                                                                                        |
| (5) | $h(x; \nu)$                                                                               | $h(y; \mu+\nu)$                                                                                                                                                                      |
| (6) | $x^{-\lambda} \quad 0 < \operatorname{Re} \mu < \operatorname{Re} \lambda$                | $\frac{\Gamma(\lambda-\mu)}{\Gamma(\lambda)} y^{\mu-\lambda}$                                                                                                                        |
| (7) | $(x+a)^{-\lambda} \quad 0 < \operatorname{Re} \mu < \operatorname{Re} \lambda$            | $\frac{\Gamma(\lambda-\mu)}{\Gamma(\lambda)} (y+a)^{\mu-\lambda}$<br>$ \arg(y/a)  < \pi$                                                                                             |
| (8) | $x^{-\lambda} (x+a)^\nu \quad 0 < \operatorname{Re} \mu < \operatorname{Re}(\lambda-\nu)$ | $y^{\mu+\nu-\lambda} \frac{\Gamma(\lambda-\mu-\nu)}{\Gamma(\lambda-\nu)}$<br>$\times {}_2F_1(-\nu, \lambda-\mu-\nu; \lambda-\nu; -a/y)$<br>$ \arg(a/y)  < \pi \text{ or }  a/y  < 1$ |

## Weyl integrals (cont'd)

|      | $f(x)$                                                                                                                                   | $[\Gamma(\mu)]^{-1} \int_y^\infty f(x) (x-y)^{\mu-1} dx$                                                                                                                                                                                                                                                                                 |
|------|------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (9)  | $x^{-\lambda} (x^2 + a^2)^\nu$<br>$0 < \operatorname{Re} \mu < \operatorname{Re}(\lambda - 2\nu)$                                        | $\frac{\Gamma(\lambda - \mu - 2\nu)}{\Gamma(\lambda - \mu)} y^{\mu - \lambda + 2\nu}$ $\times {}_3F_2 \left( -\nu, \frac{\lambda - \mu}{2} - \nu, \frac{1 + \lambda - \mu}{2} - \nu; \right.$ $\left. \frac{\lambda}{2} - \nu, \frac{1 + \lambda}{2} - \nu; -\frac{a^2}{y^2} \right)$ $ y  >  a  \text{ or } \operatorname{Re}(a/y) > 0$ |
| (10) | $(x^2 - 1)^{-\frac{1}{2}} [(x+1)^{\frac{1}{2}} - (x-1)^{\frac{1}{2}}]^{2\nu}$<br>$0 < \operatorname{Re} \mu < 1 + \operatorname{Re} \nu$ | $2^{\nu + \frac{1}{2}} \pi^{-\frac{1}{2}} e^{(\mu - \frac{1}{2})\pi i}$ $\times (y^2 - 1)^{\frac{1}{2}\mu - \frac{1}{2}} Q_{\nu - \frac{1}{2}}^{-\mu}(y)$ $ \arg(y - 1)  < \pi$                                                                                                                                                          |
| (11) | $e^{-ax}$<br>$\operatorname{Re} \mu > 0$                                                                                                 | $a^{-\mu} e^{-ay}$<br>$\operatorname{Re}(ay) > 0$                                                                                                                                                                                                                                                                                        |
| (12) | $x^{\mu-1} e^{-ax}$<br>$\operatorname{Re} \mu > 0$                                                                                       | $\pi^{-\frac{1}{2}} (y/a)^{\mu - \frac{1}{2}} \exp(-\frac{1}{2}ay)$<br>$\times K_{\mu - \frac{1}{2}}(\frac{1}{2}ay)$<br>$\operatorname{Re}(ay) > 0$                                                                                                                                                                                      |
| (13) | $x^{-\lambda} e^{-ax}$<br>$\operatorname{Re} \mu > 0$                                                                                    | $a^{\nu - \frac{1}{2}} y^{-\nu - \frac{1}{2}} e^{-\frac{1}{2}ay} W_{\kappa, \nu}(ay)$<br>$2\kappa = 1 - \lambda - \mu, \quad 2\nu = \lambda - \mu$<br>$\operatorname{Re}(ay) > 0$                                                                                                                                                        |
| (14) | $x^{-2\mu} \exp(a/x)$<br>$\operatorname{Re} \mu > 0$                                                                                     | $(\pi/y)^{\frac{1}{2}} a^{\frac{1}{2} - \mu} \exp\left(\frac{a}{2y}\right)$<br>$\times I_{\mu - \frac{1}{2}}\left(\frac{a}{2y}\right)$                                                                                                                                                                                                   |



**Weyl integrals (cont'd)**

|      | $f(x)$                                                                              | $[\Gamma(\mu)]^{-1} \int_y^\infty f(x) (x-y)^{\mu-1} dx$                                                                                                       |
|------|-------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (15) | $x^{-\lambda} \exp(a/x)$<br>$0 < \operatorname{Re} \mu < \operatorname{Re} \lambda$ | $\frac{\Gamma(\lambda - \mu)}{\Gamma(\lambda)} y^{\mu-\lambda} {}_1F_1(\lambda - \mu; \lambda; a/y)$                                                           |
| (16) | $\exp(-ax^{1/2})$ $\operatorname{Re} \mu > 0$                                       | $2^{\mu+1/2} \pi^{-1/2} a^{1/2-\mu} y^{1/2\mu+1/4} K_{\mu+1/2}(ay^{1/2})$<br>$\operatorname{Re}(ay^{1/2}) > 0$                                                 |
| (17) | $x^{-1/2} \exp(-ax^{1/2})$ $\operatorname{Re} \mu > 0$                              | $2^{\mu+1/2} \pi^{-1/2} a^{1/2-\mu} y^{1/2\mu-1/4} K_{\mu-1/2}(ay^{1/2})$<br>$\operatorname{Re}(ay^{1/2}) > 0$                                                 |
| (18) | $x^{-\lambda} \log x$ $0 < \operatorname{Re} \mu < \operatorname{Re} \lambda$       | $\frac{\Gamma(\lambda - \mu)}{\Gamma(\lambda)} y^{\mu-\lambda} [\log y + \psi(\lambda) - \psi(\lambda - \mu)]$                                                 |
| (19) | $\sin(ax)$ $a > 0, \quad 0 < \operatorname{Re} \mu < 1$                             | $a^{-\mu} \sin(ay + \frac{1}{2}\mu\pi)$                                                                                                                        |
| (20) | $x^{\mu-1} \sin(ax)$<br>$a > 0, \quad 0 < \operatorname{Re} \mu < \frac{1}{2}$      | $\frac{1}{2} \pi^{1/2} (y/a)^{\mu-1/2} [\cos(\frac{1}{2}ay) J_{\frac{1}{2}-\mu}(\frac{1}{2}ay)$<br>$- \sin(\frac{1}{2}ay) Y_{\frac{1}{2}-\mu}(\frac{1}{2}ay)]$ |
| (21) | $x^{-2\mu} \sin(a/x)$ $\operatorname{Re} \mu > 0$                                   | $\left(\frac{\pi}{y}\right)^{1/2} a^{1/2-\mu} \sin\left(\frac{a}{2y}\right)$<br>$\times J_{\mu-1/2}\left(\frac{a}{2y}\right)$                                  |
| (22) | $\sin(ax^{1/2})$<br>$a > 0, \quad 0 < \operatorname{Re} \mu < \frac{1}{2}$          | $2^{\mu-1/2} \pi^{1/2} a^{1/2-\mu} y^{1/2\mu+1/4} Y_{-\frac{1}{2}-\mu}(ay^{1/2})$                                                                              |
| (23) | $x^{-1/2} \sin(ax^{1/2})$<br>$a > 0, \quad 0 < \operatorname{Re} \mu < 1$           | $2^{\mu-1/2} \pi^{1/2} a^{1/2-\mu} y^{1/2\mu-1/4} J_{\frac{1}{2}-\mu}(ay^{1/2})$                                                                               |

## Weyl integrals (cont'd)

|      | $f(x)$                                                                                                                                                     | $[\Gamma(\mu)]^{-1} \int_y^\infty f(x) (x-y)^{\mu-1} dx$                                                                                                                                                                                                                                          |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (24) | $\cos(ax) \quad a > 0, \quad 0 < \operatorname{Re} \mu < 1$                                                                                                | $a^{-\mu} \cos(ay + \frac{1}{2}\mu\pi)$                                                                                                                                                                                                                                                           |
| (25) | $x^{\mu-1} \cos(ay)$<br>$a > 0, \quad 0 < \operatorname{Re} \mu < \frac{1}{2}$                                                                             | $-\frac{1}{2}\pi^{\frac{1}{2}} (y/a)^{\mu-\frac{1}{2}} [\sin(\frac{1}{2}ay) J_{\frac{1}{2}-\mu}(\frac{1}{2}ay)$<br>$+ \cos(\frac{1}{2}ay) Y_{\frac{1}{2}-\mu}(\frac{1}{2}ay)]$                                                                                                                    |
| (26) | $x^{-2\mu} \cos(a/x) \quad \operatorname{Re} \mu > 0$                                                                                                      | $(\pi/y)^{\frac{1}{2}} a^{\frac{1}{2}-\mu} \cos(a/2y) J_{\mu-\frac{1}{2}}(a/2y)$                                                                                                                                                                                                                  |
| (27) | $\cos(ax^{\frac{1}{2}})$<br>$a > 0, \quad 0 < \operatorname{Re} \mu < \frac{1}{2}$                                                                         | $2^{\mu-\frac{1}{2}} \pi^{\frac{1}{2}} a^{\frac{1}{2}-\mu} y^{\frac{1}{2}\mu+\frac{1}{4}} J_{-\frac{1}{2}-\mu}(ay^{\frac{1}{2}})$                                                                                                                                                                 |
| (28) | $x^{-\frac{1}{2}} \cos(ax^{\frac{1}{2}})$<br>$a > 0, \quad 0 < \operatorname{Re} \mu < 1$                                                                  | $-2^{\mu-\frac{1}{2}} \pi^{\frac{1}{2}} a^{\frac{1}{2}-\mu} y^{\frac{1}{2}\mu-\frac{1}{4}} Y_{\frac{1}{2}-\mu}(ay^{\frac{1}{2}})$                                                                                                                                                                 |
| (29) | $Q_\nu(x) \quad 0 < \operatorname{Re} \mu < 1 + \operatorname{Re} \nu$                                                                                     | $e^{\mu\pi i} (y^2 - 1)^{\frac{1}{2}\mu} Q_\nu^{-\mu}(y)$<br>$ \arg(y-1)  < \pi$                                                                                                                                                                                                                  |
| (30) | $(x^2 - 1)^{\frac{1}{2}\lambda} Q_\nu^{-\lambda}(x)$<br>$0 < \operatorname{Re} \mu < 1 + \operatorname{Re}(\nu - \lambda)$                                 | $e^{\mu\pi i} (y^2 - 1)^{\frac{1}{2}\lambda+\frac{1}{2}\mu} Q_\nu^{-\lambda-\mu}(y)$<br>$ \arg(y-1)  < \pi$                                                                                                                                                                                       |
| (31) | $x^{-\nu} e^{iax} J_\nu(ax)$<br>$a > 0, \quad 0 < \operatorname{Re} \mu < \frac{1}{2} + \operatorname{Re} \nu$                                             | $\frac{e^{\frac{1}{2}i\mu\pi} (2a)^{\nu-\mu} \Gamma(\frac{1}{2}-\mu+\nu)}{\pi^{\frac{1}{2}} \Gamma(1-\mu+2\nu)}$<br>$\times {}_1F_1(\frac{1}{2}-\mu+\nu; 1-\mu+2\nu; 2aiy)$<br>$y > 0$                                                                                                            |
| (32) | $x^{\frac{1}{2}\nu-\frac{1}{2}} J_\nu(ax^{\frac{1}{2}})$<br>$\mu = \nu + \frac{1}{2}, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$<br>$a > 0$ | $-\frac{\pi^{\frac{1}{2}}}{2} \left(\frac{2y}{a}\right)^\nu \left[ J_\nu\left(\frac{ay^{\frac{1}{2}}}{2}\right) Y_{-\nu}\left(\frac{ay^{\frac{1}{2}}}{2}\right) \right.$<br>$\left. + J_{-\nu}\left(\frac{ay^{\frac{1}{2}}}{2}\right) Y_\nu\left(\frac{ay^{\frac{1}{2}}}{2}\right) \right] y > 0$ |

## Weyl integrals (cont'd)

|      | $f(x)$                                                                                                                                                         | $[\Gamma(\mu)]^{-1} \int_y^\infty f(x) (x-y)^{\mu-1} dx$                                                                                                                                                                                                                                               |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (33) | $x^{\frac{1}{2}\nu-\frac{1}{2}} J_{-\nu}(ax^{\frac{1}{2}}) \quad a > 0$<br>$\mu = \nu + \frac{1}{2}, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$ | $-\pi^{\frac{1}{2}} \left(\frac{2y}{a}\right)^\nu J_{-\nu}\left(\frac{ay^{\frac{1}{2}}}{2}\right)$<br>$\times Y_{-\nu}\left(\frac{ay^{\frac{1}{2}}}{2}\right) \quad y > 0$                                                                                                                             |
| (34) | $x^{-\frac{1}{2}\nu} J_\nu(ax^{\frac{1}{2}}) \quad a > 0$<br>$0 < \operatorname{Re} \mu < \frac{1}{2} \operatorname{Re} \nu + \frac{3}{4}$                     | $2^\mu a^{-\mu} y^{\frac{1}{2}\mu-\frac{1}{2}\nu} J_{\nu-\mu}(ay^{\frac{1}{2}}) \quad y > 0$                                                                                                                                                                                                           |
| (35) | $x^{-\frac{1}{2}\nu} J_{-\nu}(ax^{\frac{1}{2}}) \quad a > 0$<br>$0 < \operatorname{Re} \mu < \frac{1}{2} \operatorname{Re} \nu + \frac{3}{4}$                  | $2^\mu a^{-\mu} y^{\frac{1}{2}\mu-\frac{1}{2}\nu} [\cos(\nu\pi) J_{\nu-\mu}(ay^{\frac{1}{2}})$<br>$- \sin(\nu\pi) Y_{\nu-\mu}(ay^{\frac{1}{2}})] \quad y > 0$                                                                                                                                          |
| (36) | $x^\lambda J_\nu(ax^{\frac{1}{2}}) \quad a > 0$<br>$0 < \operatorname{Re} \mu < \frac{1}{4} - \operatorname{Re} \lambda$                                       | $2^{2\lambda} a^{-2\lambda} y^\mu$<br>$\times G_{13}^{20}\left(\frac{a^2 y}{4} \middle  \begin{matrix} 0 \\ -\mu, \lambda + \frac{1}{2}\nu, \lambda - \frac{1}{2}\nu \end{matrix}\right) \quad y > 0$                                                                                                  |
| (37) | $x^{-\nu} [J_\nu(ax^{\frac{1}{2}})]^2 \quad a > 0$<br>$\mu = \nu - \frac{1}{2}, \quad \operatorname{Re} \nu > \frac{1}{2}$                                     | $\pi^{-\frac{1}{2}} a^{-\nu} y^{-\frac{1}{2}\nu-\frac{1}{2}} \mathbf{H}_\nu(2ay^{\frac{1}{2}}) \quad y > 0$                                                                                                                                                                                            |
| (38) | $x^{\frac{1}{2}\nu-\frac{1}{2}} Y_\nu(ax^{\frac{1}{2}}) \quad a > 0$<br>$\mu = \nu + \frac{1}{2}, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$    | $\frac{\pi^{\frac{1}{2}}}{2} \left(\frac{2y}{a}\right)^\nu \left[ J_\nu\left(\frac{ay^{\frac{1}{2}}}{2}\right) J_{-\nu}\left(\frac{ay^{\frac{1}{2}}}{2}\right) \right.$<br>$\left. - Y_\nu\left(\frac{ay^{\frac{1}{2}}}{2}\right) Y_{-\nu}\left(\frac{ay^{\frac{1}{2}}}{2}\right) \right] \quad y > 0$ |

**Weyl integrals (cont'd)**

|      | $f(x)$                                                                                                                                        | $[\Gamma(\mu)]^{-1} \int_y^\infty f(x) (x-y)^{\mu-1} dx$                                                                                                                                                                        |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (39) | $x^{\frac{1}{2}\nu} Y_\nu(ax^{\frac{1}{2}})$<br>$0 < \operatorname{Re} \mu < \frac{3}{4} - \frac{1}{2} \operatorname{Re} \nu$<br>$a > 0$      | $2^\mu a^{-\mu} y^{\frac{1}{2}\mu + \frac{1}{2}\nu}$<br>$\times [\cos(\nu\pi) Y_{-\mu-\nu}(ay^{\frac{1}{2}})$<br>$- \sin(\nu\pi) J_{-\mu-\nu}(ay^{\frac{1}{2}})]$<br>$y > 0$                                                    |
| (40) | $x^{-\frac{1}{2}\nu} Y_\nu(ax^{\frac{1}{2}})$<br>$0 < \operatorname{Re} \mu < \frac{1}{2} \operatorname{Re} \nu + \frac{3}{4}$<br>$a > 0$     | $2^\mu a^{-\mu} y^{\frac{1}{2}\mu - \frac{1}{2}\nu} Y_{\nu-\mu}(ay^{\frac{1}{2}})$<br>$y > 0$                                                                                                                                   |
| (41) | $x^{-\nu} J_\nu(ax^{\frac{1}{2}}) Y_\nu(ax^{\frac{1}{2}})$<br>$\mu = \nu - \frac{1}{2}, \quad \operatorname{Re} \nu > \frac{1}{2}$<br>$a > 0$ | $-\pi^{-\frac{1}{2}} a^{-\nu} y^{-\frac{1}{2}\nu - \frac{1}{2}} J_\nu(2ay^{\frac{1}{2}})$                                                                                                                                       |
| (42) | $x^{-\nu} [Y_\nu(ax^{\frac{1}{2}})]^2$<br>$\mu = \nu - \frac{1}{2}, \quad \operatorname{Re} \nu > \frac{1}{2}$<br>$a > 0$                     | $\pi^{-\frac{1}{2}} a^{-\nu} y^{-\frac{1}{2}\nu - \frac{1}{2}} [\mathbf{H}_\nu(2ay^{\frac{1}{2}})$<br>$- 2 Y_\nu(2ay^{\frac{1}{2}})]$<br>$y > 0$                                                                                |
| (43) | $x^{\frac{1}{2}\nu - \frac{1}{2}} H_\nu^{(1)}(ax^{\frac{1}{2}})$<br>$\mu = \nu + \frac{1}{2}, \quad \operatorname{Re} \nu > -\frac{1}{2}$     | $\frac{\pi^{\frac{1}{2}} i}{2} \left(\frac{2y}{a}\right)^\nu H_\nu^{(1)}\left(\frac{ay^{\frac{1}{2}}}{2}\right)$<br>$\times H_{-\nu}^{(1)}\left(\frac{ay^{\frac{1}{2}}}{2}\right)$<br>$\operatorname{Im}(ay^{\frac{1}{2}}) > 0$ |
| (44) | $x^{\frac{1}{2}\nu - \frac{1}{2}} H_{-\nu}^{(1)}(ax^{\frac{1}{2}})$<br>$\mu = \nu + \frac{1}{2}, \quad \operatorname{Re} \nu > -\frac{1}{2}$  | $\frac{\pi^{\frac{1}{2}} i}{2} \left(\frac{2y}{a}\right)^\nu \left[ H_{-\nu}^{(1)}\left(\frac{ay^{\frac{1}{2}}}{2}\right) \right]^2$<br>$\operatorname{Im}(ay^{\frac{1}{2}}) > 0$                                               |
| (45) | $x^{-\frac{1}{2}\nu} H_\nu^{(1)}(ax^{\frac{1}{2}})$<br>$\operatorname{Re} \mu > 0$                                                            | $2^\mu a^{-\mu} y^{\frac{1}{2}\mu - \frac{1}{2}\nu} H_{\nu-\mu}^{(1)}(ay^{\frac{1}{2}})$<br>$\operatorname{Im}(ay^{\frac{1}{2}}) > 0$                                                                                           |
| (46) | $x^{\frac{1}{2}\nu - \frac{1}{2}} H_\nu^{(2)}(ax^{\frac{1}{2}})$<br>$\mu = \nu + \frac{1}{2}, \quad \operatorname{Re} \nu > -\frac{1}{2}$     | $\frac{\pi^{\frac{1}{2}} i}{2} \left(\frac{2y}{a}\right)^\nu H_\nu^{(2)}\left(\frac{ay^{\frac{1}{2}}}{2}\right)$<br>$\times H_{-\nu}^{(2)}\left(\frac{ay^{\frac{1}{2}}}{2}\right)$<br>$\operatorname{Im}(ay^{\frac{1}{2}}) < 0$ |

## Weyl integrals (cont'd)

|      | $f(x)$                                                                                                                              | $[\Gamma(\mu)]^{-1} \int_y^\infty f(x) (x-y)^{\mu-1} dx$                                                                                                                                                                                     |
|------|-------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (47) | $x^{\frac{1}{2}\nu-\frac{1}{2}} H_{-\nu}^{(2)}(ax^{\frac{1}{2}})$<br>$\mu = \nu + \frac{1}{2}, \quad \text{Re } \nu > -\frac{1}{2}$ | $\frac{\pi^{\frac{1}{2}} i}{2} \left(\frac{2y}{a}\right)^\nu \left[ H_{-\nu}^{(2)}\left(\frac{ay^{\frac{1}{2}}}{2}\right) \right]^2$<br>$\text{Im}(ay^{\frac{1}{2}}) < 0$                                                                    |
| (48) | $x^{-\frac{1}{2}\nu} H_\nu^{(2)}(ax^{\frac{1}{2}}) \quad \text{Re } \mu > 0$                                                        | $2^\mu a^{-\mu} y^{\frac{1}{2}\mu-\frac{1}{2}\nu} H_{\nu-\mu}^{(2)}(ay^{\frac{1}{2}})$<br>$\text{Im}(ay^{\frac{1}{2}}) < 0$                                                                                                                  |
| (49) | $x^{-\nu} e^{-ax} I_\nu(ax)$<br>$0 < \text{Re } \mu < \frac{1}{2} + \text{Re } \nu$                                                 | $\frac{(2a)^{\nu-\mu} \Gamma(\frac{1}{2}-\mu+\nu)}{\pi^{\frac{1}{2}} \Gamma(1-\mu+2\nu)}$<br>$\times {}_1F_1(\frac{1}{2}-\mu+\nu; 1-\mu+2\nu; -2ay)$<br>$\text{Re}(ay) > 0$                                                                  |
| (50) | $x^{-\lambda} e^{-ax} I_\nu(ax)$<br>$0 < \text{Re } \mu < \frac{1}{2} + \text{Re } \lambda$                                         | $\pi^{-\frac{1}{2}} (2a)^\lambda y^\mu$<br>$\times G_{23}^{21} \left( 2ay \left  \begin{matrix} \frac{1}{2}-\lambda, 0 \\ -\mu, \nu-\lambda, -\nu-\lambda \end{matrix} \right. \right)$<br>$\text{Re}(ay) > 0$                               |
| (51) | $x^{-\mu-\frac{1}{2}} e^{-ax} K_\nu(ax) \quad \text{Re } \mu > 0$                                                                   | $\pi^{\frac{1}{2}} (2a)^{-\frac{1}{2}} y^{-1} e^{-ay} W_{-\mu, \nu}(2ay)$<br>$\text{Re}(ay) > 0$                                                                                                                                             |
| (52) | $x^{-\nu} e^{ax} K_\nu(ax)$<br>$0 < \text{Re } \mu < \frac{1}{2} + \text{Re } \nu$                                                  | $\frac{\pi^{\frac{1}{2}} y^{\frac{1}{2}\mu-\nu-\frac{1}{2}} \Gamma(\frac{1}{2}-\mu+\nu)}{(2a)^{\frac{1}{2}\mu+\frac{1}{2}} \Gamma(\frac{1}{2}+\nu)}$<br>$\times e^{ay} W_{\frac{1}{2}\mu, \nu-\frac{1}{2}\mu}(2ay)$<br>$ \arg(ay)  < 3\pi/2$ |

## Weyl integrals (cont'd)

|      | $f(x)$                                                                                                                             | $[\Gamma(\mu)]^{-1} \int_y^\infty f(x) (x-y)^{\mu-1} dx$                                                                                                                                                                       |
|------|------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (53) | $x^{-\nu} e^{-ax} K_\nu(ax) \quad \operatorname{Re} \mu > 0$                                                                       | $\pi^{\frac{1}{2}} (2a)^{-\frac{1}{2}\mu-\frac{1}{2}} y^{\frac{1}{2}\mu-\nu-\frac{1}{2}} \times e^{-ay} W_{-\frac{1}{2}\mu, \nu-\frac{1}{2}\mu}(2ay) \quad \operatorname{Re}(ay) > 0$                                          |
| (54) | $x^{-\lambda} e^{ax} K_\nu(ax) \quad 0 < \operatorname{Re} \mu < \frac{1}{2} + \operatorname{Re} \lambda$                          | $\pi^{-\frac{1}{2}} (2a)^\lambda y^\mu \cos(i\pi) \times G_{23}^{31} \left( 2ay \left  \begin{array}{c} \frac{1}{2} - \lambda, 0 \\ -\mu, \nu - \lambda, -\nu - \lambda \end{array} \right. \right) \quad  \arg(ay)  < 3\pi/2$ |
| (55) | $x^{-\lambda} e^{-ax} K_\nu(ax) \quad \operatorname{Re} \mu > 0$                                                                   | $\pi^{\frac{1}{2}} (2a)^\lambda y^\mu \times G_{23}^{30} \left( 2ay \left  \begin{array}{c} 0, \frac{1}{2} - \lambda \\ -\mu, \nu - \lambda, -\nu - \lambda \end{array} \right. \right) \quad \operatorname{Re}(ay) > 0$       |
| (56) | $x^{-\frac{1}{2}} K_{2\nu}(ax^{\frac{1}{2}}) \quad \mu = \frac{1}{2}$                                                              | $\pi^{-\frac{1}{2}} [K_\nu(\frac{1}{2}ay^{\frac{1}{2}})]^2 \quad \operatorname{Re}(ay^{\frac{1}{2}}) > 0$                                                                                                                      |
| (57) | $x^{\frac{1}{2}\nu-\frac{1}{2}} K_\nu(ax^{\frac{1}{2}}) \quad \mu = \nu + \frac{1}{2}, \quad \operatorname{Re} \nu > -\frac{1}{2}$ | $\pi^{-\frac{1}{2}} \left( \frac{2y}{a} \right)^\nu \left[ K_\nu \left( \frac{ay^{\frac{1}{2}}}{2} \right) \right]^2 \quad \operatorname{Re}(ay^{\frac{1}{2}}) > 0$                                                            |
| (58) | $x^{\frac{1}{2}\nu-\frac{1}{2}} K_\nu(ax^{\frac{1}{2}}) \quad \mu = \nu - \frac{1}{2}, \quad \operatorname{Re} \nu > \frac{1}{2}$  | $\pi^{-\frac{1}{2}} (2/a)^{\nu-1} y^{\nu-\frac{1}{2}} \times K_\nu(\frac{1}{2}ay^{\frac{1}{2}}) K_{\nu-1}(\frac{1}{2}ay^{\frac{1}{2}}) \quad \operatorname{Re}(ay^{\frac{1}{2}}) > 0$                                          |

## Weyl integrals (cont'd)

|      | $f(x)$                                                                                                                                                                  | $[\Gamma(\mu)]^{-1} \int_y^\infty f(x) (x-y)^{\mu-1} dx$                                                                                                                                                                                 |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (59) | $x^{-\frac{1}{2}\nu} K_\nu(ax^{\frac{1}{2}}) \quad \operatorname{Re} \mu > 0$                                                                                           | $2^\mu a^{-\mu} y^{\frac{1}{2}\mu-\frac{1}{2}\nu} K_{\nu-\mu}(ay^{\frac{1}{2}})$<br>$\operatorname{Re}(ay^{\frac{1}{2}}) > 0$                                                                                                            |
| (60) | $x^{-\lambda} K_\nu(ax^{\frac{1}{2}}) \quad \operatorname{Re} \mu > 0$                                                                                                  | $2^{-2\lambda-1} a^{2\lambda} y^\mu$<br>$\times G_{13}^{30} \left( \frac{a^2 y}{4} \middle  \begin{matrix} 0 \\ -\mu, \frac{1}{2}\nu-\lambda, -\frac{1}{2}\nu-\lambda \end{matrix} \right)$<br>$\operatorname{Re}(ay^{\frac{1}{2}}) > 0$ |
| (61) | $x^{-\nu} I_\nu(ax^{\frac{1}{2}}) K_\nu(ax^{\frac{1}{2}})$<br>$\mu = \nu - \frac{1}{2}, \quad \operatorname{Re} \nu > \frac{1}{2}$                                      | $\frac{1}{2} \pi^{\frac{1}{2}} a^{-\nu} y^{-\frac{1}{2}\nu-\frac{1}{2}}$<br>$\times [I_\nu(2ay^{\frac{1}{2}}) - \mathbf{L}_\nu(2ay^{\frac{1}{2}})]$<br>$\operatorname{Re}(ay^{\frac{1}{2}}) > 0$                                         |
| (62) | $x^{-\nu} [K_\nu(ax^{\frac{1}{2}})]^2$<br>$\mu = \nu - \frac{1}{2}, \quad \operatorname{Re} \nu > \frac{1}{2}$                                                          | $\pi^{\frac{1}{2}} a^{-\nu} y^{-\frac{1}{2}\nu-\frac{1}{2}} K_\nu(2ay^{\frac{1}{2}})$<br>$\operatorname{Re}(ay^{\frac{1}{2}}) > 0$                                                                                                       |
| (63) | $x^{\frac{1}{2}\mu} \mathbf{H}_{-\mu}(ax^{\frac{1}{2}})$<br>$a > 0, \quad 0 < \operatorname{Re} \mu < \frac{1}{2}$                                                      | $\left(\frac{2y}{a}\right)^\mu \left[ Y_{-2\mu}(ay^{\frac{1}{2}}) \right.$<br>$\left. + \frac{2}{\pi} S_{0, 2\mu}(ay^{\frac{1}{2}}) \right] \quad y > 0$                                                                                 |
| (64) | $x^{\frac{1}{2}\nu-\frac{1}{2}} \mathbf{H}_{-\nu}(ax^{\frac{1}{2}}) \quad a > 0$<br>$\mu = \nu + \frac{1}{2}, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$ | $\pi^{\frac{1}{2}} \left(\frac{2y}{a}\right)^\nu \left[ J_{-\nu}\left(\frac{ay^{\frac{1}{2}}}{2}\right) \right]^2$<br>$y > 0$                                                                                                            |

## Weyl integrals (cont'd)

|      | $f(x)$                                                                                                                                                                                                       | $[\Gamma(\mu)]^{-1} \int_y^\infty f(x) (x-y)^{\mu-1} dx$                                                                                                                                                                                                           |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (65) | $x^{\frac{1}{2}\nu} \mathbf{H}_\nu(ax^{\frac{1}{2}})$<br>$a > 0$<br>$\operatorname{Re} \mu > 0, \quad \operatorname{Re}(\mu + \nu) < \frac{1}{2}$<br>$\operatorname{Re}(\mu + \frac{1}{2}\nu) < \frac{3}{4}$ | $\frac{(2/a)^\mu y^{\frac{1}{2}\nu + \frac{1}{2}\mu}}{\cos[(\mu + \nu)\pi]} [\cos(\nu\pi) \mathbf{H}_{\mu+\nu}(ay^{\frac{1}{2}})$<br>$+ \sin(\mu\pi) J_{-\mu-\nu}(ay^{\frac{1}{2}})] \quad y > 0$                                                                  |
| (66) | $x^{\frac{1}{2}\mu} [\mathbf{H}_{-\mu}(ax^{\frac{1}{2}}) - Y_{-\mu}(ax^{\frac{1}{2}})]$<br>$0 < \operatorname{Re} \mu < \frac{1}{2}$                                                                         | $\frac{2}{\pi} \left(\frac{2y}{a}\right)^\mu S_{0,2\mu}(ay^{\frac{1}{2}})$<br>$ \arg(ay^{\frac{1}{2}})  < \pi$                                                                                                                                                     |
| (67) | $x^{\frac{1}{2}\nu - \frac{1}{2}} [\mathbf{H}_{-\nu}(ax^{\frac{1}{2}}) - Y_{-\nu}(ax^{\frac{1}{2}})]$<br>$\mu = \nu + \frac{1}{2}, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                 | $\frac{\pi^{\frac{1}{2}}}{2} \left(\frac{2y}{a}\right)^\nu \left\{ \left[ J_\nu\left(\frac{ay^{\frac{1}{2}}}{2}\right) \right]^2 \right.$<br>$\left. + \left[ Y_\nu\left(\frac{ay^{\frac{1}{2}}}{2}\right) \right]^2 \right\}$<br>$ \arg(ay^{\frac{1}{2}})  < \pi$ |
| (68) | $x^{\frac{1}{2}\nu} [\mathbf{H}_\nu(ax^{\frac{1}{2}}) - Y_\nu(ax^{\frac{1}{2}})]$<br>$0 < \operatorname{Re} \mu < \frac{1}{2} - \operatorname{Re} \nu$                                                       | $\frac{(2/a)^\mu \cos(\nu\pi)}{\cos[(\mu + \nu)\pi]} y^{\frac{1}{2}\nu + \frac{1}{2}\mu}$<br>$\times [\mathbf{H}_{\mu+\nu}(ay^{\frac{1}{2}}) - Y_{\mu+\nu}(ay^{\frac{1}{2}})]$<br>$ \arg(ay^{\frac{1}{2}})  < \pi$                                                 |
| (69) | $x^{\frac{1}{2}\nu - \frac{1}{2}} [I_{-\nu}(ax^{\frac{1}{2}}) - \mathbf{L}_{-\nu}(ax^{\frac{1}{2}})]$<br>$\mu = \nu + \frac{1}{2}, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                 | $\frac{2}{\pi^{\frac{1}{2}}} \left(\frac{2y}{a}\right)^\nu I_{-\nu}\left(\frac{ay^{\frac{1}{2}}}{2}\right)$<br>$\times K_\nu\left(\frac{ay^{\frac{1}{2}}}{2}\right) \quad \operatorname{Re}(ay^{\frac{1}{2}}) > 0$                                                 |



## Weyl integrals (cont'd)

|      | $f(x)$                                                                                                                                                    | $[\Gamma(\mu)]^{-1} \int_y^\infty f(x) (x-y)^{\mu-1} dx$                                                                                                                                                                                                                  |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (70) | $x^{\frac{1}{2}\nu} [I_{-\nu}(ax^{\frac{1}{2}}) - \mathbf{L}_\nu(ax^{\frac{1}{2}})]$<br>$0 < \operatorname{Re} \mu < \frac{1}{2} - \operatorname{Re} \nu$ | $\frac{\cos(\nu\pi)}{\cos[(\mu+\nu)\pi]} \left(\frac{2}{a}\right)^\mu y^{\frac{1}{2}\mu+\frac{1}{2}\nu}$<br>$\times [I_{-\mu-\nu}(ay^{\frac{1}{2}}) - \mathbf{L}_{\mu+\nu}(ay^{\frac{1}{2}})]$<br>$\operatorname{Re}(ay^{\frac{1}{2}}) > 0$                               |
| (71) | $x^{\frac{1}{2}\nu} S_{\lambda,\nu}(ax^{\frac{1}{2}})$<br>$0 < 2\operatorname{Re} \mu < 1 - \operatorname{Re}(\lambda + \nu)$                             | $\frac{\Gamma(\frac{1}{2} - \frac{1}{2}\lambda - \mu - \frac{1}{2}\nu)}{\Gamma(\frac{1}{2} - \frac{1}{2}\lambda - \frac{1}{2}\nu)} a^{-\mu} y^{\frac{1}{2}\mu+\frac{1}{2}\nu}$<br>$\times S_{\lambda+\mu, \mu+\nu}(ay^{\frac{1}{2}})$<br>$ \arg(ay^{\frac{1}{2}})  < \pi$ |
| (72) | $x^{\lambda-\frac{1}{2}} e^{\frac{1}{2}ax} W_{\kappa,\lambda}(ax)$<br>$0 < \operatorname{Re} \mu < \frac{1}{2} - \operatorname{Re}(\kappa + \lambda)$     | $\frac{\Gamma(\frac{1}{2} - \kappa - \lambda - \mu)}{\Gamma(\frac{1}{2} - \kappa - \lambda)} a^{-\frac{1}{2}\mu} y^{\frac{1}{2}\mu+\lambda-\frac{1}{2}}$<br>$\times e^{\frac{1}{2}ay} W_{\kappa+\frac{1}{2}\mu, \lambda+\frac{1}{2}\mu}(ay)$<br>$ \arg(ay)  < 3\pi/2$     |
| (73) | $x^{\kappa-\mu-1} e^{-\frac{1}{2}ax} W_{\kappa,\lambda}(ax)$<br>$\operatorname{Re} \mu > 0$                                                               | $y^{\kappa-1} e^{-\frac{1}{2}ay} W_{\kappa-\mu,\lambda}(ay)$<br>$\operatorname{Re}(ay) > 0$                                                                                                                                                                               |
| (74) | $x^{\lambda-\frac{1}{2}} e^{-\frac{1}{2}ax} W_{\kappa,\lambda}(ax)$<br>$\operatorname{Re} \mu > 0$                                                        | $a^{-\frac{1}{2}\mu} y^{\frac{1}{2}\mu+\lambda-\frac{1}{2}} e^{-\frac{1}{2}ay}$<br>$\times W_{\kappa-\frac{1}{2}\mu, \lambda-\frac{1}{2}\mu}(ay)$<br>$\operatorname{Re}(ay) > 0$                                                                                          |

## Weyl integrals (cont'd)

|      | $f(x)$                                                                                                                                                                                                                     | $[\Gamma(\mu)]^{-1} \int_y^\infty f(x) (x-y)^{\mu-1} dx$                                                                                                                                                                                                                                               |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (75) | $x^{-\rho} e^{-\frac{1}{2}\alpha x} W_{\kappa, \lambda}(\alpha x)$<br>$0 < \operatorname{Re} \mu < \operatorname{Re}(\rho - \kappa)$                                                                                       | $\frac{y^{\mu-\rho}}{\Gamma(\frac{1}{2} + \lambda - \kappa) \Gamma(\frac{1}{2} - \lambda - \kappa)} \times G_{23}^{31} \left( \alpha y \left  \begin{matrix} 1 + \kappa, \rho \\ \rho - \mu, \frac{1}{2} + \lambda, \frac{1}{2} - \lambda \end{matrix} \right. \right)$<br>$ \arg(\alpha y)  < 3\pi/2$ |
| (76) | $x^{-\rho} e^{-\frac{1}{2}\alpha x} W_{\kappa, \lambda}(\alpha x)$<br>$\operatorname{Re} \mu > 0$                                                                                                                          | $y^{\mu-\rho} G_{23}^{30} \left( \alpha y \left  \begin{matrix} \rho, 1 - \kappa \\ \rho - \mu, \frac{1}{2} + \lambda, \frac{1}{2} - \lambda \end{matrix} \right. \right)$<br>$\operatorname{Re}(\alpha y) > 0$                                                                                        |
| (77) | $x^{-\lambda} {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; -\alpha/x)$<br>$0 < \operatorname{Re} \mu < \operatorname{Re} \lambda, \quad p \leq q + 1$                                                                         | $\frac{\Gamma(\lambda - \mu)}{\Gamma(\lambda)} y^{\mu-\lambda} \times {}_{p+1}F_{q+1}(\lambda - \mu, a_1, \dots, a_p; \lambda, b_1, \dots, b_q; -\alpha/y)$<br>$ y  >  \alpha  \quad \text{or} \quad  \arg(\alpha/y)  < \pi$<br>$\text{if } p = q + 1$                                                 |
| (78) | $G_{pq}^{mn} \left( \alpha x \left  \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) \quad p \geq q$<br>$0 < \operatorname{Re} \mu < 1 - \operatorname{Re} a_j \quad j = 1, \dots, n$        | $y^\mu G_{p+1, q+1}^{m+1, n} \left( \alpha y \left  \begin{matrix} a_1, \dots, a_p, 0 \\ -\mu, b_1, \dots, b_q \end{matrix} \right. \right)$<br>$ \alpha y  > 1 \quad \text{if } p = q$                                                                                                                |
| (79) | $G_{pq}^{mn} \left( \alpha x \left  \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right)$<br>$p + q < 2(m + n)$<br>$0 < \operatorname{Re} \mu < 1 - \operatorname{Re} a_j \quad j = 1, \dots, n$ | $y^\mu G_{p+1, q+1}^{m+1, n} \left( \alpha y \left  \begin{matrix} a_1, \dots, a_p, 0 \\ -\mu, b_1, \dots, b_q \end{matrix} \right. \right)$<br>$ \arg(\alpha y)  < (m + n - \frac{1}{2}p - \frac{1}{2}q)\pi$                                                                                          |

## CHAPTER XIV

### STIELTJES TRANSFORMS

We call

$$g(y) = \mathfrak{G}\{f(x); y\} = \int_0^\infty f(x) (x + y)^{-1} dx$$

the *Stieltjes transform* of  $f(x)$ . Here integration is over the positive real  $x$ -axis, and  $y$  is a complex variable ranging over the complex  $y$ -plane cut along the negative real axis.

Stieltjes transforms are iterated Laplace transforms,

$$\mathfrak{G}\{f(x); y\} = \mathfrak{L}\{\mathfrak{L}\{f(x); t\}; y\}$$

and accordingly, information about Stieltjes transforms is found in works on Laplace transforms, in particular in Widder (1941, Chapter VIII) and Titchmarsh (1937, sections 11.8, 11.9). Stieltjes transforms are also connected with the moment problem for the semi-infinite interval (Shohat and Tamarkin, 1943) and hence with certain continued fractions.

We also give a brief list of *generalized Stieltjes transforms* of order  $\rho$

$$g(y; \rho) = \mathfrak{G}_\rho\{f(x); y\} = \int_0^\infty f(x) (x + y)^{-\rho} dx$$

where  $x$  and  $y$  are as before, and  $\rho$  is a complex parameter. For the theory see Widder (1941, Chapter VIII). Generalized Stieltjes transforms of different orders are connected with each other, and with Stieltjes transforms, by fractional integration according to the formulas

$$\Gamma(\rho) \mathfrak{I}_{\rho-1} \mathfrak{G}_\rho = \mathfrak{G}$$

$$\Gamma(\rho) \mathfrak{I}_\mu \mathfrak{G}_\rho = \Gamma(\rho - \mu) \mathfrak{G}_{\rho-\mu}.$$

From the transform pairs given in the tables, further integrals may be derived by the methods mentioned in the introduction to vol. I, by the general formulas given in the tables, and by using the above formulas in connection with tables of Laplace transforms and fractional integrals.

## REFERENCES

- Shohat, J.A. and J.D. Tamarkin, 1943: *The problem of moments*. Amer. Math. Soc. New York.
- Titchmarsh, E.C., 1937: *Introduction to the theory of Fourier integrals*. Oxford.
- Widder, D.V., 1941: *The Laplace transform*. Princeton University Press, Princeton, N.J.

## STIELTJES TRANSFORMS

### 14.1. General formulas

|     | $f(x)$                                  | $\int_0^\infty f(x) (x+y)^{-1} dx \quad  \arg y  < \pi$                                   |
|-----|-----------------------------------------|-------------------------------------------------------------------------------------------|
| (1) | $f(x)$                                  | $g(y)$                                                                                    |
| (2) | $xf(x)$                                 | $\int_0^\infty f(x) dx - yg(y)$                                                           |
| (3) | $(x+a)^{-1} f(x) \quad  \arg a  < \pi$  | $(y-a)^{-1} [g(a) - g(y)]$                                                                |
| (4) | $\frac{f(x) - f(a)}{x - a} \quad a > 0$ | $(y+a)^{-1} [\frac{1}{2}g(ae^{i\pi}) + \frac{1}{2}g(ae^{-i\pi}) - f(a) \log(y/a) - g(y)]$ |
| (5) | $g(xe^{i\pi}) - g(xe^{-i\pi})$          | $2\pi i g(y)$                                                                             |
| (6) | $f(ax) \quad a > 0$                     | $g(ay)$                                                                                   |
| (7) | $x^{-1} f(a/x) \quad a > 0$             | $y^{-1} g(a/y)$                                                                           |
| (8) | $f(x^{\frac{1}{2}})$                    | $g(iy^{\frac{1}{2}}) + g(-iy^{\frac{1}{2}})$                                              |
| (9) | $f'(x)$                                 | $-y^{-1} f(0) - g'(y)$                                                                    |

## 14.2. Elementary functions

|     | $f(x)$                                                                                       | $\int_0^\infty f(x) (x+y)^{-1} dx \quad  \arg y  < \pi$                                                                                                                         |
|-----|----------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $-1 \quad 2n < x < 2n+1$<br>$1 \quad 2n+1 < x < 2n+2$<br>$n = 0, 1, 2, \dots$                | $\log\{\frac{1}{2}y [\Gamma(\frac{1}{2}y)/\Gamma(\frac{1}{2}y + \frac{1}{2})]^2\}$                                                                                              |
| (2) | $(a+x)^{-1} \quad  \arg a  < \pi$                                                            | $(a-y)^{-1} \log(a/y)$                                                                                                                                                          |
| (3) | $\frac{1}{a^2 + x^2} \quad \operatorname{Re} a > 0$                                          | $\frac{1}{a^2 + y^2} \left[ \frac{\pi y}{2a} - \log\left(\frac{y}{a}\right) \right]$                                                                                            |
| (4) | $\frac{x}{a^2 + x^2} \quad \operatorname{Re} a > 0$                                          | $\frac{1}{a^2 + y^2} \left[ \frac{\pi a}{2} + y \log\left(\frac{y}{a}\right) \right]$                                                                                           |
| (5) | $x^\nu \quad -1 < \operatorname{Re} \nu < 0$                                                 | $-\pi y^\nu \csc(\pi\nu)$                                                                                                                                                       |
| (6) | $\frac{x^\nu}{a+x}$<br>$ \arg a  < \pi, \quad -1 < \operatorname{Re} \nu < 1$                | $\frac{\pi(a^\nu - y^\nu)}{(a-y) \sin(\nu\pi)}$                                                                                                                                 |
| (7) | $\frac{x^\nu}{a^2 + x^2}$<br>$\operatorname{Re} a > 0, \quad -1 < \operatorname{Re} \nu < 2$ | $\frac{\pi}{a^2 + y^2} \left[ \frac{a^{\nu-1} y}{2 \cos(\frac{1}{2}\nu\pi)} \right.$<br>$\left. + \frac{a^\nu}{2 \sin(\frac{1}{2}\nu\pi)} - \frac{y^\nu}{\sin(\nu\pi)} \right]$ |
| (8) | $\frac{x^\nu - a^\nu}{x - a}$<br>$-1 < \operatorname{Re} \nu < 1$                            | $\frac{\pi}{a+y} \left[ \frac{y^\nu}{\sin(\nu\pi)} - a^\nu \cotn(\nu\pi) \right.$<br>$\left. + \frac{a^\nu}{\pi} \log\left(\frac{a}{y}\right) \right]$                          |

## Elementary functions (cont'd)

|      | $f(x)$                                                                                                      | $\int_0^\infty f(x) (x+y)^{-1} dx \quad  \arg y  < \pi$                                                           |
|------|-------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------|
| (9)  | $x^{\nu-1} (a+x)^{1-\mu}$<br>$ \arg a  < \pi, \quad 0 < \operatorname{Re} \nu < \operatorname{Re} \mu$      | $\frac{\Gamma(\nu) \Gamma(\mu-\nu) y^{\nu-1}}{\Gamma(\mu) a^{\mu-1}}$<br>$\times {}_2F_1(\mu-1, \nu; \mu; 1-y/a)$ |
| (10) | $x^{-\rho} (a+x)^{-\sigma}$<br>$ \arg a  < \pi$<br>$-\operatorname{Re} \sigma < \operatorname{Re} \rho < 1$ | $\pi \csc(\rho\pi) y^{-\rho} (a-y)^{-\sigma} I_{1-y/a}(\sigma, \rho)$                                             |
| (11) | $e^{-ax}$<br>$\operatorname{Re} a > 0$                                                                      | $-e^{ay} \operatorname{Ei}(-ay)$                                                                                  |
| (12) | $e^{-ax}$<br>0                                                                                              | $0 < x < b$<br>$b < x < \infty$<br>$e^{ay} [\operatorname{Ei}(-ab-ay) - \operatorname{Ei}(-ay)]$                  |
| (13) | 0<br>$e^{-ax}$                                                                                              | $0 < x < b$<br>$b < x < \infty$<br>$\operatorname{Re} a > 0$<br>$-e^{ay} \operatorname{Ei}(-ab-ay)$               |
| (14) | $x^n e^{-ax}$<br>$\operatorname{Re} a > 0$                                                                  | $(-1)^{n+1} y^n e^{ay} \operatorname{Ei}(-ay)$<br>$+ \sum_{r=1}^n (-1)^{n-r} (r-1)! a^{-r} y^{n-r}$               |
| (15) | $x^{-1/2} e^{-ax}$<br>$\operatorname{Re} a > 0$                                                             | $\pi y^{-1/2} e^{ay} \operatorname{Erfc}(a^{1/2} y^{1/2})$                                                        |
| (16) | $x^{1/2} e^{-ax}$<br>$\operatorname{Re} a > 0$                                                              | $\pi^{1/2} a^{-1/2} - \pi y^{1/2} e^{ay} \operatorname{Erfc}(a^{1/2} y^{1/2})$                                    |
| (17) | $x^{-\nu} e^{-ax}$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu < 1$                            | $\Gamma(1-\nu) y^{-\nu} e^{ay} \Gamma(\nu, ay)$                                                                   |
| (18) | $x^{-1} (1-e^{-ax})$<br>$\operatorname{Re} a > 0$                                                           | $y^{-1} [\log(ay) - e^{ay} \operatorname{Ei}(-ay)]$                                                               |

**Elementary functions (cont'd)**

|      | $f(x)$                                                                                         | $\int_0^\infty f(x) (x+y)^{-1} dx \quad  \arg y  < \pi$                                                                                                                       |
|------|------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (19) | $x^{\nu-1} e^{-a/x}$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu < 1$             | $\Gamma(1-\nu) y^{\nu-1} e^{a/y} \Gamma(\nu; a/y)$                                                                                                                            |
| (20) | $\exp(-ax^{1/2})$<br>$\operatorname{Re} a > 0$                                                 | $2 \cos(ay^{1/2}) \operatorname{ci}(ay^{1/2})$<br>$- 2 \sin(ay^{1/2}) \operatorname{si}(ay^{1/2})$                                                                            |
| (21) | $x^{-1/2} \exp(-ax^{1/2})$<br>$\operatorname{Re} a > 0$                                        | $-2y^{-1/2} [\sin(ay^{1/2}) \operatorname{ci}(ay^{1/2})$<br>$+ \cos(ay^{1/2}) \operatorname{si}(ay^{1/2})]$                                                                   |
| (22) | $x^\lambda \exp(-ax^{1/2})$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \lambda > -1$ | $\Gamma(2\lambda+1) y^\lambda [\exp(iay^{1/2} + \lambda\pi i)$<br>$\times \Gamma(-2\lambda, iay^{1/2})$<br>$+ \exp(-iay^{1/2} - \lambda\pi i) \Gamma(-2\lambda, -iay^{1/2})]$ |
| (23) | $[\exp(ax^{1/2}) - 1]^{-1}$<br>$\operatorname{Re} a > 0$                                       | $\log(ay^{1/2}) - (2ay^{1/2})^{-1} - \psi(ay^{1/2})$                                                                                                                          |
| (24) | $(a+x)^{-1} \log x$<br>$ \arg a  < \pi$                                                        | $\frac{1}{2}(y-a)^{-1} [(\log y)^2 - (\log a)^2]$                                                                                                                             |
| (25) | $(a+x)^{-1} \log(x/a)$<br>$ \arg a  < \pi$                                                     | $\frac{1}{2(y-a)} \left[ \log\left(\frac{y}{a}\right) \right]^2$                                                                                                              |
| (26) | $(x-a)^{-1} \log(x/a)$<br>$a > 0$                                                              | $\frac{1}{2}(y+a)^{-1} \{ \pi^2 + [\log(y/a)]^2 \}$                                                                                                                           |
| (27) | $x^{-1/2} \log(ax + \beta)$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \beta > 0$    | $2\pi y^{-1/2} \log(a^{1/2} y^{1/2} + \beta^{1/2})$                                                                                                                           |
| (28) | $x^\nu \log x$<br>$-1 < \operatorname{Re} \nu < 0$                                             | $-\pi y^\nu \csc(\nu\pi) [\log y - \pi \operatorname{ctn}(\nu\pi)]$                                                                                                           |



## Elementary functions (cont'd)

|      | $f(x)$                                                                                 | $\int_0^\infty f(x) (x+y)^{-1} dx \quad  \arg y  < \pi$                                                                              |
|------|----------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------|
| (29) | $x^\nu (a+x)^{-1} \log x$<br>$ \arg a  < \pi, \quad -1 < \operatorname{Re} \nu < 1$    | $-\pi \csc(\nu\pi) (y-a)^{-1} [a^\nu \log a$<br>$-y^\nu \log y - \pi \operatorname{ctn}(\nu\pi) (a^\nu - y^\nu)]$                    |
| (30) | $x^\nu (a+x)^{-1} \log(x/a)$<br>$ \arg a  < \pi, \quad -1 < \operatorname{Re} \nu < 1$ | $\pi \csc(\nu\pi) (y-a)^{-1} [y^\nu \log(y/a)$<br>$+ \pi \operatorname{ctn}(\nu\pi) (a^\nu - y^\nu)]$                                |
| (31) | $\sin(ax) \quad a > 0$                                                                 | $-\sin(ay) \operatorname{ci}(ay) - \cos(ay) \operatorname{si}(ay)$                                                                   |
| (32) | $x^{1/2} \sin(ax) \quad a > 0$                                                         | $\pi y^{1/2} [\sin(ay) - 2^{1/2} \sin(ay + 1/4\pi) C(ay)$<br>$+ 2^{1/2} \cos(ay + 1/4\pi) S(ay)]$<br>$- 2^{-1/2} \pi^{1/2} a^{-1/2}$ |
| (33) | $x^{-1/2} \sin(ax) \quad a > 0$                                                        | $\pi y^{-1/2} [2^{1/2} \sin(ay + 1/4\pi) C(ay)$<br>$- 2^{1/2} \cos(ay + 1/4\pi) S(ay) - \sin(ay)]$                                   |
| (34) | $x^{-\nu} \sin(ax)$<br>$a > 0, \quad -1 < \operatorname{Re} \nu < 2$                   | $1/2 i \Gamma(1-\nu) y^{-\nu} [e^{-iay} \Gamma(\nu, -iay)$<br>$- e^{iay} \Gamma(\nu, iay)]$                                          |
| (35) | $0 \quad 0 < x < a$<br>$(x^2 - a^2)^{-1/2} \sin(bx) \quad a < x < \infty$              | See Erdélyi, Arthur, 1939: <i>Proc. Edinburgh Math. Soc.</i> (2) 6, 94-104.                                                          |
| (36) | $\sin(ax^{1/2}) \quad a > 0$                                                           | $\pi \exp(-ay^{1/2})$                                                                                                                |
| (37) | $x^{-1} \sin(ax^{1/2}) \quad a > 0$                                                    | $\pi y^{-1} [1 - \exp(-ay^{1/2})]$                                                                                                   |
| (38) | $x^{-1/2} \sin(ax^{1/2}) \quad a > 0$                                                  | $y^{-1/2} [\exp(-ay^{1/2}) \overline{\operatorname{Ei}}(ay^{1/2})$<br>$- \exp(ay^{1/2}) \operatorname{Ei}(-ay^{1/2})]$               |

**Elementary functions (cont'd)**

|      | $f(x)$                                                                                      | $\int_0^\infty f(x) (x+y)^{-1} dx \quad  \arg y  < \pi$                                                                                                                                         |
|------|---------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (39) | $x^\lambda \sin(ax^{1/2})$<br>$a > 0, \quad -3/2 < \operatorname{Re} \lambda < 1/2$         | $-\pi y^\lambda \sec(\lambda\pi) \sinh(ay^{1/2})$<br>$-a^{-2\lambda} \Gamma(2\lambda) \sin(\lambda\pi)$<br>$\times [{}_1F_1(1; 1-2\lambda; ay^{1/2})$<br>$+ {}_1F_1(1; 1-2\lambda; -ay^{1/2})]$ |
| (40) | $(x+\beta)^{-1} \sin(ax^{1/2})$<br>$a > 0, \quad  \arg \beta  < \pi$                        | $\pi(y-\beta)^{-1} [\exp(-a\beta^{1/2}) - \exp(-ay^{1/2})]$                                                                                                                                     |
| (41) | $x^{-\beta} \sin(ax^{1/2} + \beta\pi)$<br>$a > 0, \quad -1/2 < \operatorname{Re} \beta < 1$ | $\pi y^{-\beta} \exp(-ay^{1/2})$                                                                                                                                                                |
| (42) | $\sin(ax^{1/2} - bx^{-1/2}) \quad a, b > 0$                                                 | $\pi \exp(-ay^{1/2} - by^{-1/2})$                                                                                                                                                               |
| (43) | $x^{-1/2} [\sin(ax^{1/2})]^2 \quad a > 0$                                                   | $1/2 \pi y^{-1/2} [1 - \exp(-2ay^{1/2})]$                                                                                                                                                       |
| (44) | $x^{-1/2} \sin(ax^{1/2}) \sin(bx^{1/2})$<br>$a > 0, \quad b > 0$                            | $1/2 \pi y^{-1/2} \{ \exp(- a-b y^{1/2})$<br>$- \exp[-(a+b)y^{1/2}] \}$                                                                                                                         |
| (45) | $x^{-1/2} \sin(ax^{1/2}) \sin(bx^{1/2})$<br>$a \geq b > 0$                                  | $\pi y^{-1/2} \exp(-ay^{1/2}) \sinh(by^{1/2})$                                                                                                                                                  |
| (46) | $\log(\beta x) \sin(ax^{1/2})$<br>$a > 0, \quad  \arg \beta  < \pi$                         | $\pi [\log(\beta y) \exp(-ay^{1/2})$<br>$- \exp(ay^{1/2}) \operatorname{Ei}(-ay^{1/2})$<br>$- \exp(-ay^{1/2}) \operatorname{Ei}(ay^{1/2})]$                                                     |
| (47) | $x^{-1/2} \log  \sin(ax^{1/2})  \quad a > 0$                                                | $\pi y^{-1/2} \log [1/2 - 1/2 \exp(-2ay^{1/2})]$                                                                                                                                                |

## Elementary functions (cont'd)

|      | $f(x)$                                                                                            | $\int_0^\infty f(x) (x+y)^{-1} dx \quad  \arg y  < \pi$                                                                                                                                                                  |
|------|---------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (48) | $\cos(ax) \quad a > 0$                                                                            | $\cos(ay) \operatorname{ci}(ay) - \sin(ay) \operatorname{si}(ay)$                                                                                                                                                        |
| (49) | $x^{-1} [\cos(bx) - \cos(ax)]$<br>$a, b > 0$                                                      | $y^{-1} [-\operatorname{ci}(by) \cos(by) + \operatorname{si}(by) \sin(by)$<br>$+ \operatorname{ci}(ay) \cos(ay) - \operatorname{si}(ay) \sin(ay)$<br>$+ \log(ab^{-1})]$                                                  |
| (50) | $x^{\frac{1}{2}} \cos(ax) \quad a > 0$                                                            | $2^{-\frac{1}{2}} \pi^{\frac{1}{2}} a^{-\frac{1}{2}} - \pi y^{\frac{1}{2}} [\cos(ay)$<br>$- 2^{\frac{1}{2}} \cos(ay + \frac{1}{4}\pi) C(ay)$<br>$- 2^{\frac{1}{2}} \sin(ay + \frac{1}{4}\pi) S(ay)]$                     |
| (51) | $x^{-\frac{1}{2}} \cos(ax) \quad a > 0$                                                           | $\pi y^{-\frac{1}{2}} [\cos(ay) - 2^{\frac{1}{2}} \cos(ay + \frac{1}{4}\pi) C(ay)$<br>$- 2^{\frac{1}{2}} \sin(ay + \frac{1}{4}\pi) S(ay)]$                                                                               |
| (52) | $x^{-\nu} \cos(ax)$<br>$a > 0, \quad -1 < \operatorname{Re} \nu < 1$                              | $\frac{1}{2} \Gamma(1-\nu) y^{-\nu} [e^{iay} \Gamma(\nu, iay)$<br>$+ e^{-iay} \Gamma(\nu, -iay)]$                                                                                                                        |
| (53) | 0<br>$(x^2 - a^2)^{-\frac{1}{2}} \cos(bx) \quad 0 < x < a$<br>$a < x < \infty$                    | See Erdélyi, Arthur, 1939: <i>Proc. Edinburgh Math. Soc.</i> (2) 6, 94-104.                                                                                                                                              |
| (54) | $\cos(ax^{\frac{1}{2}}) \quad a > 0$                                                              | $-\exp(-ay^{\frac{1}{2}}) \overline{\operatorname{Ei}}(ay^{\frac{1}{2}})$<br>$-\exp(ay^{\frac{1}{2}}) \operatorname{Ei}(-ay^{\frac{1}{2}})$                                                                              |
| (55) | $x^{-\frac{1}{2}} \cos(ax^{\frac{1}{2}}) \quad a > 0$                                             | $\pi y^{-\frac{1}{2}} \exp(-ay^{\frac{1}{2}})$                                                                                                                                                                           |
| (56) | $x^\lambda \cos(ax^{\frac{1}{2}})$<br>$a > 0, \quad -1 < \operatorname{Re} \lambda < \frac{1}{2}$ | $-\pi y^\lambda \csc(\lambda\pi) \cosh(ay^{\frac{1}{2}})$<br>$- a^{-2\lambda} \cos(\lambda\pi) \Gamma(2\lambda)$<br>$\times [{}_1F_1(1; 1-2\lambda; ay^{\frac{1}{2}})$<br>$+ {}_1F_1(1; 1-2\lambda; -ay^{\frac{1}{2}})]$ |

## Elementary functions (cont'd)

|      | $f(x)$                                                                                                         | $\int_0^\infty f(x) (x+y)^{-1} dx \quad  \arg y  < \pi$                                                                                                         |
|------|----------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (57) | $x^{-1/2} (x+\beta)^{-1} \cos(ax^{1/2})$<br>$a > 0, \quad  \arg \beta  < \pi$                                  | $\pi (y-\beta)^{-1} [\beta^{-1/2} \exp(-a\beta^{1/2})$<br>$- y^{-1/2} \exp(-ay^{1/2})]$                                                                         |
| (58) | $x^{-1/2} \cos(ax^{1/2} - bx^{-1/2})$<br>$a, b > 0$                                                            | $\pi y^{-1/2} \exp(-ay^{1/2} - by^{1/2})$                                                                                                                       |
| (59) | $x^{-3/2} [\cos(ax^{1/2}) - \cos(bx^{1/2})]$<br>$a > 0, \quad b > 0$                                           | $\pi y^{-3/2} [(b-a)y^{1/2}$<br>$+ \exp(-by^{1/2}) - \exp(-ay^{1/2})]$                                                                                          |
| (60) | $x^{-1/2} [\cos(ax^{1/2})]^2$<br>$a > 0$                                                                       | $\frac{1}{2} \pi y^{-1/2} [1 - \exp(-2ay^{1/2})]$                                                                                                               |
| (61) | $x^{-1/2} \cos(ax^{1/2}) \cos(bx^{1/2})$<br>$a > 0, \quad b > 0$                                               | $\frac{1}{2} \pi y^{-1/2} \{ \exp(- a-b y^{1/2})$<br>$+ \exp[-(a+b)y^{1/2}] \}$                                                                                 |
| (62) | $x^{-1/2} \cos(ax^{1/2}) \cos(bx^{1/2})$<br>$a \geq b > 0$                                                     | $\pi y^{-1/2} \exp(-ay^{1/2}) \cosh(by^{1/2})$                                                                                                                  |
| (63) | $\frac{x^{-1/2}}{[\beta \sin(ax^{1/2})]^2 + [\gamma \cos(ax^{1/2})]^2}$<br>$ \arg(\beta/\gamma)  < \pi$        | $\frac{\pi y^{-1/2} [\frac{1}{2}(\beta/\gamma - \gamma/\beta) \sinh(2ay^{1/2}) - 1]}{[\beta \sinh(ay^{1/2})]^2 - [\gamma \cosh(ay^{1/2})]^2}$                   |
| (64) | $\frac{\sin(2ax^{1/2})}{[\beta \sin(ax^{1/2})]^2 + [\gamma \cos(ax^{1/2})]^2}$<br>$ \arg(\beta/\gamma)  < \pi$ | $\frac{\pi [(\beta-\gamma)/(\beta+\gamma) - \exp(-2ay^{1/2})]}{[\beta \sinh(ay^{1/2})]^2 - [\gamma \cosh(ay^{1/2})]^2}$                                         |
| (65) | $x^{-1/2} \log(\beta x) \cos(ax^{1/2})$<br>$a > 0, \quad  \arg \beta  < \pi$                                   | $\pi y^{-1/2} [\log(\beta y) \exp(-ay^{1/2})$<br>$+ \exp(ay^{1/2}) \operatorname{Ei}(-ay^{1/2})$<br>$- \exp(-ay^{1/2}) \overline{\operatorname{Ei}}(ay^{1/2})]$ |

**Elementary functions (cont'd)**

|      | $f(x)$                                                                                                                                                                                                                                  | $\int_0^\infty f(x) (x+y)^{-1} dx \quad  \arg y  < \pi$                                                              |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------|
| (66) | $x^{-\frac{1}{2}} \log  \cos(ax^{\frac{1}{2}})  \quad a > 0$                                                                                                                                                                            | $\pi y^{-\frac{1}{2}} \log [\frac{1}{2} + \frac{1}{2} \exp(-2ay^{\frac{1}{2}})]$                                     |
| (67) | $x^{-\frac{1}{2}} \log[1 + 2\beta \cos(ax^{\frac{1}{2}}) + \beta^2] \quad a > 0, \quad  \beta  < 1$                                                                                                                                     | $2\pi y^{-\frac{1}{2}} \log[1 + \beta \exp(-ay^{\frac{1}{2}})]$                                                      |
| (68) | $x^{-\frac{1}{2}} \log\{[b \sin(ax^{\frac{1}{2}})]^2 + [c \cos(ax^{\frac{1}{2}})]^2\} \quad a, b, c > 0$                                                                                                                                | $2\pi y^{-\frac{1}{2}} \log[b \sinh(ay^{\frac{1}{2}}) + c \cosh(ay^{\frac{1}{2}})] - 2\pi a$                         |
| (69) | $[\cos(ax^{\frac{1}{2}})]^n \sin(nax^{\frac{1}{2}}) \quad a > 0, \quad n = 1, 2, \dots$                                                                                                                                                 | $2^{-n} \pi \{[1 + \exp(-2ay^{\frac{1}{2}})]^n - 1\}$                                                                |
| (70) | $x^{-\frac{1}{2}} \log  \tan(ax^{\frac{1}{2}})  \quad a > 0$                                                                                                                                                                            | $\pi y^{-\frac{1}{2}} \log [\tanh(ay^{\frac{1}{2}})]$                                                                |
| (71) | $x^{-\frac{1}{2}} \log\{1 + [b \tan(ax^{\frac{1}{2}})]^2\} \quad a, b > 0$                                                                                                                                                              | $2\pi y^{-\frac{1}{2}} \log[1 + b \tanh(ay^{\frac{1}{2}})]$                                                          |
| (72) | $x^{-\frac{1}{2}} \log\{1 + [b \operatorname{ctn}(ax^{\frac{1}{2}})]^2\} \quad a, b > 0$                                                                                                                                                | $2\pi y^{-\frac{1}{2}} \log[1 + b \operatorname{ctnh}(ay^{\frac{1}{2}})]$                                            |
| (73) | $\operatorname{csch}(\pi x^{\frac{1}{2}})$                                                                                                                                                                                              | $-y^{-\frac{1}{2}} + \psi(\frac{1}{2}y^{\frac{1}{2}} + \frac{1}{2}) - \psi(\frac{1}{2}y^{\frac{1}{2}})$              |
| (74) | $x^{-\frac{1}{2}} \operatorname{sech}(\pi x^{\frac{1}{2}})$                                                                                                                                                                             | $y^{-\frac{1}{2}} [\psi(\frac{1}{2}y^{\frac{1}{2}} + \frac{3}{4}) - \psi(\frac{1}{2}y^{\frac{1}{2}} + \frac{1}{4})]$ |
| (75) | $\frac{x^{-\frac{1}{2}} \sin(ax^{\frac{1}{2}})}{\sinh(bx^{\frac{1}{2}})}$<br>$\frac{x^{-\frac{1}{2}} \cos(ax^{\frac{1}{2}})}{\cosh(bx^{\frac{1}{2}})}$<br>$\frac{x^{-\frac{1}{2}} \cos(ax^{\frac{1}{2}})}{c + \cosh(bx^{\frac{1}{2}})}$ | See Ramanujan, Srinivasa, 1914:<br><i>Messenger of Math.</i> 44, 75-85.                                              |

## 14.3. Higher transcendental functions

|     | $f(x)$                                                                                               | $\int_0^\infty f(x) (x+y)^{-1} dx \quad  \arg y  < \pi$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |
|-----|------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $\text{ci}(ax) \quad a > 0$                                                                          | $\frac{1}{2} [\text{ci}(ay)]^2 + \frac{1}{2} [\text{si}(ay)]^2$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                            |
| (2) | $J_\nu(ax) \quad a > 0, \quad \text{Re } \nu > -1$                                                   | $\pi \csc(\nu\pi) [\mathbf{J}_\nu(ay) - J_\nu(ay)]$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        |
| (3) | $x^\nu J_\nu(ax) \quad a > 0, \quad -1/2 < \text{Re } \nu < 3/2$                                     | $\frac{1}{2} \pi y^\nu \sec(\nu\pi) [\mathbf{H}_{-\nu}(ay) - Y_{-\nu}(ay)]$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                |
| (4) | $x^{\nu+1} J_\nu(ax) \quad a > 0, \quad -1 < \text{Re } \nu < \frac{1}{2}$                           | $2^\nu \pi^{-\frac{1}{2}} a^{-\nu-1} \Gamma(\nu + \frac{1}{2})$<br>$+ \frac{1}{2} \pi \sec(\nu\pi) y^{\nu+1}$<br>$\times [Y_{-\nu}(ay) - \mathbf{H}_{-\nu}(ay)]$                                                                                                                                                                                                                                                                                                                                                                                                           |
| (5) | $x^{-\nu} J_\nu(ax) \quad a > 0, \quad \text{Re } \nu > -3/2$                                        | $\frac{1}{2} \pi y^{-\nu} [\mathbf{H}_\nu(ay) - Y_\nu(ay)]$<br>$- \frac{2^{1-\nu} y^{-\nu}}{\Gamma(\nu)} S_{\nu-1, \nu}(ay)$                                                                                                                                                                                                                                                                                                                                                                                                                                               |
| (6) | $x^\lambda J_\nu(ax) \quad a > 0, \quad \text{Re } \lambda < 3/2$<br>$\text{Re}(\lambda + \nu) > -1$ | $-\pi y^\lambda \csc[(\lambda + \nu)\pi] J_\nu(ay)$<br>$+ \frac{2^{\lambda-1} a^{-\lambda} \Gamma(\frac{1}{2}\lambda + \frac{1}{2}\nu)}{\Gamma(1 - \frac{1}{2}\lambda + \frac{1}{2}\nu)}$<br>$\times {}_1F_2\left(1; 1 - \frac{\lambda + \nu}{2}, 1 - \frac{\lambda - \nu}{2}; -\frac{a^2 y^2}{4}\right)$<br>$- \frac{2^{\lambda-2} a^{1-\lambda} y \Gamma(\frac{1}{2}\lambda + \frac{1}{2}\nu - \frac{1}{2})}{\Gamma(3/2 - \frac{1}{2}\lambda + \frac{1}{2}\nu)}$<br>$\times {}_1F_2\left(1; \frac{3-\lambda-\nu}{2}, \frac{3-\lambda+\nu}{2}; -\frac{a^2 y^2}{4}\right)$ |

**Higher transcendental functions (cont'd)**

|      | $f(x)$                                                                                                                                              | $\int_0^\infty f(x) (x+y)^{-1} dx \quad  \arg y  < \pi$                                                                                                                                                                                                                                                 |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (7)  | $x^\nu \sin(ax) J_\nu(ax)$<br>$a > 0, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}$                                                               | $\frac{1}{2} \pi y^\nu \sec(\nu\pi) [\cos(ay - \nu\pi) J_\nu(ay) + \sin(ay - \nu\pi) Y_\nu(ay)]$                                                                                                                                                                                                        |
| (8)  | $x^\nu \cos(ax) J_\nu(ax)$<br>$a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                                                     | $\frac{1}{2} \pi y^\nu \sec(\nu\pi) [\sin(ay - \nu\pi) J_\nu(ay) - \cos(ay - \nu\pi) Y_\nu(ay)]$                                                                                                                                                                                                        |
| (9)  | $x^\nu \cos(ax + \beta) J_\nu(ax)$<br>$a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                                             | $\frac{1}{2} \pi y^\nu \sec(\nu\pi) [\sin(ay - \nu\pi - \beta) J_\nu(ay) - \cos(ay - \nu\pi - \beta) Y_\nu(ay)]$                                                                                                                                                                                        |
| (10) | $x^{\frac{1}{2}\nu+k} J_\nu(ax^{\frac{1}{2}})$<br>$a > 0, \quad k = 0, 1, 2$<br>$-k - 1 < \operatorname{Re} \nu < -2k + 3/2$                        | $2(-1)^k y^{\frac{1}{2}\nu+k} K_\nu(ay^{\frac{1}{2}})$                                                                                                                                                                                                                                                  |
| (11) | $x^{\frac{1}{2}\nu+k-\frac{1}{2}} J_\nu(ax^{\frac{1}{2}})$<br>$a > 0, \quad k = 0, 1, 2$<br>$-k - \frac{1}{2} < \operatorname{Re} \nu < -2k + 5/2$  | $(-1)^k \pi \sec(\nu\pi) y^{\frac{1}{2}\nu+k-\frac{1}{2}} \times [I_\nu(ay^{\frac{1}{2}}) - \mathbf{L}_{-\nu}(ay^{\frac{1}{2}})]$                                                                                                                                                                       |
| (12) | $x^{k-\frac{1}{2}\nu-\frac{1}{2}} J_\nu(ax^{\frac{1}{2}})$<br>$a > 0, \quad k = 0, 1, 2, \dots$<br>$\operatorname{Re} \nu > 2k - 5/2$               | $\pi y^{k-\frac{1}{2}\nu-\frac{1}{2}} [I_\nu(ay^{\frac{1}{2}}) - \mathbf{L}_\nu(ay^{\frac{1}{2}})]$                                                                                                                                                                                                     |
| (13) | $x^\lambda J_\nu(ax^{\frac{1}{2}})$<br>$a > 0, \quad \operatorname{Re}(\lambda + \frac{1}{2}\nu) > -1$<br>$\operatorname{Re} \lambda < \frac{3}{4}$ | $\left(\frac{2}{a}\right)^{2\lambda} \frac{\Gamma(\lambda + \frac{1}{2}\nu)}{\Gamma(1 - \lambda + \frac{1}{2}\nu)} \times {}_1F_2\left(1; 1 - \lambda - \frac{\nu}{2}, 1 - \lambda + \frac{\nu}{2}; \frac{a^2 y}{4}\right) - \pi \csc[(\lambda + \frac{1}{2}\nu)\pi] y^\lambda I_\nu(ay^{\frac{1}{2}})$ |

**Higher transcendental functions (cont'd)**

|      | $f(x)$                                                                                                                                  | $\int_0^\infty f(x) (x+y)^{-1} dx \quad  \arg y  < \pi$                                                                            |
|------|-----------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------|
| (14) | $\sin(ax^{\frac{1}{2}}) J_0(bx^{\frac{1}{2}}) \quad 0 < b < a$                                                                          | $\pi \exp(-ay^{\frac{1}{2}}) I_0(by^{\frac{1}{2}})$                                                                                |
| (15) | $x^{-\frac{1}{2}} \sin(ax^{\frac{1}{2}}) J_0(bx^{\frac{1}{2}}) \quad 0 < a < b$                                                         | $2y^{-\frac{1}{2}} \sinh(ay^{\frac{1}{2}}) K_0(by^{\frac{1}{2}})$                                                                  |
| (16) | $\cos(ax^{\frac{1}{2}}) J_0(bx^{\frac{1}{2}}) \quad 0 < a < b$                                                                          | $2 \cosh(ay^{\frac{1}{2}}) K_0(by^{\frac{1}{2}})$                                                                                  |
| (17) | $x^{-\frac{1}{2}} \cos(ax^{\frac{1}{2}}) J_0(bx^{\frac{1}{2}}) \quad 0 < b < a$                                                         | $\pi y^{-\frac{1}{2}} \exp(-ay^{\frac{1}{2}}) I_0(by^{\frac{1}{2}})$                                                               |
| (18) | $x^{\frac{1}{2}\nu-\frac{1}{2}} \sin(ax^{\frac{1}{2}}) J_\nu(bx^{\frac{1}{2}}) \quad 0 < a < b, \quad -1 < \operatorname{Re} \nu < 3/2$ | $2y^{\frac{1}{2}\nu-\frac{1}{2}} \sinh(ay^{\frac{1}{2}}) K_\nu(by^{\frac{1}{2}})$                                                  |
| (19) | $x^{-\frac{1}{2}\nu} \sin(ax^{\frac{1}{2}}) J_\nu(bx^{\frac{1}{2}}) \quad 0 < b < a, \quad \operatorname{Re} \nu > -\frac{1}{2}$        | $\pi y^{-\frac{1}{2}\nu} \exp(-ay^{\frac{1}{2}}) I_\nu(by^{\frac{1}{2}})$                                                          |
| (20) | $x^{\frac{1}{2}\nu} \cos(ax^{\frac{1}{2}}) J_\nu(bx^{\frac{1}{2}}) \quad 0 < a < b, \quad -1 < \operatorname{Re} \nu < \frac{1}{2}$     | $2y^{\frac{1}{2}\nu} \cosh(ay^{\frac{1}{2}}) K_\nu(by^{\frac{1}{2}})$                                                              |
| (21) | $x^{-\frac{1}{2}\nu-\frac{1}{2}} \cos(ax^{\frac{1}{2}}) J_\nu(bx^{\frac{1}{2}}) \quad 0 < b < a, \quad \operatorname{Re} \nu > -3/2$    | $\pi y^{-\frac{1}{2}\nu-\frac{1}{2}} \exp(-ay^{\frac{1}{2}}) I_\nu(by^{\frac{1}{2}})$                                              |
| (22) | $[J_\nu(ax)]^2 \quad a > 0$                                                                                                             | $2 I_\nu(ay^{\frac{1}{2}}) K_\nu(ay^{\frac{1}{2}})$                                                                                |
| (23) | $J_\nu(ax^{\frac{1}{2}}) J_\nu(bx^{\frac{1}{2}}) \quad a, b > 0, \quad \operatorname{Re} \nu > -1$                                      | $2 I_\nu(ay^{\frac{1}{2}}) K_\nu(by^{\frac{1}{2}}) \quad b > a$<br>$2 I_\nu(by^{\frac{1}{2}}) K_\nu(ay^{\frac{1}{2}}) \quad b < a$ |



**Higher transcendental functions (cont'd)**

|      | $f(x)$                                                                                                                                                                                                                                       | $\int_0^\infty f(x) (x + \gamma)^{-1} dx \quad  \arg \gamma  < \pi$                                                                                                                                                                                                                                                                                                                    |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (24) | $x^{\frac{1}{2}\nu - \frac{1}{2}\mu} J_\nu(bx^{\frac{1}{2}}) J_\mu(ax^{\frac{1}{2}})$<br>$0 < a < b$<br>$2 + \operatorname{Re} \mu > \operatorname{Re} \nu > -1$                                                                             | $2\gamma^{\frac{1}{2}\nu - \frac{1}{2}\mu} I_\mu(ay^{\frac{1}{2}}) K_\nu(by^{\frac{1}{2}})$                                                                                                                                                                                                                                                                                            |
| (25) | $x^\lambda J_\mu(ax^{\frac{1}{2}}) J_\nu(ax^{\frac{1}{2}})$<br>$a > 0, \quad \operatorname{Re} \lambda < 1$<br>$\operatorname{Re}(2\lambda + \mu + \nu) > -2$                                                                                | $a^{-2\lambda} \pi^{-\frac{1}{2}} G_{35}^{23} \left( a^2 \gamma \left  \begin{matrix} 0, \lambda, \lambda + \frac{1}{2} \\ 0, p, q, r, s \end{matrix} \right. \right)$<br>$p = \lambda + \frac{1}{2}\mu + \frac{1}{2}\nu, \quad q = \lambda + \frac{1}{2}\mu - \frac{1}{2}\nu$<br>$r = \lambda - \frac{1}{2}\mu + \frac{1}{2}\nu, \quad s = \lambda - \frac{1}{2}\mu - \frac{1}{2}\nu$ |
| (26) | $x^{\frac{1}{2}\mu + n} (x + \gamma)^{-\frac{1}{2}\nu} J_\mu(ax^{\frac{1}{2}})$<br>$\times J_\nu[b(x + \gamma)^{\frac{1}{2}}]$<br>$a > b > 0, \quad n = 0, 1, 2, \dots$<br>$-1 - n < \operatorname{Re} \mu < 2 - 2n + \operatorname{Re} \nu$ | $2(-1)^n \gamma^{\frac{1}{2}\mu + n} (\gamma - \gamma)^{-\frac{1}{2}\nu}$<br>$\times K_\mu(ay^{\frac{1}{2}}) I_\mu[b(y - \gamma)^{\frac{1}{2}}]$                                                                                                                                                                                                                                       |
| (27) | $x^{-\frac{1}{2}} [\sin(ax) J_\nu(ax)$<br>$+ \cos(ax) Y_\nu(ax)]$<br>$a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                                                                                                       | $\pi \sec(\nu\pi) \gamma^{-\frac{1}{2}} [-\sin(ay) J_\nu(ay)$<br>$+ \cos(ay) Y_\nu(ay)]$                                                                                                                                                                                                                                                                                               |
| (28) | $x^{-\frac{1}{2}} [\cos(ax) J_\nu(ax)$<br>$- \sin(ax) Y_\nu(ax)]$<br>$a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                                                                                                       | $\pi \sec(\nu\pi) \gamma^{-\frac{1}{2}} [\cos(ay) J_\nu(ay)$<br>$+ \sin(ay) Y_\nu(ay)]$                                                                                                                                                                                                                                                                                                |

## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                                                 | $\int_0^\infty f(x) (x+y)^{-1} dx \quad  \arg y  < \pi$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |
|------|----------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (29) | $x^\lambda Y_\nu(ax) \quad a > 0$<br>$-1 +  \operatorname{Re} \nu  < \operatorname{Re} \lambda < 3/2$                                  | $-\frac{\pi \cot(\nu\pi) y^\lambda}{\sin[(\nu+\lambda)\pi]} J_\nu\left(\frac{ay}{2}\right)$ $-\frac{\pi y^\lambda}{\sin(\nu\pi) \sin[(\nu-\lambda)\pi]} J_{-\nu}\left(\frac{ay}{2}\right)$ $-2^{\lambda-1} \pi^{-1} a^{-\lambda} \cos[\frac{1}{2}(\lambda-\nu)\pi]$ $\times \Gamma\left(\frac{\lambda-\nu}{2}\right) \Gamma\left(\frac{\lambda+\nu}{2}\right)$ $\times {}_1F_2\left(1; 1-\frac{\lambda-\nu}{2}, 1-\frac{\lambda+\nu}{2}; -\frac{a^2 y^2}{4}\right)$ $+ 2^{\lambda-2} \pi^{-1} a^{1-\lambda} y \sin\left(\frac{\lambda-\nu}{2} \pi\right)$ $\times \Gamma\left(\frac{\lambda-\nu-1}{2}\right) \Gamma\left(\frac{\lambda+\nu-1}{2}\right)$ $\times {}_1F_2\left(1; \frac{3-\lambda+\nu}{2}, \frac{3-\lambda-\nu}{2}; -\frac{a^2 y^2}{4}\right)$ |
| (30) | $x^{\frac{1}{2}\nu-\frac{1}{2}} Y_\nu(ax^{\frac{1}{2}})$<br>$a > 0, \quad -1/2 < \operatorname{Re} \nu < 5/2$                          | $-2y^{\frac{1}{2}\nu-\frac{1}{2}} K_\nu(ay^{\frac{1}{2}})$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |
| (31) | $x^{\frac{1}{2}\nu+\frac{1}{2}} Y_\nu(ax^{\frac{1}{2}})$<br>$a > 0, \quad -3/2 < \operatorname{Re} \nu < 1/2$                          | $2y^{\frac{1}{2}\nu+\frac{1}{2}} K_\nu(ay^{\frac{1}{2}})$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     |
| (32) | $x^\lambda Y_\nu(ax^{\frac{1}{2}}) \quad a > 0$<br>$-1 + \frac{1}{2} \operatorname{Re} \nu  < \operatorname{Re} \lambda < \frac{3}{4}$ | $y^\lambda G_{24}^{31}\left(\frac{a^2 y}{4} \left  \begin{array}{c} -\lambda, -\frac{\nu+1}{2} \\ -\lambda, -\frac{\nu}{2}, \frac{\nu}{2}, -\frac{\nu+1}{2} \end{array} \right. \right)$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      |

**Higher transcendental functions (cont'd)**

|      | $f(x)$                                                                                                                                                                                                                                                                                                                     | $\int_0^\infty f(x) (x+y)^{-1} dx \quad  \arg y  < \pi$                                                                                                                        |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (33) | $x^{\lambda-1} \{ \cos[(\lambda - \frac{1}{2}\nu)\pi] J_\nu(ax^{\frac{1}{2}})$<br>$+ \sin[(\lambda - \frac{1}{2}\nu)\pi] Y_\nu(ax^{\frac{1}{2}}) \}$<br>$a > 0, \quad  \operatorname{Re} \nu  < 2 \operatorname{Re} \lambda < 7/2$                                                                                         | $-2y^{\lambda-1} K_\nu(ay^{\frac{1}{2}})$                                                                                                                                      |
| (34) | $x^{\frac{1}{2}\mu+n-\frac{1}{2}} (x+y)^{-\frac{1}{2}\nu} Y_\mu(ax^{\frac{1}{2}})$<br>$\times J_\nu[b(x+y)^{\frac{1}{2}}]$<br>$a > b > 0, \quad n = 0, 1, 2, \dots$<br>$-\frac{1}{2} - n < \operatorname{Re} \mu < 3 - 2n + \operatorname{Re} \nu$                                                                         | $2(-1)^{n+1} y^{\frac{1}{2}\mu+n-\frac{1}{2}} (y-\gamma)^{-\frac{1}{2}\nu}$<br>$\times K_\mu(ay^{\frac{1}{2}}) I_\nu[b(y-\gamma)^{\frac{1}{2}}]$                               |
| (35) | $x^{\lambda-1} (x+y)^{-\frac{1}{2}\mu} J_\mu[b(x+y)^{\frac{1}{2}}]$<br>$\times \{ \cos[(\lambda - \frac{1}{2}\nu)\pi] J_\nu(ax^{\frac{1}{2}})$<br>$+ \sin[(\lambda - \frac{1}{2}\nu)\pi] Y_\nu(ax^{\frac{1}{2}}) \}$<br>$a > b > 0$<br>$ \operatorname{Re} \nu  < 2 \operatorname{Re} \lambda < 4 + \operatorname{Re} \mu$ | $-2y^{\lambda-1} (y-\gamma)^{-\frac{1}{2}\mu}$<br>$\times I_\mu[b(y-\gamma)^{\frac{1}{2}}] K_\nu(ay^{\frac{1}{2}})$                                                            |
| (36) | $x^\nu e^{-ax} I_\nu(ax)$<br>$\operatorname{Re} a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                                                                                                                                                                                                           | $y^\nu \sec(\nu\pi) e^{ay} K_\nu(ay)$                                                                                                                                          |
| (37) | $x^\lambda e^{-ax} I_\nu(ax)$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \lambda < \frac{1}{2}$<br>$\operatorname{Re}(\lambda + \nu) > -1$                                                                                                                                                                       | $\pi^{-\frac{1}{2}} y^\lambda G_{23}^{22} \left( 2ay \left  \begin{matrix} -\lambda, \frac{1}{2} \\ -\lambda, \nu, -\nu \end{matrix} \right. \right)$                          |
| (38) | $x^\lambda e^{ax} K_\nu(ax)$<br>$ \arg a  < 3\pi/2$<br>$\operatorname{Re} \lambda -  \operatorname{Re} \nu  > -1$                                                                                                                                                                                                          | $\pi^{-\frac{1}{2}} \cos(\nu\pi) y^\lambda$<br>$\times G_{23}^{32} \left( 2ay \left  \begin{matrix} -\lambda, \frac{1}{2} \\ -\lambda, \nu, -\nu \end{matrix} \right. \right)$ |

## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                                                                                                | $\int_0^\infty f(x) (x+y)^{-1} dx \quad  \arg y  < \pi$                                                                                                                                                                                                                                                                   |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (39) | $x^{-\frac{1}{2}} e^{-ax} K_\nu(ax)$<br>$\operatorname{Re} a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                                                           | $\pi \sec(\nu\pi) y^{-\frac{1}{2}} e^{ay} K_\nu(ay)$                                                                                                                                                                                                                                                                      |
| (40) | $x^\lambda e^{-ax} K_\nu(ax)$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \lambda -  \operatorname{Re} \nu  > -1$                                                            | $\pi^{\frac{1}{2}} y^\lambda G_{23}^{31} \left( 2ay \left  \begin{matrix} -\lambda, \frac{1}{2} \\ -\lambda, \nu, -\nu \end{matrix} \right. \right)$                                                                                                                                                                      |
| (41) | $x^{-\frac{1}{2}\nu - \frac{1}{2}} K_\nu(ax^{\frac{1}{2}})$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu < \frac{1}{2}$                                                   | $\frac{1}{2} \pi^2 y^{-\frac{1}{2}\nu - \frac{1}{2}} \sec(\nu\pi)$<br>$\times [\mathbf{H}_\nu(ay^{\frac{1}{2}}) - Y_\nu(ay^{\frac{1}{2}})]$                                                                                                                                                                               |
| (42) | $x^\lambda K_\nu(ax^{\frac{1}{2}})$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \lambda > \frac{1}{2}  \operatorname{Re} \nu  - 1$                                           | $2^{2\lambda+1} y^\lambda \Gamma(1+\lambda+\frac{1}{2}\nu) \Gamma(1+\lambda-\frac{1}{2}\nu)$<br>$\times S_{-2\lambda-1, \nu}(ay^{\frac{1}{2}})$                                                                                                                                                                           |
| (43) | $x^{-\frac{1}{2}} [2\pi^{-1} K_0(ax^{\frac{1}{2}}) - Y_0(ax^{\frac{1}{2}})]$<br>$ \arg a  < \frac{1}{4}\pi$                                                                           | $4y^{-\frac{1}{4}} \ker(ay^{\frac{1}{4}})$                                                                                                                                                                                                                                                                                |
| (44) | $x^{\frac{1}{2}\nu} \mathbf{H}_\nu(ax^{\frac{1}{2}})$<br>$a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                                                            | $\pi \sec(\nu\pi) y^{\frac{1}{2}\nu} [I_{-\nu}(ay^{\frac{1}{2}}) - \mathbf{L}_\nu(ay^{\frac{1}{2}})]$                                                                                                                                                                                                                     |
| (45) | $x^{-\frac{1}{2}\nu} \mathbf{H}_\nu(ax^{\frac{1}{2}})$<br>$a > 0, \quad \operatorname{Re} \nu > -3/2$                                                                                 | $\pi y^{-\frac{1}{2}\nu} [I_\nu(ay^{\frac{1}{2}}) - \mathbf{L}_\nu(ay^{\frac{1}{2}})]$                                                                                                                                                                                                                                    |
| (46) | $x^\lambda \mathbf{H}_\nu(ax^{\frac{1}{2}}) \quad a > 0, \quad \operatorname{Re} \lambda < \frac{3}{4}$<br>$-\frac{1}{2} < \operatorname{Re}(\lambda + \frac{1}{2}\nu) < \frac{1}{2}$ | $\frac{\pi}{\cos[(\lambda + \frac{1}{2}\nu)\pi]} \left[ \frac{(2/a)^{2\lambda}}{\Gamma(1-\lambda+\frac{1}{2}\nu) \Gamma(1-\lambda-\frac{1}{2}\nu)} \right.$<br>$\times {}_1F_2(1; 1-\lambda+\frac{1}{2}\nu, 1-\lambda-\frac{1}{2}\nu; \frac{1}{4}a^2 y)$<br>$\left. - y^\lambda \mathbf{L}_\nu(ay^{\frac{1}{2}}) \right]$ |

## Higher transcendental functions (cont'd)

|                                                                                                                                                                            | $f(x)$                                                                                                                                                                        | $\int_0^\infty f(x) (x+y)^{-1} dx \quad  \arg y  < \pi$                                                                                                                                                                                               |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (47)                                                                                                                                                                       | $x^{-1/2} [\cos(\frac{1}{2}\nu\pi) J_\nu(ax^{1/2})$<br>$+ \sin(\frac{1}{2}\nu\pi) \mathbf{H}_\nu(ax^{1/2})]$<br>$a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < 2$       | $\pi y^{-1/2} [I_\nu(ay^{1/2}) - \mathbf{L}_\nu(ay^{1/2})]$                                                                                                                                                                                           |
| (48)                                                                                                                                                                       | $x^{-1/2} [I_\nu(ax^{1/2}) - \mathbf{L}_\nu(ax^{1/2})]$<br>$\operatorname{Re} a > 0, \quad -1 < \operatorname{Re} \nu < 2$                                                    | $\pi a^{-1} \csc(\frac{1}{2}\nu\pi) [\mathbf{H}_\nu(ay^{1/2}) + \mathbf{E}_\nu(ay^{1/2})]$                                                                                                                                                            |
| (49)                                                                                                                                                                       | $x^\lambda [I_\nu(ax^{1/2}) - \mathbf{L}_\nu(ax^{1/2})]$<br>$\operatorname{Re} a > 0$<br>$-2 < \operatorname{Re}(2\lambda + \nu) < 1$                                         | $\frac{y^\lambda}{\pi} G_{24}^{32} \left( \frac{a^2 y}{4} \left  \begin{array}{c} -\lambda, \frac{1}{2}\nu + \frac{1}{2} \\ -\lambda, \frac{1}{2}\nu, \frac{1}{2}\nu + \frac{1}{2}, -\frac{1}{2}\nu \end{array} \right. \right)$                      |
| For other integrals with Bessel functions see Watson, G.N., 1922:<br><i>A treatise on the theory of Bessel functions</i> , Cambridge, in particular sections 13.5 to 13.6. |                                                                                                                                                                               |                                                                                                                                                                                                                                                       |
| (50)                                                                                                                                                                       | $x^{\mu-1/2} e^{-1/2 ax} M_{\kappa, \mu}(ax)$<br>$\operatorname{Re} a > 0$<br>$-\frac{1}{2} < \operatorname{Re} \mu < \operatorname{Re} \kappa + \frac{1}{2}$                 | $\Gamma(2\mu+1) \Gamma(\kappa - \mu + \frac{1}{2}) y^{\mu-1/2}$<br>$\times e^{1/2 ax} W_{-\kappa, \mu}(ax)$                                                                                                                                           |
| (51)                                                                                                                                                                       | $x^\lambda e^{-1/2 ax} M_{\kappa, \mu}(ax) \quad \operatorname{Re} a > 0$<br>$-3/2 - \operatorname{Re} \mu < \operatorname{Re} \lambda < \operatorname{Re} \kappa$            | $\frac{\Gamma(2\mu+1) y^\lambda}{\Gamma(\kappa + \mu + \frac{1}{2})} G_{23}^{22} \left( ax \left  \begin{array}{c} -\lambda, 1-\kappa \\ -\lambda, \mu + \frac{1}{2}, \frac{1}{2}-\mu \end{array} \right. \right)$                                    |
| (52)                                                                                                                                                                       | $x^\lambda e^{1/2 ax} W_{\kappa, \mu}(ax)$<br>$ \arg a  < 3\pi/2$<br>$\operatorname{Re}(\kappa + \lambda) < 0$<br>$\operatorname{Re} \lambda >  \operatorname{Re} \mu  - 3/2$ | $y^\lambda [\Gamma(\frac{1}{2} + \mu - \kappa) \Gamma(\frac{1}{2} - \mu - \kappa)]^{-1}$<br>$\times G_{23}^{32} \left( ay \left  \begin{array}{c} -\lambda, \kappa + 1 \\ -\lambda, \frac{1}{2} + \mu, \frac{1}{2} - \mu \end{array} \right. \right)$ |

**Higher transcendental functions (cont'd)**

|      | $f(x)$                                                                                                                                                                                                                                                                                                | $\int_0^\infty f(x) (x+y)^{-1} dx \quad  \arg y  < \pi$                                                                                                                    |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (53) | $x^{\kappa-1} e^{-\frac{1}{2}ax} W_{\kappa, \mu}(ax)$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \kappa >  \operatorname{Re} \mu  - \frac{1}{2}$                                                                                                                                            | $\Gamma(\kappa + \mu + \frac{1}{2}) \Gamma(\kappa - \mu + \frac{1}{2}) y^{\kappa-1}$<br>$\times e^{\frac{1}{2}ay} W_{-\kappa, \mu}(ay)$                                    |
| (54) | $x^\lambda e^{-\frac{1}{2}ax} W_{\kappa, \mu}(ax) \quad \operatorname{Re} a > 0$<br>$\operatorname{Re} \lambda >  \operatorname{Re} \mu  - 3/2$                                                                                                                                                       | $y^\lambda G_{23}^{31} \left( ay \left  \begin{matrix} -\lambda, 1-\kappa \\ -\lambda, \frac{1}{2}+\mu, \frac{1}{2}-\mu \end{matrix} \right. \right)$                      |
| (55) | $G_{pq}^{mn} \left( ax \left  \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right)$<br>$p+q < 2(m+n)$<br>$ \arg a  < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$<br>$\operatorname{Re} a_j < 1 \quad j=1, \dots, n$<br>$\operatorname{Re} b_j > -1 \quad j=1, \dots, m$             | $G_{p+1, q+1}^{m+1, n+1} \left( ay \left  \begin{matrix} 0, a_1, \dots, a_p \\ 0, b_1, \dots, b_q \end{matrix} \right. \right)$                                            |
| (56) | $G_{pq}^{mn} \left( ax^2 \left  \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right)$<br>$p+q < 2(m+n)$<br>$ \arg a  < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$<br>$\operatorname{Re} a_j < 1 \quad j=1, \dots, n$<br>$\operatorname{Re} b_j > -\frac{1}{2} \quad j=1, \dots, m$ | $\frac{1}{2\pi} G_{p+2, q+2}^{m+2, n+2} \left( ay^2 \left  \begin{matrix} 0, \frac{1}{2}, a_1, \dots, a_p \\ 0, \frac{1}{2}, b_1, \dots, b_q \end{matrix} \right. \right)$ |

## 14.4. Generalized Stieltjes transforms

|      | $f(x)$                                                                                                     | $\int_0^\infty f(x) (x+y)^{-\rho} dx \quad  \arg y  < \pi$                                                                                                                             |
|------|------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1)  | $f(x)$                                                                                                     | $g(y; \rho)$                                                                                                                                                                           |
| (2)  | $x f(x)$                                                                                                   | $g(y; \rho-1) - y g(y; \rho)$                                                                                                                                                          |
| (3)  | $f(ax) \quad a > 0$                                                                                        | $a^{\rho-1} g(ay; \rho)$                                                                                                                                                               |
| (4)  | $x^{\rho-2} f(a/x) \quad a > 0$                                                                            | $a^{\rho-1} y^{-\rho} g(a/y)$                                                                                                                                                          |
| (5)  | $f'(x)$                                                                                                    | $\rho g(y; \rho+1) - y^{-\rho} f(0)$                                                                                                                                                   |
| (6)  | $\int_0^x f(t) dt$                                                                                         | $(\rho-1)^{-1} g(y; \rho-1) \quad \operatorname{Re} \rho > 1$                                                                                                                          |
| (7)  | $[\Gamma(\mu)]^{-1} \int_0^x f(t)(x-t)^{\mu-1} dt$<br>$0 < \operatorname{Re} \mu < \operatorname{Re} \rho$ | $\frac{\Gamma(\rho-\mu)}{\Gamma(\rho)} g(y; \rho-\mu)$                                                                                                                                 |
| (8)  | $x^{\nu-1} \quad \operatorname{Re} \nu > 0$                                                                | $\frac{\Gamma(\nu) \Gamma(\rho-\nu)}{\Gamma(\rho)} y^{\nu-\rho} \quad \operatorname{Re} \rho > \operatorname{Re} \nu$                                                                  |
| (9)  | $x^{\nu-1} (a+x)^{-\mu}$<br>$ \arg a  < \pi, \quad \operatorname{Re} \nu > 0$                              | $\frac{\Gamma(\nu) \Gamma(\mu-\nu+\rho) y^{\nu-\rho}}{\Gamma(\mu+\rho) a^\mu}$<br>$\times {}_2F_1(\mu, \nu; \mu+\rho; 1-y/a)$<br>$\operatorname{Re} \rho > \operatorname{Re}(\nu-\mu)$ |
| (10) | $e^{-ax} \quad \operatorname{Re} a > 0$                                                                    | $a^{\rho-1} e^{ay} \Gamma(1-\rho, ay)$                                                                                                                                                 |
| (11) | $x^{-\rho} e^{-ax} \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \rho < 1$                        | $\pi^{-\frac{1}{2}} \Gamma(1-\rho)(a/y)^{\rho-\frac{1}{2}} e^{\frac{1}{2}ay}$<br>$\times K_{\rho-\frac{1}{2}}(\frac{1}{2}ay)$                                                          |

## Generalized Stieltjes transforms (cont'd)

|      | $f(x)$                                                                                                 | $\int_0^\infty f(x) (x+y)^{-\rho} dx \quad  \arg y  < \pi$                                                                                                                                                                                                                                               |
|------|--------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (12) | $x^\lambda e^{-ax}$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \lambda > -1$                 | $\Gamma(\lambda+1) a^{\frac{1}{2}\rho - \frac{1}{2}\lambda - 1} y^{\frac{1}{2}\lambda - \frac{1}{2}\rho}$<br>$\times e^{\frac{1}{2}ay} W_{k,m}(ay)$<br>$2k = -\lambda - \rho, \quad 2m = \lambda - \rho + 1$                                                                                             |
| (13) | $x^\lambda \exp(-a/x)$<br>$\operatorname{Re} a > 0$                                                    | $\Gamma(\rho - \lambda - 1) a^{\frac{1}{2}\lambda} y^{-\frac{1}{2}\lambda - 1}$<br>$\times \exp\left(\frac{a}{2y}\right) W_{k,m}\left(\frac{a}{y}\right)$<br>$k = \frac{1}{2}\lambda - \rho + 1, \quad m = \frac{1}{2}\lambda + \frac{1}{2}$<br>$\operatorname{Re} \rho > \operatorname{Re} \lambda + 1$ |
| (14) | $x^{-\frac{1}{2}} \exp(-ax^{\frac{1}{2}})$<br>$\operatorname{Re} a > 0$                                | $\pi^{\frac{1}{2}} (2y^{\frac{1}{2}}/a)^{\frac{1}{2}-\rho} \Gamma(1-\rho)$<br>$\times [\mathbf{H}_{\frac{1}{2}-\rho}(ay^{\frac{1}{2}}) - Y_{\frac{1}{2}-\rho}(ay^{\frac{1}{2}})]$                                                                                                                        |
| (15) | $x^\lambda \exp(-ax^{\frac{1}{2}})$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \lambda > -1$ | $\frac{y^{\lambda-\rho+1}}{\pi^{\frac{1}{2}} \Gamma(\rho)} G_{13}^{31}\left(\frac{a^2 y}{4} \left  \begin{matrix} -\lambda \\ \rho - \lambda - 1, 0, \frac{1}{2} \end{matrix} \right.\right)$                                                                                                            |
| (16) | $\sin(ax^{\frac{1}{2}})$<br>$a > 0$                                                                    | $\frac{2\pi^{\frac{1}{2}} y^{\frac{1}{2}}}{\Gamma(\rho)} \left(\frac{2y^{\frac{1}{2}}}{a}\right)^{\frac{1}{2}-\rho} K_{\rho-3/2}(ay^{\frac{1}{2}})$<br>$\operatorname{Re} \rho > \frac{1}{2}$                                                                                                            |
| (17) | $x^\lambda \sin(ax^{\frac{1}{2}})$<br>$a > 0, \quad \operatorname{Re} \lambda > -\frac{1}{2}$          | $\frac{\pi^{\frac{1}{2}} y^{\lambda-\rho+1}}{\Gamma(\rho)} G_{13}^{21}\left(\frac{a^2 y}{4} \left  \begin{matrix} -\lambda \\ \rho - \lambda - 1, \frac{1}{2}, 0 \end{matrix} \right.\right)$<br>$\operatorname{Re} \rho > \operatorname{Re} \lambda + \frac{1}{2}$                                      |
| (18) | $x^{-\frac{1}{2}} \cos(ax^{\frac{1}{2}})$<br>$a > 0$                                                   | $\frac{2\pi^{\frac{1}{2}}}{\Gamma(\rho)} \left(\frac{2y^{\frac{1}{2}}}{a}\right)^{\frac{1}{2}-\rho} K_{\rho-\frac{1}{2}}(ay^{\frac{1}{2}})$<br>$\operatorname{Re} \rho > 0$                                                                                                                              |



## Generalized Stieltjes transforms (cont'd)

|      | $f(x)$                                                                                                            | $\int_0^\infty f(x) (x+y)^{-\rho} dx \quad  \arg y  < \pi$                                                                                                                                                                                                                                                     |
|------|-------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (19) | $x^\lambda \cos(ax^{1/2})$<br>$a > 0, \quad \operatorname{Re} \lambda > -1$                                       | $\frac{\pi^{1/2} y^{\lambda-\rho+1}}{\Gamma(\rho)} G_{13}^{21} \left( \frac{a^2 y}{4} \left  \begin{matrix} -\lambda \\ \rho-\lambda-1, 0, \frac{1}{2} \end{matrix} \right. \right)$<br>$\operatorname{Re} \rho > \operatorname{Re} \lambda + \frac{1}{2}$                                                     |
| (20) | $x^{1/2\nu} J_\nu(ax^{1/2})$<br>$a > 0, \quad \operatorname{Re} \nu > -1$                                         | $\frac{a^{\rho-1}}{2^\rho \Gamma(\rho)} y^{1/2\nu+1/2-1/2\rho} K_{\nu-\rho+1}(ay^{1/2})$<br>$\operatorname{Re} \rho > \frac{1}{2} \operatorname{Re} \nu + \frac{1}{4}$                                                                                                                                         |
| (21) | $x^\lambda J_\nu(ax^{1/2})$<br>$a > 0, \quad \operatorname{Re}(\lambda + \frac{1}{2}\nu) > -1$                    | $\frac{2^{2\lambda} y^{1-\rho}}{a^{2\lambda} \Gamma(\rho)} G_{13}^{21} \left( \frac{a^2 y}{4} \left  \begin{matrix} 0 \\ \rho-1, \lambda+\frac{1}{2}\nu, \lambda-\frac{1}{2}\nu \end{matrix} \right. \right)$<br>$\operatorname{Re} \rho > \operatorname{Re} \lambda + \frac{1}{4}$                            |
| (22) | $x^\nu e^{-ax} I_\nu(ax)$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$                | $\frac{\Gamma(\nu+\frac{1}{2}) \Gamma(\rho-\nu-\frac{1}{2})}{\pi^{1/2} \Gamma(\rho)} (2a)^{1/2\rho-1}$<br>$\times y^{\nu-1/2\rho} e^{ay} W_{k,m}(2ay)$<br>$k = \frac{1}{2} - \frac{1}{2}\rho, \quad m = \frac{1}{2} - \frac{1}{2}\rho + \nu$<br>$\operatorname{Re} \rho > \operatorname{Re} \nu + \frac{1}{2}$ |
| (23) | $x^\lambda e^{-ax} I_\nu(ax)$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re}(\lambda + \nu) > -1$           | $\frac{y^{\lambda+1-\rho}}{\pi^{1/2} \Gamma(\rho)} G_{23}^{22} \left( 2ay \left  \begin{matrix} -\lambda, \frac{1}{2} \\ \rho-\lambda-1, \nu, -\nu \end{matrix} \right. \right)$<br>$\operatorname{Re} \rho > \operatorname{Re} \lambda + \frac{1}{2}$                                                         |
| (24) | $x^\lambda e^{ax} K_\nu(ax) \quad  \arg a  < 3\pi/2$<br>$\operatorname{Re} \lambda >  \operatorname{Re} \nu  - 1$ | $\frac{\cos(\nu\pi)}{\pi^{1/2} \Gamma(\rho)} y^{\lambda+1-\rho}$<br>$\times G_{23}^{32} \left( 2ay \left  \begin{matrix} -\lambda, \frac{1}{2} \\ \rho-\lambda-1, \nu, -\nu \end{matrix} \right. \right)$<br>$\operatorname{Re} \rho > \operatorname{Re} \lambda + \frac{1}{2}$                                |

## Generalized Stieltjes transforms (cont'd)

|      | $f(x)$                                                                                                                              | $\int_0^\infty f(x) (x+y)^{-\rho} dx \quad  \arg y  < \pi$                                                                                                                                                                                                                                                                                    |
|------|-------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (25) | $x^{\rho-3/2} e^{-ax} K_\nu(ax)$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \rho >  \operatorname{Re} \nu  + \frac{1}{2}$ | $\frac{\pi^{1/2} \Gamma(\rho + \nu - \frac{1}{2}) \Gamma(\rho - \nu - \frac{1}{2})}{(2a)^{1/2} \gamma \Gamma(\rho)} \times e^{ay} W_{1-\rho, \nu}(2ay)$                                                                                                                                                                                       |
| (26) | $x^\lambda e^{-ax} K_\nu(ax)$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \lambda >  \operatorname{Re} \nu  - 1$           | $\frac{\pi^{1/2}}{\Gamma(\rho)} \gamma^{\lambda+1-\rho} \times G_{23}^{31} \left( 2ay \left  \begin{array}{c} -\lambda, \frac{1}{2} \\ \rho - \lambda - 1, \nu, -\nu \end{array} \right. \right)$                                                                                                                                             |
| (27) | $x^\lambda K_\nu(ax^{1/2})$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \lambda > \frac{1}{2}  \operatorname{Re} \nu  - 1$ | $\frac{\gamma^{\lambda-\rho+1}}{2\Gamma(\rho)} G_{13}^{31} \left( \frac{a^2 y}{4} \left  \begin{array}{c} -\lambda \\ \rho - \lambda - 1, \frac{1}{2}\nu, -\frac{1}{2}\nu \end{array} \right. \right)$                                                                                                                                        |
| (28) | $x^{\mu-1/2} e^{-1/2 ax} M_{\kappa, \mu}(ax)$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re} \mu > -\frac{1}{2}$              | $\Gamma(2\mu+1) \Gamma(\kappa + \rho - \mu - \frac{1}{2}) [\Gamma(\rho)]^{-1} \times a^{1/2 \rho - 1/2} \gamma^{\lambda+1/2 - 1/2 \rho} e^{1/2 ay} W_{k, m}(ay)$<br>$k = \frac{1}{2} - \frac{1}{2} \rho - \kappa, \quad m = \frac{1}{2} - \frac{1}{2} \rho + \mu$<br>$\operatorname{Re} \rho > \operatorname{Re}(\mu - \kappa) + \frac{1}{2}$ |
| (29) | $x^\lambda e^{-1/2 ax} M_{\kappa, \mu}(ax)$<br>$\operatorname{Re} a > 0, \quad \operatorname{Re}(\lambda + \mu) > -3/2$             | $\frac{\Gamma(2\mu+1) \gamma^{\lambda+1-\rho}}{\Gamma(\rho) \Gamma(\kappa + \mu + \frac{1}{2})} \times G_{23}^{22} \left( ay \left  \begin{array}{c} -\lambda, 1-\kappa \\ \rho - \lambda - 1, \frac{1}{2} + \mu, \frac{1}{2} - \mu \end{array} \right. \right)$<br>$\operatorname{Re} \rho > \operatorname{Re}(\lambda - \kappa) + 1$        |

## Generalized Stieltjes transforms (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                   | $\int_0^\infty f(x) (x+y)^{-\rho} dx \quad  \arg y  < \pi$                                                                                                                                                                                                                                                                          |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (30) | $x^\lambda e^{\frac{1}{2}\alpha x} W_{\kappa, \mu}(ax)$<br>$ \arg a  < 3\pi/2$<br>$\operatorname{Re} \lambda >  \operatorname{Re} \mu  - 3/2$                                                                                            | $\frac{y^{\lambda+1-\rho}}{\Gamma(\rho) \Gamma(\frac{1}{2}-\kappa+\mu) \Gamma(\frac{1}{2}-\kappa-\mu)}$ $\times G_{23}^{32} \left( ay \left  \begin{matrix} -\lambda, 1+\kappa \\ \rho-\lambda-1, \frac{1}{2}+\mu, \frac{1}{2}-\mu \end{matrix} \right. \right)$ $\operatorname{Re} \rho > \operatorname{Re}(\lambda + \kappa) + 1$ |
| (31) | $x^{\kappa+\rho-2} e^{-\frac{1}{2}\alpha x} W_{\kappa, \mu}(ax)$<br>$\operatorname{Re} \alpha > 0$<br>$\operatorname{Re} \rho >  \operatorname{Re} \mu  - \operatorname{Re} \kappa + \frac{1}{2}$                                        | $\Gamma(\kappa+\rho+\mu-\frac{1}{2}) \Gamma(\kappa+\rho-\mu-\frac{1}{2}) [\Gamma(\rho)]^{-1}$ $\times y^{\kappa-1} e^{\frac{1}{2}\alpha y} W_{1-\kappa-\rho, \mu}(ay)$                                                                                                                                                              |
| (32) | $x^\lambda e^{-\frac{1}{2}\alpha x} W_{\kappa, \mu}(ax) \quad \operatorname{Re} \alpha > 0$<br>$\operatorname{Re} \lambda >  \operatorname{Re} \mu  - 3/2$                                                                               | $\frac{y^{\lambda+1-\rho}}{\Gamma(\rho)} G_{23}^{31} \left( ay \left  \begin{matrix} -\lambda, 1-\kappa \\ \rho-\lambda-1, \frac{1}{2}+\mu, \frac{1}{2}-\mu \end{matrix} \right. \right)$                                                                                                                                           |
| (33) | $G_{pq}^{mn} \left( ax \left  \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right)$<br>$p+q < 2(m+n)$<br>$ \arg a  < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi$<br>$\operatorname{Re} b_j > -1 \quad j = 1, \dots, m$ | $\frac{y^{1-\rho}}{\Gamma(\rho)} G_{p+1, q+1}^{m+1, n+1} \left( ay \left  \begin{matrix} 0, a_1, \dots, a_p \\ \rho-1, b_1, \dots, b_q \end{matrix} \right. \right)$ $\operatorname{Re} \rho > \operatorname{Re} a_j \quad j = 1, \dots, n$                                                                                         |



## CHAPTER XV

### HILBERT TRANSFORMS

We call

$$g(y) = \pi^{-1} \int_{-\infty}^{\infty} f(x) (x - y)^{-1} dx$$

the *Hilbert transform* of  $f(x)$ . Here  $x$  and  $y$  are real variables, and

$$\int_{-\infty}^{\infty} = \lim_{\epsilon \rightarrow +0} \left( \int_{-\infty}^{y-\epsilon} + \int_{y+\epsilon}^{\infty} \right)$$

is the Cauchy Principal Value of  $\int_{-\infty}^{\infty}$ .

For the theory of Hilbert transforms see chapter V of Titchmarsh's book (1937) and the references given there. Additional references to papers which appeared after the publication of Titchmarsh's book are given below. The finite Hilbert transform,

$$g(y) = \pi^{-1} \int_a^b f(x) (x - y)^{-1} dx$$

and its application to airfoil theory was discussed recently by Tricomi (1951 a, b) and Nickel (1951, 1953); the latter author gives references to earlier work on this subject.

In the above relation,  $g(x)$  is said to be *conjugate* to  $f(x)$ : the relationship is *skew-reciprocal*, i.e.,  $-f(x)$  is conjugate to  $g(x)$ . For the relation of Hilbert transforms to Fourier integrals see Titchmarsh (1937) and Kober (1942, 1943 a, b). The connection with Laplace transforms may be expressed by stating that, formally, the imaginary part of a Laplace transform evaluated on a line parallel to the imaginary axis is conjugate to the real part of that Laplace transform evaluated on the same line. Hilbert transforms may be evaluated by means of tables of Stieltjes transforms (Chapter XIV) using the formulas

$$g(y) = \pi^{-1} \mathfrak{G}\{f(x); -y\} - (2\pi)^{-1} \mathfrak{G}\{f(-x); |y| e^{i\pi}\} \\ - (2\pi)^{-1} \mathfrak{G}\{f(-x); |y| e^{-i\pi}\} \quad -\infty < y < 0$$

$$g(y) = (2\pi)^{-1} \mathfrak{G}\{f(x); ye^{i\pi}\} + (2\pi)^{-1} \mathfrak{G}\{f(x); ye^{-i\pi}\} \\ - \pi^{-1} \mathfrak{G}\{f(-x); y\} \quad 0 < y < \infty$$

Related transforms are

$$\int_{-\pi}^{\pi} f(x) \operatorname{ctn} [1/2(x-y)] dx$$

$$\int_0^{\pi} f(x) (\cos x - \cos y)^{-1} dx.$$

These can be reduced to Hilbert transforms by a change of the variables of integration.

From the transform pairs given in the tables, further transform pairs may be derived by the methods mentioned in the introduction to vol. I, by the general formulas given in sec. 15.1, and by exploiting the connection with other transforms (see above).

## REFERENCES

- Cossar, James, 1939: *Proc. London Math. Soc.* (2) 45, 369-381.
- Kober, Hermann, 1942: *Bull. Amer. Math. Soc.* 48, 421-426.
- Kober, Hermann, 1943 a: *J. London Math. Soc.* 18, 66-71.
- Kober, Hermann, 1943 b: *Quart. J. Math. Oxford Ser.* 14, 49-54.
- Nickel, Karl, 1951: *Math. Z.* 54, 81-96.
- Nickel, Karl, 1953: *Math. Z.* 58, 49-62.
- Titchmarsh, E.C., 1937: *Introduction to the theory of Fourier integrals.* Oxford.
- Tricomi, F.G., 1951a: *Quart. J. Math. Oxford Ser.* (2) 2, 199-211.
- Tricomi, F.G., 1951b: *Z. Angew. Math. Physik* 2, 402-406.





## HILBERT TRANSFORMS

### 15.1. General formulas

|     | $f(x)$                                                            | $\pi^{-1} \int_{-\infty}^{\infty} f(x)(x-y)^{-1} dx^*$ |
|-----|-------------------------------------------------------------------|--------------------------------------------------------|
| (1) | $f(x)$                                                            | $g(y)$                                                 |
| (2) | $g(x)$                                                            | $-f(y)$                                                |
| (3) | $f(a+x)$ <span style="float: right;"><math>a</math> real</span>   | $g(a+y)$                                               |
| (4) | $f(ax)$ <span style="float: right;"><math>a &gt; 0</math></span>  | $g(ay)$                                                |
| (5) | $f(-ax)$ <span style="float: right;"><math>a &gt; 0</math></span> | $-g(-ay)$                                              |
| (6) | $xf(x)$                                                           | $yg(y) + \pi^{-1} \int_{-\infty}^{\infty} f(x) dx$     |
| (7) | $(x+a)f(x)$                                                       | $(y+a)g(y) + \pi^{-1} \int_{-\infty}^{\infty} f(x) dx$ |
| (8) | $f'(x)$                                                           | $g'(y)$                                                |

### 15.2. Elementary functions

|     |                                                                                                                                                                                                                         |                                                     |
|-----|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------|
| (1) | 1                                                                                                                                                                                                                       | 0                                                   |
| (2) | <div style="display: flex; justify-content: space-between;"> <div> 0<br/>1<br/>0 </div> <div> <math>-\infty &lt; x &lt; a</math><br/> <math>a &lt; x &lt; b</math><br/> <math>b &lt; x &lt; \infty</math> </div> </div> | $\frac{1}{\pi} \log \left  \frac{b-y}{a-y} \right $ |

\*  $y$  is real, and the integral is a Cauchy Principal Value.

## Elementary functions (cont'd)

|     | $f(x)$                                                                                             | $\pi^{-1} \int_{-\infty}^{\infty} f(x) (x-y)^{-1} dx$                                                             |
|-----|----------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------|
| (3) | $0$<br>$x^{-1}$<br>$-\infty < x < a$<br>$a < x < \infty$<br>$a > 0$                                | $\frac{1}{\pi y} \log \left  \frac{a}{a-y} \right $<br>$y \neq 0, \quad y \neq a$                                 |
| (4) | $x^{-1}$<br>$0$<br>$x^{-1}$<br>$-\infty < x < a$<br>$a < x < b$<br>$b < x < \infty$<br>$a < 0 < b$ | $\frac{1}{\pi y} \log \left  \frac{(y-a)b}{a(b-y)} \right $<br>$y \neq 0, a, b$                                   |
| (5) | $0$<br>$x^{-2}$<br>$-\infty < x < a$<br>$a < x < \infty$<br>$a > 0$                                | $\frac{1}{\pi y^2} \log \left  \frac{a}{a-y} \right  - \frac{1}{\pi ay}$<br>$y \neq 0, \quad y \neq a$            |
| (6) | $(x+a)^{-1}$<br>$\text{Im } a > 0$                                                                 | $i(y+a)^{-1}$                                                                                                     |
| (7) | $(x+a)^{-1}$<br>$\text{Im } a < 0$                                                                 | $-i(y+a)^{-1}$                                                                                                    |
| (8) | $0$<br>$(ax+b)^{-1}$<br>$-\infty < x < 0$<br>$0 < x < \infty$<br>$a, b > 0$                        | $\frac{1}{\pi(ay+b)} \log \left  \frac{b}{ay} \right $<br>$y \neq -b/a, \quad y \neq 0$                           |
| (9) | $0$<br>$(ax+b)^{-2}$<br>$-\infty < x < 0$<br>$0 < x < \infty$<br>$a, b > 0$                        | $\frac{1}{\pi(ay+b)^2} \log \left  \frac{b}{ay} \right  - \frac{1}{\pi b(ay+b)}$<br>$y \neq 0, \quad y \neq -b/a$ |

$y$  is real, and the integral is a Cauchy Principal Value.

**Elementary functions (cont'd)**

|      | $f(x)$                                                                                                          | $\pi^{-1} \int_{-\infty}^{\infty} f(x) (x-y)^{-1} dx$                                                                                                |
|------|-----------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------|
| (10) | $(x^2 + a^2)^{-1} \quad \text{Re } a > 0$                                                                       | $-\frac{y}{a(y^2 + a^2)}$                                                                                                                            |
| (11) | $\frac{x}{x^2 + a^2} \quad \text{Re } a > 0$                                                                    | $\frac{a}{y^2 + a^2}$                                                                                                                                |
| (12) | $\frac{\lambda x + \mu a}{x^2 + a^2} \quad \text{Re } a > 0$                                                    | $\frac{\lambda a - \mu y}{y^2 + a^2}$                                                                                                                |
| (13) | $0 \quad -\infty < x < 0$<br>$\frac{cx + d}{(ax + b)^2} \quad 0 < x < \infty$<br>$a, b > 0$                     | $\frac{cy + d}{\pi(ay + b)^2} \log \left  \frac{b}{ay} \right $<br>$-\frac{ad - bc}{\pi ab(ay + b)}$<br>$y \neq 0, \quad y \neq -b/a$                |
| (14) | $(a-x)^{1/2} - (b-x)^{1/2} \quad -\infty < x < a$<br>$-(b-x)^{1/2} \quad a < x < b$<br>$0 \quad b < x < \infty$ | $0 \quad -\infty < y < a$<br>$(y-a)^{1/2} \quad a < y < b$<br>$(y-a)^{1/2} - (y-b)^{1/2} \quad b < y < \infty$                                       |
| (15) | $0 \quad -\infty < x < a$<br>$(x-a)^{1/2} \quad a < x < b$<br>$(x-a)^{1/2} - (x-b)^{1/2} \quad b < x < \infty$  | $(b-y)^{1/2} - (a-y)^{1/2} \quad -\infty < y < a$<br>$(b-y)^{1/2} \quad a < y < b$<br>$0 \quad b < y < \infty$                                       |
| (16) | $ a-x ^{1/2} -  b-x ^{1/2}$<br>$a > 0, \quad b > 0$                                                             | $(b-y)^{1/2} - (a-y)^{1/2} \quad -\infty < y < a$<br>$(b-y)^{1/2} + (y-a)^{1/2} \quad a < y < b$<br>$(y-a)^{1/2} - (y-b)^{1/2} \quad b < y < \infty$ |

$y$  is real, and the integral is a Cauchy Principal Value.

**Elementary functions (cont'd)**

|      | $f(x)$                                                                                                 | $\pi^{-1} \int_{-\infty}^{\infty} f(x)(x-y)^{-1} dx$                                                                                                                                                                                                                                                                                                                                                             |
|------|--------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (17) | $0 \quad -\infty < x < 0$<br>$(ax+b)^{-\frac{1}{2}} \quad 0 < x < \infty$<br>$a, b > 0$                | $2\pi^{-1}(-ay-b)^{-\frac{1}{2}} \times \tan^{-1}\{[-(ay+b)/b]^{\frac{1}{2}}\} \quad -\infty < y < -b/a$<br>$2\pi^{-1} b^{-1} \quad y = -b/a$<br>$\frac{1}{(ay+b)^{\frac{1}{2}}} \log \left  \frac{b^{\frac{1}{2}} + (ay+b)^{\frac{1}{2}}}{b^{\frac{1}{2}} - (ay+b)^{\frac{1}{2}}} \right  \quad -b/a < y < \infty$                                                                                              |
| (18) | $0 \quad -\infty < x < 0$<br>$(a^2 - x^2)^{\frac{1}{2}} \quad 0 < x < a$<br>$0 \quad a < x < \infty$   | $-\pi^{-1} a - \frac{1}{2}y - \pi^{-1}(y^2 - a^2)^{\frac{1}{2}} \times \cos^{-1}(-a/y) \quad -\infty < y < -a$<br>$-\pi^{-1} a - \frac{1}{2}y + \pi^{-1}(a^2 - y^2)^{\frac{1}{2}} \times \log \left  \frac{a + (a^2 - y^2)^{\frac{1}{2}}}{-y} \right  \quad -a < y < a$<br>$-\pi^{-1} a - \frac{1}{2}y + \pi^{-1}(y^2 - a^2)^{\frac{1}{2}} \times \cos^{-1}(-a/y) \quad a < y < \infty$<br>$0 < \cos^{-1} < \pi$ |
| (19) | $0 \quad -\infty < x < -a$<br>$(a^2 - x^2)^{\frac{1}{2}} \quad -a < x < a$<br>$0 \quad a < x < \infty$ | $-y - (y^2 - a^2)^{\frac{1}{2}} \quad -\infty < y < -a$<br>$-y \quad -a < y < a$<br>$-y + (y^2 - a^2)^{\frac{1}{2}} \quad a < y < \infty$                                                                                                                                                                                                                                                                        |

$y$  is real, and the integral is a Cauchy Principal Value.

## Elementary functions (cont'd)

|      | $f(x)$                                                                                                  | $\pi^{-1} \int_{-\infty}^{\infty} f(x)(x-y)^{-1} dx$                                                                                                                                                                                                                                                                                                                                   |
|------|---------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (20) | $0 \quad -\infty < x < 0$<br>$(a^2 - x^2)^{-\frac{1}{2}} \quad 0 < x < a$<br>$0 \quad a < x < \infty$   | $\frac{\cos^{-1}(-a/y)}{\pi(y^2 - a^2)^{\frac{1}{2}}} \quad -\infty < y < -a$<br>$\frac{1}{\pi(a^2 - y^2)^{\frac{1}{2}}} \log \left  \frac{a + (a^2 - y^2)^{\frac{1}{2}}}{-y} \right  \quad -a < y < a$<br>$-\frac{\cos^{-1}(-a/y)}{\pi(y^2 - a^2)^{\frac{1}{2}}} \quad a < y < \infty$<br>$0 < \cos^{-1} < \pi$                                                                       |
| (21) | $0 \quad -\infty < x < -a$<br>$(a^2 - x^2)^{-\frac{1}{2}} \quad -a < x < a$<br>$0 \quad a < x < \infty$ | $(y^2 - a^2)^{-\frac{1}{2}} \quad -\infty < y < -a$<br>$0 \quad -a < y < a$<br>$-(y^2 - a^2)^{-\frac{1}{2}} \quad a < y < \infty$                                                                                                                                                                                                                                                      |
| (22) | $0 \quad -\infty < x < a$<br>$(x^2 - a^2)^{-\frac{1}{2}} \quad a < x < \infty$<br>$a > 0$               | $\frac{1}{\pi(y^2 - a^2)^{\frac{1}{2}}} \log \left  \frac{-y + (y^2 - a^2)^{\frac{1}{2}}}{a} \right  \quad -\infty < y < -a$<br>$\frac{1}{\pi(a^2 - y^2)^{\frac{1}{2}}} \cos^{-1} \left( \frac{-y}{a} \right) \quad -a < y < a$<br>$\frac{1}{\pi(y^2 - a^2)^{\frac{1}{2}}} \log \left  \frac{-y + (y^2 - a^2)^{\frac{1}{2}}}{a} \right  \quad a < y < \infty$<br>$0 < \cos^{-1} < \pi$ |

$y$  is real, and the integral is a Cauchy Principal Value.

## Elementary functions (cont'd)

|      | $f(x)$                                                                                                            | $\pi^{-1} \int_{-\infty}^{\infty} f(x) (x-y)^{-1} dx$                                                                                                                                                                                                                                                                                                                                                                                |
|------|-------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (23) | $-(x^2 - a^2)^{-1/2} \quad -\infty < x < -a$<br>$0 \quad -a < x < a$<br>$(x^2 - a^2)^{-1/2} \quad a < x < \infty$ | $0 \quad -\infty < y < -a$<br>$(a^2 - y^2)^{-1/2} \quad -a < y < a$<br>$0 \quad a < y < \infty$                                                                                                                                                                                                                                                                                                                                      |
| (24) | $0 \quad -\infty < x < 0$<br>$(a-x)^{1/2} (a+x)^{-1/2} \quad 0 < x < a$<br>$0 \quad a < x < \infty$               | $-\frac{1}{2} + \frac{1}{\pi} \left  \frac{a-y}{a+y} \right ^{1/2} \cos^{-1} \left( -\frac{a}{y} \right) \quad -\infty < y < -a$<br>$-\frac{1}{2} + \frac{1}{\pi} \left( \frac{a-y}{a+y} \right)^{1/2} \log \left  \frac{a+(a^2-y^2)^{1/2}}{-y} \right  \quad -a < y < a$<br>$-\frac{1}{2} + \frac{1}{\pi} \left( \frac{y-a}{y+a} \right)^{1/2} \cos^{-1} \left( -\frac{a}{y} \right) \quad a < y < \infty$<br>$0 < \cos^{-1} < \pi$ |
| (25) | $0 \quad -\infty < x < -a$<br>$(a-x)^{1/2} (a+x)^{-1/2} \quad -a < x < a$<br>$0 \quad a < x < \infty$             | $-1 + (a-y)^{1/2}  y+a ^{-1/2} \quad -\infty < y < -a$<br>$-1 \quad -a < y < a$<br>$-1 + (y-a)^{1/2} (y+a)^{-1/2} \quad a < y < \infty$                                                                                                                                                                                                                                                                                              |

$y$  is real, and the integral is a Cauchy Principal Value.

## Elementary functions (cont'd)

|      | $f(x)$                                                                                                 | $\pi^{-1} \int_{-\infty}^{\infty} f(x) (x-y)^{-1} dx$                                                                                                                                                                                                                                                                                                                                                                             |
|------|--------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (26) | $0 \quad -\infty < x < 0$<br>$(a+x)^{1/2} (a-x)^{-1/2} \quad 0 < x < a$<br>$0 \quad a < x < \infty$    | $\frac{1}{2} - \frac{1}{\pi} \left  \frac{a+y}{a-y} \right ^{1/2} \cos^{-1} \left( -\frac{a}{y} \right) \quad -\infty < y < -a$<br>$\frac{1}{2} + \frac{1}{\pi} \left( \frac{a+y}{a-y} \right)^{1/2} \log \left  \frac{a+(a^2-y^2)^{1/2}}{-y} \right  \quad -a < y < a$<br>$\frac{1}{2} - \frac{1}{\pi} \left( \frac{y+a}{y-a} \right)^{1/2} \cos^{-1} \left( -\frac{a}{y} \right) \quad a < y < \infty$<br>$0 < \cos^{-1} < \pi$ |
| (27) | $0 \quad -\infty < x < -a$<br>$x(a-x)^{1/2} (a+x)^{-1/2} \quad -a < x < a$<br>$0 \quad a < x < \infty$ | $a - y + y \left  \frac{a-y}{a+y} \right ^{1/2} \quad -\infty < y < -a$<br>$a - y \quad -a < y < a$<br>$a - y + y \left( \frac{y-a}{y+a} \right)^{1/2} \quad a < y < \infty$                                                                                                                                                                                                                                                      |
| (28) | $0 \quad -\infty < x < 0$<br>$x^{\nu-1} \quad 0 < x < \infty$<br>$0 < \operatorname{Re} \nu < 1$       | $\csc(\nu\pi) (-y)^{\nu-1} \quad -\infty < y < 0$<br>$-\operatorname{ctn}(\nu\pi) y^{\nu-1} \quad 0 < y < \infty$                                                                                                                                                                                                                                                                                                                 |
| (29) | $ x ^{\nu-1} \quad 0 < \operatorname{Re} \nu < 1$                                                      | $-\operatorname{ctn}(\frac{1}{2}\nu\pi) \operatorname{sgn} y  y ^{\nu-1}$                                                                                                                                                                                                                                                                                                                                                         |
| (30) | $\operatorname{sgn} x  x ^{\nu-1} \quad 0 < \operatorname{Re} \nu < 1$                                 | $\tan(\frac{1}{2}\nu\pi)  y ^{\nu-1}$                                                                                                                                                                                                                                                                                                                                                                                             |

$y$  is real, and the integral is a Cauchy Principal Value.

## Elementary functions (cont'd)

|      | $f(x)$                                                                                                                                                                         | $\pi^{-1} \int_{-\infty}^{\infty} f(x) (x-y)^{-1} dx$                                                                                                                                                                                                                                                                                                                                                                                                                                         |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (31) | $0 \quad -\infty < x < a$<br>$(x-a)^{\nu} (b-x)^{-\nu} \quad a < x < b$<br>$0 \quad b < x < \infty$<br>$ \operatorname{Re} \nu  < 1$                                           | $\csc(\nu\pi) \left[ 1 - \left( \frac{a-y}{b-y} \right)^{\nu} \right] \quad -\infty < y < a$<br>$\csc(\nu\pi) \left[ 1 - \cos(\nu\pi) \left( \frac{y-a}{b-y} \right)^{\nu} \right] \quad a < y < b$<br>$\csc(\nu\pi) \left[ 1 - \left( \frac{y-a}{y-b} \right)^{\nu} \right] \quad b < y < \infty$                                                                                                                                                                                            |
| (32) | $0 \quad -\infty < x < a$<br>$(x-a)^{\nu-1} (b-x)^{-\nu} \quad a < x < b$<br>$0 \quad b < x < \infty$<br>$0 < \operatorname{Re} \nu < 1$                                       | $\frac{\csc(\nu\pi)}{b-y} \left  \frac{a-y}{b-y} \right ^{\nu-1} \quad -\infty < y < a \quad \text{or} \quad b < y < \infty$<br>$-(y-a)^{\nu-1} (b-y)^{-\nu} \operatorname{ctn}(\nu\pi) \quad a < y < b$                                                                                                                                                                                                                                                                                      |
| (33) | $0 \quad -\infty < x < a$<br>$(x-a)^{\rho-1} (b-x)^{\sigma-1} \quad a < x < b$<br>$0 \quad b < x < \infty$<br>$\operatorname{Re} \rho > 0, \quad \operatorname{Re} \sigma > 0$ | $\frac{\Gamma(\rho) \Gamma(\sigma) (b-a)^{\rho+\sigma-1}}{(b-y) \pi \Gamma(\rho+\sigma)}$<br>$\times {}_2F_1 \left( 1, \sigma; \rho+\sigma; \frac{b-a}{b-y} \right) \quad -\infty < y < a \quad \text{or} \quad b < y < \infty$<br>$(y-a)^{\rho-1} (b-y)^{\sigma-1} \operatorname{ctn}(\sigma\pi)$<br>$-\frac{\Gamma(\rho) \Gamma(\sigma-1)}{\pi \Gamma(\rho+\sigma-1)} (b-a)^{\rho+\sigma-2}$<br>$\times {}_2F_1 \left( 2-\rho-\sigma, 1; 2-\sigma; \frac{b-y}{b-a} \right) \quad a < y < b$ |

$y$  is real, and the integral is a Cauchy Principal Value.



## Elementary functions (cont'd)

|      | $f(x)$                                                                                                                                                                       | $\pi^{-1} \int_{-\infty}^{\infty} f(x) (x-y)^{-1} dx$                                                                                                                                                                                                                                                                                                                                                                                  |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (34) | $  \begin{aligned}  &0 & -\infty < x < 0 \\  &x^{\nu-1} (x+a)^{1-\mu} & 0 < x < \infty \\  &a > 0, \quad 0 < \operatorname{Re} \nu < \operatorname{Re} \mu  \end{aligned}  $ | $  \begin{aligned}  &\frac{\Gamma(\mu-\nu) \Gamma(\nu) (-y)^{\nu-1}}{\pi \Gamma(\mu) a^{\mu-1}} \\  &\times {}_2F_1(\mu-1, \nu; \mu; 1+y/a) \\  &\quad -\infty < y < 0 \\  &y^{\nu-1} (y+a)^{1-\mu} \operatorname{ctn}[(\mu-\nu)\pi] \\  &- \frac{\Gamma(\mu-\nu-1) \Gamma(\nu) a^{1-\mu+\nu}}{(y+a) \pi \Gamma(\mu-1)} \\  &\times {}_2F_1\left(2-\mu, 1; 2-\mu+\nu; \frac{a}{y+a}\right) \\  &\quad 0 < y < \infty  \end{aligned}  $ |
| (35) | $\exp(-a x ) \quad a > 0$                                                                                                                                                    | $  \begin{aligned}  &\pi^{-1} \operatorname{sgn} y [\exp(a y ) \operatorname{Ei}(-a y ) \\  &\quad - \exp(-a y ) \overline{\operatorname{Ei}}(a y )]  \end{aligned}  $                                                                                                                                                                                                                                                                 |
| (36) | $\operatorname{sgn} x \exp(-a x ) \quad a > 0$                                                                                                                               | $  \begin{aligned}  &-\pi^{-1} [\exp(a y ) \operatorname{Ei}(-a y ) \\  &\quad + \exp(-a y ) \overline{\operatorname{Ei}}(a y )]  \end{aligned}  $                                                                                                                                                                                                                                                                                     |
| (37) | $  \begin{aligned}  &0 & -\infty < x < a \\  &e^{-bx} & a < x < \infty \\  && b > 0  \end{aligned}  $                                                                        | $  \begin{aligned}  &-\pi^{-1} e^{-by} \operatorname{Ei}(by-ab) & -\infty < y < a \\  &-\pi^{-1} e^{-by} \overline{\operatorname{Ei}}(by-ab) & a < y < \infty  \end{aligned}  $                                                                                                                                                                                                                                                        |
| (38) | $e^{iax} \quad a > 0$                                                                                                                                                        | $ie^{iay}$                                                                                                                                                                                                                                                                                                                                                                                                                             |

$y$  is real, and the integral is a Cauchy Principal Value.

## Elementary functions (cont'd)

|      | $f(x)$                                                                         | $\pi^{-1} \int_{-\infty}^{\infty} f(x) (x-y)^{-1} dx$                                                                                                                                                                                                                                               |
|------|--------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (39) | $0 \quad -\infty < x < 0$<br>$\exp(-ax^{1/2}) \quad 0 < x < \infty$<br>$a > 0$ | $2\pi^{-1} \cos(a y ^{1/2}) \operatorname{ci}(a y ^{1/2})$<br>$-2\pi^{-1} \sin(a y ^{1/2}) \operatorname{si}(a y ^{1/2})$<br>$-\infty < y < 0$<br>$-\pi^{-1} \exp(ay^{1/2}) \operatorname{Ei}(-ay^{1/2})$<br>$-\pi^{-1} \exp(-ay^{1/2}) \overline{\operatorname{Ei}}(ay^{1/2})$<br>$0 < y < \infty$ |
| (40) | $\log \left  \frac{b-x}{x-a} \right  \quad a < b$                              | $0 \quad -\infty < y < a$<br>$-\pi \quad a < y < b$<br>$0 \quad b < y < \infty$                                                                                                                                                                                                                     |
| (41) | $\frac{1}{x} \log \left  \frac{1+ax}{1-bx} \right  \quad a > 0, \quad b > 0$   | $-\pi y^{-1} \quad -\infty < y < -a^{-1}$<br>$0 \quad -a^{-1} < y < b^{-1}$<br>$-\pi y^{-1} \quad -b^{-1} < y < \infty$                                                                                                                                                                             |
| (42) | $\log \left  \frac{x^2 - a^2}{x^2 - b^2} \right  \quad 0 < a < b$              | $-\pi \quad -b < y < -a$<br>$\pi \quad a < y < b$<br>$0 \quad \text{elsewhere}$                                                                                                                                                                                                                     |
| (43) | $\sin(ax) \quad a > 0$                                                         | $\cos(ay)$                                                                                                                                                                                                                                                                                          |
| (44) | $\frac{\sin(ax)}{x} \quad a > 0$                                               | $\frac{\cos(ay) - 1}{y}$                                                                                                                                                                                                                                                                            |
| (45) | $0 \quad -\infty < x < 0$<br>$\sin(ax^{1/2}) \quad 0 < x < \infty$<br>$a > 0$  | $\exp(-a y ^{1/2}) \quad -\infty < y < 0$<br>$\cos(ay^{1/2}) \quad 0 < y < \infty$                                                                                                                                                                                                                  |

$y$  is real, and the integral is a Cauchy Principal Value.

## Elementary functions (cont'd)

|      | $f(x)$                                                      | $\pi^{-1} \int_{-\infty}^{\infty} f(x) (x-y)^{-1} dx$  |
|------|-------------------------------------------------------------|--------------------------------------------------------|
| (46) | $\operatorname{sgn} x \sin(a x ^{\frac{1}{2}}) \quad a > 0$ | $\cos(a y ^{\frac{1}{2}}) + \exp(-a y ^{\frac{1}{2}})$ |
| (47) | $\cos(ax) \quad a > 0$                                      | $-\sin(ay)$                                            |
| (48) | $\frac{1 - \cos(ax)}{x} \quad a > 0$                        | $\frac{\sin(ay)}{y}$                                   |

## 15.3. Higher transcendental functions

|     |                                                                                                                           |                                                                                                          |
|-----|---------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------|
| (1) | $e^{-ax} \operatorname{Ei}(ax) \quad -\infty < x < 0$<br>$e^{-ax} \overline{\operatorname{Ei}}(ax) \quad 0 < x < \infty$  | $0 \quad -\infty < y < 0$<br>$\pi e^{-ay} \quad 0 < y < \infty$                                          |
| (2) | $\operatorname{ci}(a x ) \quad a > 0$                                                                                     | $\operatorname{sgn} y \operatorname{si}(a y )$                                                           |
| (3) | $\operatorname{sgn} x \operatorname{si}(a x ) \quad a > 0$                                                                | $\operatorname{Ci}(a y )$                                                                                |
| (4) | $\cos(ax) \operatorname{ci}(a x )$<br>$-\sin(a x ) \operatorname{si}(a x ) \quad a > 0$                                   | $\operatorname{sgn} y \cos(ay) \operatorname{si}(a y )$<br>$+\sin(ay) \operatorname{ci}(a y )$           |
| (5) | $\sin(ax) \operatorname{ci}(a x )$<br>$+ \operatorname{sgn} x \cos(ax) \operatorname{si}(a x ) \quad a > 0$               | $\sin(a y ) \operatorname{si}(a y ) - \cos(ay) \operatorname{ci}(a y )$                                  |
| (6) | $0 \quad -\infty < x < -1, \quad 1 < x < \infty$<br>$P_n(x) \quad -1 < x < 1$<br>$n = 0, 1, 2, \dots$                     | $-2\pi^{-1} Q_n(y) \quad -\infty < y < -1, \quad 1 < y < \infty$<br>$-2\pi^{-1} Q_n(y) \quad -1 < y < 1$ |
| (7) | $0 \quad -\infty < x < -1, \quad 1 < x < \infty$<br>$(1-x^2)^{-\frac{1}{2}} T_n(x) \quad -1 < x < 1$<br>$n = 1, 2, \dots$ | $U_{n-1}(y) \quad -1 < y < 1$                                                                            |

$y$  is real, and the integral is a Cauchy Principal Value.

**Higher transcendental functions (cont'd)**

|      | $f(x)$                                                                                                                                                                                             | $\pi^{-1} \int_{-\infty}^{\infty} f(x) (x-y)^{-1} dx$                                                                                                                                                |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (8)  | $0 \quad -\infty < x < -1, \quad 1 < x < \infty$<br>$(1-x^2)^{\frac{1}{2}} U_n(x) \quad -1 < x < 1$<br>$n = 0, 1, 2, \dots$                                                                        | $-T_{n+1}(y) \quad -1 < y < 1$                                                                                                                                                                       |
| (9)  | $0 \quad -\infty < x < -1, \quad 1 < x < \infty$<br>$(1-x)^{\alpha} (1+x)^{\beta} P_n^{(\alpha, \beta)}(x)$<br>$-1 < x < 1$<br>$\operatorname{Re} \alpha > -1, \quad \operatorname{Re} \beta > -1$ | $-2\pi^{-1} (y-1)^{\alpha} (y+1)^{\beta} Q_n^{(\alpha, \beta)}(y)$<br>$-\infty < y < -1, \quad 1 < y < \infty$<br>$-2\pi^{-1} (1-y)^{\alpha} (1+y)^{\beta} Q_n^{(\alpha, \beta)}(y)$<br>$-1 < y < 1$ |
| (10) | $0 \quad -\infty < x < 0$<br>$J_{\nu}(ax) \quad 0 < x < \infty$<br>$a > 0, \quad \operatorname{Re} \nu > -1$                                                                                       | $\csc(\nu\pi) [\mathbf{J}_{\nu}(-ay) - J_{\nu}(-ay)]$<br>$-\infty < y < 0$<br>$\csc(\nu\pi) [\mathbf{J}_{\nu}(-ay) - \cos(\nu\pi) J_{\nu}(ay)]$<br>$0 < y < \infty$                                  |
| (11) | $-J_{-\nu}(-ax) \quad -\infty < x < 0$<br>$J_{\nu}(ax) \quad 0 < x < \infty$<br>$a > 0, \quad -1 < \operatorname{Re} \nu < 1$                                                                      | $-Y_{-\nu}(-ay) \quad -\infty < y < 0$<br>$-Y_{\nu}(ay) \quad 0 < y < \infty$                                                                                                                        |
| (12) | $0 \quad -\infty < x < 0$<br>$x^{\nu} J_{\nu}(ax) \quad 0 < x < \infty$<br>$a > 0, \quad -1/2 < \operatorname{Re} \nu < 3/2$                                                                       | $\frac{1}{2}  y ^{\nu} [\tan(\nu\pi) \operatorname{sgn} y J_{\nu}(a y )$<br>$- Y_{\nu}(a y ) - \sec(\nu\pi) \operatorname{sgn} y \mathbf{H}_{-\nu}(a y )]$                                           |
| (13) | $ x ^{\nu} J_{\nu}(a x )$<br>$a > 0, \quad -1/2 < \operatorname{Re} \nu < 3/2$                                                                                                                     | $\operatorname{sgn} y  y ^{\nu} [\tan(\nu\pi) J_{\nu}(a y )$<br>$- \sec(\nu\pi) \mathbf{H}_{-\nu}(a y )]$                                                                                            |
| (14) | $\operatorname{sgn} x  x ^{\nu} J_{\nu}(a x )$<br>$a > 0, \quad -1/2 < \operatorname{Re} \nu < 3/2$                                                                                                | $- y ^{\nu} Y_{\nu}(a y )$                                                                                                                                                                           |

$y$  is real, and the integral is a Cauchy Principal Value.

**Higher transcendental functions (cont'd)**

|      | $f(x)$                                                                                                                                                   | $\pi^{-1} \int_{-\infty}^{\infty} f(x) (x-y)^{-1} dx$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (15) | $ x ^{-\nu} J_{\nu}(a x )$<br>$a > 0, \quad \operatorname{Re} \nu > -3/2$                                                                                | $-\operatorname{sgn} y  y ^{-\nu} \mathbf{H}_{\nu}(a y )$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               |
| (16) | $0 \quad -\infty < x < 0$<br>$x^{\lambda} J_{\nu}(ax) \quad 0 < x < \infty$<br>$a > 0$<br>$-1 - \operatorname{Re} \nu < \operatorname{Re} \lambda < 3/2$ | $\frac{2^{\lambda-1} \Gamma(\frac{1}{2}\lambda + \frac{1}{2}\nu)}{\pi a^{\lambda} \Gamma(1 - \frac{1}{2}\lambda + \frac{1}{2}\nu)}$ $\times {}_1F_2\left(1; 1 - \frac{\lambda + \nu}{2}, 1 - \frac{\lambda - \nu}{2}; -\frac{a^2 y^2}{4}\right)$ $+ \frac{2^{\lambda-2} y \Gamma(\frac{1}{2}\lambda + \frac{1}{2}\nu - \frac{1}{2})}{\pi a^{\lambda-1} \Gamma(3/2 - \frac{1}{2}\lambda + \frac{1}{2}\nu)}$ $\times {}_1F_2\left(1; \frac{3 - \lambda - \nu}{2}, \frac{3 - \lambda + \nu}{2}; -\frac{a^2 y^2}{4}\right)$ $- h(y)  y ^{\lambda} J_{\nu}(a y )$ $h(y) = \begin{cases} \csc [(\lambda + \nu)\pi] & -\infty < y < 0 \\ \operatorname{ctn} [(\lambda + \nu)\pi] & 0 < y < \infty \end{cases}$ |
| (17) | $\sin(ax) J_1(ax) \quad a > 0$                                                                                                                           | $\cos(ay) J_1(ay)$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      |
| (18) | $\sin(ax) J_n(by)$<br>$0 < b < a, \quad n = 0, 1, 2, \dots$                                                                                              | $\cos(ay) J_n(by)$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      |
| (19) | $\cos(ax) J_1(ax) \quad a > 0$                                                                                                                           | $-\sin(ay) J_1(ay)$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     |
| (20) | $\cos(ax) J_n(bx)$<br>$0 < b < a, \quad n = 0, 1, 2, \dots$                                                                                              | $-\sin(ay) J_n(by)$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     |

$y$  is real, and the integral is a Cauchy Principal Value.

**Higher transcendental functions (cont'd)**

|      | $f(x)$                                                                                                                                              | $\pi^{-1} \int_{-\infty}^{\infty} f(x) (x-y)^{-1} dx$                                                                                                        |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (21) | $\operatorname{sgn} x  x ^{\nu} \sin(a x  - \pi\nu)$<br>$\times J_{\nu}(a x )$<br>$a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$ | $ y ^{\nu} \cos(a y  - \pi\nu) J_{\nu}(a y )$                                                                                                                |
| (22) | $ x ^{\nu} \cos(a x  - \pi\nu) J_{\nu}(a x )$<br>$a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                                  | $-\operatorname{sgn} y  y ^{\nu} \sin(a y  - \pi\nu) J_{\nu}(a y )$                                                                                          |
| (23) | $ x ^{-\nu} \sin(ax) J_{\nu}(a x )$<br>$a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$                                                          | $ y ^{-\nu} \cos(ay) J_{\nu}(a y )$                                                                                                                          |
| (24) | $ x ^{-\nu} \cos(ax) J_{\nu}(a x )$<br>$a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$                                                          | $- y ^{-\nu} \sin(ay) J_{\nu}(a y )$                                                                                                                         |
| (25) | $ x ^{\frac{1}{2}} J_{\nu-\frac{1}{4}}(a x ) J_{-\nu-\frac{1}{4}}(a x )$<br>$a > 0$                                                                 | $-\operatorname{sgn} y  y ^{\frac{1}{2}} J_{\frac{1}{4}+\nu}(a y ) J_{\frac{1}{4}-\nu}(a y )$                                                                |
| (26) | $\operatorname{sgn} x  x ^{\frac{1}{2}} J_{\frac{1}{4}+\nu}(a x )$<br>$\times J_{\frac{1}{4}-\nu}(a x )$<br>$a > 0$                                 | $ y ^{\frac{1}{2}} J_{\nu-\frac{1}{4}}(a y ) J_{-\nu-\frac{1}{4}}(a y )$                                                                                     |
| (27) | $0 \quad -\infty < x < 0$<br>$x^{\frac{1}{2}\nu} J_{\nu}(ax^{\frac{1}{2}}) \quad 0 < x < \infty$<br>$a > 0, \quad -1 < \operatorname{Re} \nu < 3/2$ | $2\pi^{-1} (-y)^{\frac{1}{2}\nu} K_{\nu}[a(-y)^{\frac{1}{2}}] \quad -\infty < y < 0$<br>$-y^{\frac{1}{2}\nu} Y_{\nu}(ay^{\frac{1}{2}}) \quad 0 < y < \infty$ |
| (28) | $ x ^{\frac{1}{2}\nu} J_{\nu}(a x ^{\frac{1}{2}})$<br>$a > 0, \quad -1 < \operatorname{Re} \nu < 3/2$                                               | $-\operatorname{sgn} y  y ^{\frac{1}{2}\nu} [2\pi^{-1} K_{\nu}(a y ^{\frac{1}{2}})$<br>$+ Y_{\nu}(a y ^{\frac{1}{2}})]$                                      |
| (29) | $\operatorname{sgn} x  x ^{\frac{1}{2}\nu} J_{\nu}(a x ^{\frac{1}{2}})$<br>$a > 0, \quad -1 < \operatorname{Re} \nu < 3/2$                          | $2\pi^{-1}  y ^{\frac{1}{2}\nu} K_{\nu}(a y ^{\frac{1}{2}})$<br>$-  y ^{\frac{1}{2}\nu} Y_{\nu}(a y ^{\frac{1}{2}})$                                         |

$y$  is real, and the integral is a Cauchy Principal Value.

**Higher transcendental functions (cont'd)**

|      | $f(x)$                                                                                                                                                     | $\pi^{-1} \int_{-\infty}^{\infty} f(x) (x-y)^{-1} dx$                                                                                                                                                                                                                                                                  |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (30) | $0 \quad -\infty < x < 0$<br>$x^{\frac{1}{2}\nu-\frac{1}{2}} J_{\nu}(ax^{\frac{1}{2}}) \quad 0 < x < \infty$<br>$a > 0, \quad -1/2 < \text{Re } \nu < 5/2$ | $ y ^{\frac{1}{2}\nu-\frac{1}{2}} \sec(\nu\pi) [I_{\nu}(a y ^{\frac{1}{2}})$<br>$\quad - \mathbf{L}_{-\nu}(a y ^{\frac{1}{2}})] \quad -\infty < y < 0$<br>$y^{\frac{1}{2}\nu-\frac{1}{2}} [\tan(\nu\pi) J_{\nu}(ay^{\frac{1}{2}})$<br>$\quad - \sec(\nu\pi) \mathbf{H}_{-\nu}(ay^{\frac{1}{2}})] \quad 0 < y < \infty$ |
| (31) | $ x ^{\frac{1}{2}\nu-\frac{1}{2}} J_{\nu}(a x ^{\frac{1}{2}})$<br>$a > 0, \quad -1/2 < \text{Re } \nu < 5/2$                                               | $\text{sgn } y  y ^{\frac{1}{2}\nu-\frac{1}{2}} \{\tan(\nu\pi) J_{\nu}(a y ^{\frac{1}{2}})$<br>$\quad + \sec(\nu\pi) [\mathbf{L}_{-\nu}(a y ^{\frac{1}{2}}) - \mathbf{H}_{-\nu}(a y ^{\frac{1}{2}})]$<br>$\quad - I_{\nu}(a y ^{\frac{1}{2}})\}$                                                                       |
| (32) | $\text{sgn } x  x ^{\frac{1}{2}\nu-\frac{1}{2}} J_{\nu}(a x ^{\frac{1}{2}})$<br>$a > 0, \quad -1/2 < \text{Re } \nu < 5/2$                                 | $ y ^{\frac{1}{2}\nu-\frac{1}{2}} \{\tan(\nu\pi) J_{\nu}(a y ^{\frac{1}{2}})$<br>$\quad + \sec(\nu\pi) [I_{\nu}(a y ^{\frac{1}{2}})$<br>$\quad - \mathbf{L}_{-\nu}(a y ^{\frac{1}{2}}) - \mathbf{H}_{-\nu}(a y ^{\frac{1}{2}})]\}$                                                                                     |
| (33) | $0 \quad -\infty < x < 0$<br>$x^{-\frac{1}{2}\nu-\frac{1}{2}} J_{\nu}(ax^{\frac{1}{2}}) \quad 0 < x < \infty$<br>$a > 0, \quad \text{Re } \nu > -5/2$      | $ y ^{-\frac{1}{2}\nu-\frac{1}{2}} [I_{\nu}(a y ^{\frac{1}{2}}) - \mathbf{L}_{\nu}(a y ^{\frac{1}{2}})]$<br>$\quad -\infty < y < 0$<br>$-y^{-\frac{1}{2}\nu-\frac{1}{2}} \mathbf{H}_{\nu}(ay^{\frac{1}{2}}) \quad 0 < y < \infty$                                                                                      |
| (34) | $ x ^{-\frac{1}{2}\nu-\frac{1}{2}} J_{\nu}(a x ^{\frac{1}{2}})$<br>$a > 0, \quad \text{Re } \nu > -5/2$                                                    | $\text{sgn } y  y ^{-\frac{1}{2}\nu-\frac{1}{2}} [\mathbf{L}_{\nu}(a y ^{\frac{1}{2}})$<br>$\quad - \mathbf{H}_{\nu}(a y ^{\frac{1}{2}}) - I_{\nu}(a y ^{\frac{1}{2}})]$                                                                                                                                               |
| (35) | $\text{sgn } x  x ^{-\frac{1}{2}\nu-\frac{1}{2}} J_{\nu}(a x ^{\frac{1}{2}})$<br>$a > 0, \quad \text{Re } \nu > -5/2$                                      | $ y ^{-\frac{1}{2}\nu-\frac{1}{2}} [I_{\nu}(a y ^{\frac{1}{2}})$<br>$\quad - \mathbf{L}_{\nu}(a y ^{\frac{1}{2}}) - \mathbf{H}_{\nu}(a y ^{\frac{1}{2}})]$                                                                                                                                                             |

$y$  is real, and the integral is a Cauchy Principal Value.

## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                                | $\pi^{-1} \int_{-\infty}^{\infty} f(x) (x-y)^{-1} dx$                                                                                                                                                                                                                                                                                                                                                                                                            |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (36) | $0 \quad -\infty < x < 0$<br>$x^{\lambda} J_{\nu}(ax^{\frac{1}{2}}) \quad 0 < x < \infty$<br>$a > 0$<br>$-1 - \frac{1}{2} \operatorname{Re} \nu < \operatorname{Re} \lambda < \frac{3}{4}$                                                            | $\frac{2^{2\lambda} \Gamma(\lambda + \frac{1}{2} \nu)}{\pi a^{2\lambda} \Gamma(1 - \lambda + \frac{1}{2} \nu)}$ $\times {}_1F_2\left(1; 1 - \lambda - \frac{\nu}{2}, 1 - \lambda + \frac{\nu}{2}; -\frac{a^2 y}{4}\right) - h(y)$ $h(y) =  y ^{\lambda} \csc[(\lambda + \frac{1}{2} \nu)\pi] I_{\nu}(a y ^{\frac{1}{2}})$ $-\infty < y < 0$ $h(y) = y^{\lambda} \operatorname{ctn}[(\lambda + \frac{1}{2} \nu)\pi] J_{\nu}(a y ^{\frac{1}{2}})$ $0 < y < \infty$ |
| (37) | $0 \quad -a < x < a$<br>$\operatorname{sgn} x (x^2 - a^2)^{\frac{1}{2} \nu}$<br>$\times J_{\nu}[b(x^2 - a^2)^{\frac{1}{2}}]$<br>$-\infty < x < -a \quad \text{or} \quad a < x < \infty$<br>$a > 0, \quad b > 0$<br>$-1 < \operatorname{Re} \nu < 3/2$ | $2\pi^{-1} (a^2 - y^2)^{\frac{1}{2} \nu} K_{\nu}[b(a^2 - y^2)^{\frac{1}{2}}]$ $-a < y < a$ $-(y^2 - a^2)^{\frac{1}{2} \nu} Y_{\nu}[b(y^2 - a^2)^{\frac{1}{2}}]$ $-\infty < y < -a \quad \text{or} \quad a < y < \infty$                                                                                                                                                                                                                                          |
| (38) | $Y_{-\nu}(-ax) \quad -\infty < x < 0$<br>$Y_{\nu}(ax) \quad 0 < x < \infty$<br>$a > 0, \quad -1 < \operatorname{Re} \nu < 1$                                                                                                                          | $-J_{-\nu}(-ay) \quad -\infty < y < 0$ $J_{\nu}(ay) \quad 0 < y < \infty$                                                                                                                                                                                                                                                                                                                                                                                        |
| (39) | $\sin(\frac{1}{2} \nu \pi) J_{\nu}(a x )$<br>$+ \cos(\frac{1}{2} \nu \pi) Y_{\nu}(a x )$<br>$a > 0, \quad -1 < \operatorname{Re} \nu < 1$                                                                                                             | $\operatorname{sgn} y [\cos(\frac{1}{2} \nu \pi) J_{\nu}(a y )$ $- \sin(\frac{1}{2} \nu \pi) Y_{\nu}(a y )]$                                                                                                                                                                                                                                                                                                                                                     |
| (40) | $\operatorname{sgn} x [\sin(\frac{1}{2} \nu \pi) Y_{\nu}(a x )$<br>$- \cos(\frac{1}{2} \nu \pi) J_{\nu}(a x )]$<br>$a > 0, \quad -1 < \operatorname{Re} \nu < 1$                                                                                      | $\cos(\frac{1}{2} \nu \pi) Y_{\nu}(a y )$ $+ \sin(\frac{1}{2} \nu \pi) J_{\nu}(a y )$                                                                                                                                                                                                                                                                                                                                                                            |

$y$  is real, and the integral is a Cauchy Principal Value.



**Higher transcendental functions (cont'd)**

|      | $f(x)$                                                                                                                                                                                                                               | $\pi^{-1} \oint_{-\infty}^{\infty} f(x) (x-y)^{-1} dx$                                                                                                                  |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (41) | $ x ^{\nu} Y_{\nu}(a x )$<br>$a > 0, \quad -1/2 < \operatorname{Re} \nu < 3/2$                                                                                                                                                       | $\operatorname{sgn} y  y ^{\nu} J_{\nu}(a y )$                                                                                                                          |
| (42) | $\operatorname{sgn} x  x ^{-\mu} \{\sin [\frac{1}{2}(\mu+\nu)\pi]$<br>$\times Y_{\nu}(a x )$<br>$- \cos [\frac{1}{2}(\mu+\nu)\pi] J_{\nu}(a x )\}$<br>$a > 0$<br>$-1/2 < \operatorname{Re} \mu < 1 -  \operatorname{Re} \nu $        | $ y ^{-\mu} \{\cos [\frac{1}{2}(\mu+\nu)\pi] Y_{\nu}(a y )$<br>$+ \sin [\frac{1}{2}(\mu+\nu)\pi] J_{\nu}(a y )\}$                                                       |
| (43) | $ x ^{-\mu} \{\sin [\frac{1}{2}(\mu+\nu)\pi] J_{\nu}(a x )$<br>$+ \cos [\frac{1}{2}(\mu+\nu)\pi] Y_{\nu}(a x )\}$<br>$a > 0$<br>$-3/2 < \operatorname{Re} \mu < 1 -  \operatorname{Re} \nu $                                         | $\operatorname{sgn} y  y ^{-\mu} \{\cos [\frac{1}{2}(\mu+\nu)\pi] J_{\nu}(a y )$<br>$- \sin [\frac{1}{2}(\mu+\nu)\pi] Y_{\nu}(a y )\}$                                  |
| (44) | $\operatorname{sgn} x  x ^{-\mu} \{\cos [a x $<br>$- \frac{1}{2}(\mu+\nu)\pi] J_{\nu}(b x )$<br>$+ \sin [a x  - \frac{1}{2}(\mu+\nu)\pi] Y_{\nu}(b x )\}$<br>$a < b$<br>$-3/2 < \operatorname{Re} \mu < 1 -  \operatorname{Re} \nu $ | $ y ^{-\mu} \{\sin [a y  - \frac{1}{2}(\mu+\nu)\pi] J_{\nu}(b y )$<br>$- \cos [a y  - \frac{1}{2}(\mu+\nu)\pi] Y_{\nu}(b y )\}$                                         |
| (45) | $ x ^{-\mu} \{\cos [a x $<br>$- \frac{1}{2}(\mu+\nu)\pi] Y_{\nu}(b x )$<br>$- \sin [a x  - \frac{1}{2}(\mu+\nu)\pi] J_{\nu}(b x )\}$<br>$a < b$<br>$-3/2 < \operatorname{Re} \mu < 1 -  \operatorname{Re} \nu $                      | $\operatorname{sgn} y  y ^{-\mu} \{\sin [a y  - \frac{1}{2}(\mu+\nu)\pi]$<br>$\times Y_{\nu}(b y ) + \cos [a y  - \frac{1}{2}(\mu+\nu)\pi]$<br>$\times J_{\nu}(b y )\}$ |

$y$  is real, and the integral is a Cauchy Principal Value.

## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                                                                                                      | $\pi^{-1} \int_{-\infty}^{\infty} f(x) (x-y)^{-1} dx$                                                                                                                                                                                                                                                                                                                                 |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (46) | $0 \quad -\infty < x < 0$<br>$x^{\mu} \{ \cos[(\mu - \frac{1}{2}\nu)\pi] J_{\nu}(ax^{\frac{1}{2}})$<br>$+ \sin[(\mu - \frac{1}{2}\nu)\pi] Y_{\nu}(ax^{\frac{1}{2}}) \}$<br>$0 < x < \infty$<br>$a > 0, \quad  \operatorname{Re} \nu  - 1 < \operatorname{Re} \mu < \frac{1}{2}$                                             | $2\pi^{-1}  y ^{\mu} K_{\nu}(a y ^{\frac{1}{2}}) \quad -\infty < y < 0$<br>$y^{\mu} \{ \sin[(\mu - \frac{1}{2}\nu)\pi] J_{\nu}(ay^{\frac{1}{2}})$<br>$- \cos[(\mu - \frac{1}{2}\nu)\pi] Y_{\nu}(ay^{\frac{1}{2}}) \}$<br>$0 < y < \infty$                                                                                                                                             |
| (47) | $0 \quad -\infty < x < 0$<br>$e^{-ax} I_0(ax) \quad 0 < x < \infty$<br>$a > 0$                                                                                                                                                                                                                                              | $\pi^{-1} e^{-ay} K_0(a y )$                                                                                                                                                                                                                                                                                                                                                          |
| (48) | $\exp(-a x ) I_0(ax) \quad a > 0$                                                                                                                                                                                                                                                                                           | $-2\pi^{-1} \sinh(ay) K_0(a y )$                                                                                                                                                                                                                                                                                                                                                      |
| (49) | $\operatorname{sgn} x \exp(-a x ) I_0(ax) \quad a > 0$                                                                                                                                                                                                                                                                      | $2\pi^{-1} \cosh(ay) K_0(a y )$                                                                                                                                                                                                                                                                                                                                                       |
| (50) | $0 \quad -\infty < x < -a$<br>$(a^2 - x^2)^{\frac{1}{2}\nu} e^{-bx} J_{\nu}[b(a^2 - x^2)^{\frac{1}{2}}]$<br>$-a < x < a$<br>$2(x^2 - a^2)^{\frac{1}{2}\nu} \cos(\nu\pi) e^{-bx}$<br>$\times I_{\nu}[b(x^2 - a^2)^{\frac{1}{2}}] \quad a < x < \infty$<br>$a > 0, \quad b > 0$<br>$-1 < \operatorname{Re} \nu < \frac{1}{2}$ | $2\pi^{-1} (y^2 - a^2)^{\frac{1}{2}\nu} e^{-by} K_{\nu}[b(y^2 - a^2)^{\frac{1}{2}}] \quad -\infty < y < -a$<br>$-(a^2 - y^2)^{\frac{1}{2}\nu} e^{-by} Y_{\nu}[b(a^2 - y^2)^{\frac{1}{2}}] \quad -a < y < a$<br>$2(y^2 - a^2)^{\frac{1}{2}\nu} e^{-by} \{ \pi^{-1} K_{\nu}[b(y^2 - a^2)^{\frac{1}{2}}]$<br>$+ \sin(\nu\pi) I_{\nu}[b(y^2 - a^2)^{\frac{1}{2}}] \}$<br>$a < y < \infty$ |
| (51) | $e^{ax} K_0(a x ) \quad a > 0$                                                                                                                                                                                                                                                                                              | $\pi e^{ay} I_0(ay) \quad -\infty < y < 0$<br>$0 \quad 0 < y < \infty$                                                                                                                                                                                                                                                                                                                |

$y$  is real, and the integral is a Cauchy Principal Value.

**Higher transcendental functions (cont'd)**

|      | $f(x)$                                                                                                                                               | $\pi^{-1} \int_{-\infty}^{\infty} f(x) (x-y)^{-1} dx$                                                                                                                                       |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (52) | $\sinh(ax) K_0(a x ) \quad a > 0$                                                                                                                    | $\frac{1}{2} \pi \exp(-a y ) I_0(ay)$                                                                                                                                                       |
| (53) | $\cosh(ax) K_0(a x ) \quad a > 0$                                                                                                                    | $-\frac{1}{2} \pi \operatorname{sgn} y \exp(-a y ) I_0(ay)$                                                                                                                                 |
| (54) | $ x ^{-\nu} e^{ax} K_{\nu}(a x )$<br>$a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                                               | $\frac{1}{2} \pi \sec(\nu \pi)  y ^{-\nu} e^{ay}$<br>$\times [I_{\nu}(a y ) + I_{-\nu}(a y )]$<br>$-\infty < y < 0$<br>$-\pi \tan(\nu \pi) y^{-\nu} e^{ay} K_{\nu}(ay)$<br>$0 < y < \infty$ |
| (55) | $ x ^{-\nu} \sinh(ax) K_{\nu}(a x )$<br>$a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < 1$                                                      | $ y ^{-\nu} [\frac{1}{2} \pi \sec(\nu \pi) \exp(-a y ) I_{\nu}(a y )$<br>$-\tan(\nu \pi) \sinh(a y ) K_{\nu}(a y )]$                                                                        |
| (56) | $ x ^{-\nu} \cosh(ax) K_{\nu}(a x )$<br>$a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                                            | $- y ^{-\nu} \operatorname{sgn} y [\frac{1}{2} \pi \sec(\nu \pi) \exp(-a y )$<br>$\times I_{\nu}(a y ) + \tan(\nu \pi) \cosh(ay) K_{\nu}(a y )]$                                            |
| (57) | $ x ^{2\nu} \exp(-ax^2) [K_{\nu}(ax^2)$<br>$+ \pi \sin(\nu \pi) I_{\nu}(ax^2)]$<br>$a > 0, \quad -\frac{1}{4} < \operatorname{Re} \nu < \frac{1}{4}$ | $-\pi \cos(\nu \pi) \operatorname{sgn} y  y ^{2\nu}$<br>$\times \exp(-ay^2) I_{\nu}(ay^2)$                                                                                                  |
| (58) | $\operatorname{sgn} x  x ^{2\nu} \exp(-ax^2) I_{\nu}(ax^2)$<br>$a > 0, \quad -\frac{1}{4} < \operatorname{Re} \nu < \frac{1}{2}$                     | $ y ^{2\nu} \exp(-ay^2)$<br>$\times [\pi^{-1} \sec(\nu \pi) K_{\nu}(ay^2)$<br>$+ \tan(\nu \pi) I_{\nu}(ay^2)]$                                                                              |
| (59) | $\operatorname{sgn} x  x ^{-\nu} \mathbf{H}_{\nu}(a x )$<br>$a > 0, \quad \operatorname{Re} \nu > -3/2$                                              | $ y ^{-\nu} J_{\nu}(a y )$                                                                                                                                                                  |

$y$  is real, and the integral is a Cauchy Principal Value.

## Higher transcendental functions (cont'd)

|      | $f(x)$                                                                                                                                                                                                                                                                                                                                             | $\pi^{-1} \int_{-\infty}^{\infty} f(x) (x-y)^{-1} dx$                                                                                                                                                                                                                                                                                                                              |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (60) | $0 \quad -\infty < x < 0$<br>$G_{pq}^{mn} \left( ax \left  \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right)$<br>$0 < x < \infty$<br>$p+q < 2(m+n)$<br>$ \arg a  < (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi$<br>$\operatorname{Re} a_j < 1 \quad j = 1, \dots, n$<br>$\operatorname{Re} b_j > -1 \quad j = 1, \dots, m$ | $\pi^{-1} G_{p+1, q+1}^{m+1, n+1} \left( a y  \left  \begin{matrix} 0, a_1, \dots, a_p \\ 0, b_1, \dots, b_q \end{matrix} \right. \right)$<br>$-\infty < y < 0$<br>$(-1)^k G_{p+2, q+2}^{m+1, n+1} \left( ay \left  \begin{matrix} 0, a_1, \dots, a_p, \frac{1}{2}+k \\ 0, b_1, \dots, b_q, \frac{1}{2}+k \end{matrix} \right. \right)$<br>$0 < y < \infty$<br>$k \text{ integer}$ |
| (61) | $G_{pq}^{mn} \left( ax^2 \left  \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right)$<br>$p+q < 2(m+n)$<br>$ \arg a  < (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi$<br>$\operatorname{Re} a_j < 1 \quad j = 1, \dots, n$<br>$\operatorname{Re} b_j > -\frac{1}{2} \quad j = 1, \dots, m$                                      | $\operatorname{sgn} y G_{p+2, q+2}^{m+1, n+1} \left( ay^2 \left  \begin{matrix} \frac{1}{2}, a_1, \dots, a_p, 1 \\ \frac{1}{2}, b_1, \dots, b_q, 1 \end{matrix} \right. \right)$                                                                                                                                                                                                   |
| (62) | $x G_{pq}^{mn} \left( ax^2 \left  \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right)$<br>$p+q < 2(m+n)$<br>$ \arg a  < (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi$<br>$\operatorname{Re} a_j < \frac{1}{2} \quad j = 1, \dots, n$<br>$\operatorname{Re} b_j > -1 \quad j = 1, \dots, m$                                    | $ y  G_{p+2, q+2}^{m+1, n+1} \left( ay^2 \left  \begin{matrix} -\frac{1}{2}, a_1, \dots, a_p, 0 \\ -\frac{1}{2}, b_1, \dots, b_q, 0 \end{matrix} \right. \right)$                                                                                                                                                                                                                  |

$y$  is real, and the integral is a Cauchy Principal Value.

## INTEGRALS OF HIGHER TRANSCENDENTAL FUNCTIONS

This part contains mostly integrals which have not been listed in the tables of Chapters I to XV.



## CHAPTER XVI

### ORTHOGONAL POLYNOMIALS

In this chapter we list integrals involving the classical orthogonal polynomials. For the theory of these polynomials see H.T.F. vol. II, Chapter X and the literature quoted there, especially Szegő's book. The notation used in the present compilation for Hermite polynomials differs from that used in H.T.F.

Further integrals may be evaluated by the methods mentioned in the introduction to vol. I, by the use of Rodrigues' formula and its analogues (given below) followed by repeated integrations by parts, by using generating functions (see below), and also by utilizing the relations (see below) between the various systems of orthogonal polynomials and between these polynomials and Legendre functions, hypergeometric series, confluent hypergeometric functions in conjunction with tables given in other chapters of this book.

#### Tchebichef polynomials

$$T_n(x) = (-1)^n T_n(-x) = \cos(n\theta) \qquad x = \cos \theta$$

$$= \frac{(1-x^2)^{\frac{1}{2}}}{2^n (\frac{1}{2})_n} \left( -\frac{d}{dx} \right)^n [(1-x^2)^{n-\frac{1}{2}}]$$

$$= {}_2F_1(-n, n; \frac{1}{2}; \frac{1}{2} - \frac{1}{2}x)$$

$$= \frac{1}{2}n C_n^0(x) = \frac{n!}{(\frac{1}{2})_n} P_n^{(-\frac{1}{2}, -\frac{1}{2})}(x)$$

$$U_n(x) = (-1)^n U_n(-x) = \frac{\sin[(n+1)\theta]}{\sin \theta} \qquad x = \cos \theta$$

$$= \frac{(n+1)(1-x^2)^{-\frac{1}{2}}}{2^{n+1} (\frac{1}{2})_{n+1}} \left( -\frac{d}{dx} \right)^n [(1-x^2)^{n+\frac{1}{2}}]$$

$$U_n(x) = (n+1) {}_2F_1\left(-n, n+1; \frac{3}{2}; \frac{1-x}{2}\right)$$

$$= C_n^{(1)}(x) = \frac{(n+1)!}{2(\frac{1}{2})_{n+1}} P_n^{(\frac{1}{2}, \frac{1}{2})}(x)$$

$$1 + 2 \sum_{n=1}^{\infty} T_n(x) z^n = \frac{1 - z^2}{1 - 2xz + z^2}$$

$$\sum_{n=0}^{\infty} U_n(x) z^n = (1 - 2xz + z^2)^{-1}$$

For other generating functions see H.T.F. vol II, p. 186.

### Legendre polynomials

$$P_n(x) = (-1)^n P_n(-x) = \frac{1}{2^n n!} \left( \frac{d}{dx} \right)^n [(x^2 - 1)^n]$$

$$= {}_2F_1(-n, n+1; 1; \frac{1}{2} - \frac{1}{2}x)$$

$$= \frac{2^n (\frac{1}{2})_n}{n!} x^n {}_2F_1(-\frac{1}{2}n, \frac{1}{2} - \frac{1}{2}n; \frac{1}{2}; x^{-2})$$

$$= C_n^{\frac{1}{2}}(x) = P_n^{(0,0)}(x)$$

$$\sum_{n=0}^{\infty} P_n(x) z^n = (1 - 2xz + z^2)^{-\frac{1}{2}}$$

For the connection with Legendre functions see H.T.F. vol. I, p. 150 ff; for additional hypergeometric series representing Legendre polynomials see H.T.F. vol. I, p. 124-131 ( $\mu = 0$ ,  $\nu = n$ ), and vol. II, p. 180; and for other generating functions see H.T.F. vol II, p. 182.

For the definition of associated Legendre polynomials, and their properties see H.T.F. vol. I, p. 148 ff, and below.

### Gegenbauer polynomials

These polynomials are also called *ultraspherical polynomials* and are denoted by  $P_n^{(\nu)}(x)$ .



$$\begin{aligned}
 C_n^\nu(x) &= (-1)^n C_n^\nu(-x) \\
 &= \frac{2^{\nu-\frac{1}{2}} \Gamma(2\nu+n) \Gamma(\nu+\frac{1}{2})}{n! \Gamma(2\nu)} (x^2-1)^{\frac{1}{2}-\nu} P_{n+\nu-\frac{1}{2}}^{\frac{1}{2}-\nu}(x) \\
 &= \frac{(2\nu)_n (1-x^2)^{\frac{1}{2}-\nu}}{2^n n! (\nu+\frac{1}{2})_n} \left(-\frac{d}{dx}\right)^n [(1-x^2)^{n+\nu-\frac{1}{2}}] \\
 &= \frac{(2\nu)_n}{n!} {}_2F_1(-n, n+2\nu; \nu+\frac{1}{2}; \frac{1}{2}-\frac{1}{2}x) \\
 &= \frac{2^n (\nu)_n}{n!} (x-1)^n {}_2F_1\left(-n, \frac{1}{2}-n-\nu; 1-2n-2\nu; \frac{2}{1-x}\right) \\
 &= \frac{(2\nu)_n}{(\nu+\frac{1}{2})_n} P_n^{(\nu-\frac{1}{2}, \nu-\frac{1}{2})}(x)
 \end{aligned}$$

$$\begin{aligned}
 C_{2n}^\nu(x) &= (-1)^n \frac{(\nu)_n}{n!} {}_2F_1(-n, n+\nu; \frac{1}{2}; x^2) \\
 &= \frac{(\nu)_n}{(\frac{1}{2})_n} P_n^{(\nu-\frac{1}{2}, -\frac{1}{2})}(2x^2-1)
 \end{aligned}$$

$$\begin{aligned}
 C_{2n+1}^\nu(x) &= (-1)^n \frac{(\nu)_{n+1}}{n!} 2x {}_2F_1(-n, n+\nu+1; 3/2; x^2) \\
 &= \frac{(\nu)_{n+1}}{(\frac{1}{2})_{n+1}} x P_n^{(\nu-\frac{1}{2}, \frac{1}{2})}(2x^2-1)
 \end{aligned}$$

$$\sum_{n=0}^{\infty} C_n^\nu(x) z^n = (1-2xz+z^2)^{-\nu}$$

For the connection with Legendre functions, and for additional hypergeometric expansions see H.T.F. vol. I, p. 175ff, p. 124-131, Vol. II, p. 176; for other generating functions see H.T.F. vol. II, p. 177.

**Jacobi polynomials**

$$\begin{aligned}
P_n^{(\alpha, \beta)}(x) &= (-1)^n P_n^{(\beta, \alpha)}(-x) \\
&= \frac{(1-x)^{-\alpha}(1+x)^{-\beta}}{2^n n!} \left( -\frac{d}{dx} \right)^n [(1-x)^{\alpha+n} (1+x)^{\beta+n}] \\
&= \binom{n+\alpha}{n} {}_2F_1 \left( -n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2} \right) \\
&= (-1)^n \binom{n+\beta}{n} {}_2F_1 \left( -n, n+\alpha+\beta+1; \beta+1; \frac{1+x}{2} \right) \\
&= \binom{n+\alpha}{n} \left( \frac{1+x}{2} \right)^n {}_2F_1 \left( -n, -n-\beta; \alpha+1; \frac{x-1}{x+1} \right) \\
&= \binom{n+\beta}{n} \left( \frac{x-1}{2} \right)^n {}_2F_1 \left( -n, -n-\alpha; \beta+1; \frac{x+1}{x-1} \right) \\
\sum_{n=0}^{\infty} P_n^{(\alpha, \beta)}(x) z^n &= 2^{\alpha+\beta} R^{-1} (1-z+R)^{-\alpha} (1+z+R)^{-\beta} \\
&\qquad\qquad\qquad R = (1-2xz+z^2)^{\frac{1}{2}}
\end{aligned}$$

Other expansions may be obtained from those given above by means of the transformations given in H.T.F. vol. I, sec. 2.9.

**Hermite polynomials**

$$\begin{aligned}
\text{He}_n(x) &= (-1)^n \text{He}_n(-x) = 2^{-\frac{1}{2}n} H_n(2^{-\frac{1}{2}}x) \\
&= e^{\frac{1}{2}x^2} \left( -\frac{d}{dx} \right)^n [e^{-\frac{1}{2}x^2}] \\
&= x^n {}_2F_0 \left( -\frac{n}{2}, \frac{1-n}{2}; -\frac{2}{x^2} \right) \\
&= 2^{\frac{1}{2}n+\frac{1}{4}} x^{-\frac{1}{2}} e^{\frac{1}{4}x^2} W_{\frac{1}{2}n+\frac{1}{4}, -\frac{1}{4}} \left( \frac{x^2}{2} \right) \\
&= e^{\frac{1}{4}x^2} D_n(x)
\end{aligned}$$

$$\begin{aligned}\text{He}_{2n}(x) &= (-2)^n \left(\frac{1}{2}\right)_n {}_1F_1(-n; \frac{1}{2}; \frac{1}{2}x^2) \\ &= (-2)^n n! L_n^{-\frac{1}{2}}(\frac{1}{2}x^2)\end{aligned}$$

$$\begin{aligned}\text{He}_{2n+1}(x) &= (-2)^n (3/2)_n x {}_1F_1(-n; 3/2; \frac{1}{2}x^2) \\ &= (-2)^n n! x L_n^{\frac{1}{2}}(\frac{1}{2}x^2)\end{aligned}$$

$$\sum_{n=0}^{\infty} \text{He}_n(x) \frac{z^n}{n!} = \exp(-\frac{1}{2}z^2 + xz)$$

For other generating functions see H.T.F. vol. II, p. 194.

### Laguerre polynomials

$$\begin{aligned}L_n^\alpha(x) &= \frac{1}{n!} e^x x^{-\alpha} \left(\frac{d}{dx}\right)^n [e^{-x} x^{n+\alpha}] \\ &= \binom{n+\alpha}{n} {}_1F_1(-n; \alpha+1; x) \\ &= \frac{(-1)^n}{n!} x^n {}_2F_0(-n, -\alpha-n; -1/x) \\ &= \frac{(-1)^n}{n!} x^{-\frac{1}{2}\alpha-\frac{1}{2}} e^{\frac{1}{2}x} W_{\frac{1}{2}+\frac{1}{2}\alpha+n, \frac{1}{2}\alpha}(x)\end{aligned}$$

$$\sum_{n=0}^{\infty} L_n^\alpha(x) z^n = (1-z)^{-\alpha-1} \exp \frac{xz}{z-1}$$

For other generating functions see H.T.F. vol. II, p. 189.



## ORTHOGONAL POLYNOMIALS

### 16.1. Tchebichef polynomials

The integrals in this section may also be expressed as integrals of trigonometric functions.

In this section  $m$  and  $n$  are non-negative integers.

|     |                                                                                                                                                                                                                                                                                       |
|-----|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $\int_{-1}^1 (1-x)^{-\frac{1}{2}} (1+x)^{\alpha} T_n(x) dx = \frac{2^{\alpha+2n+\frac{1}{2}} \pi^{\frac{1}{2}} (n!)^2 \Gamma(\alpha+1) \Gamma(\alpha+3/2)}{(2n)! \Gamma(\alpha+n+3/2) \Gamma(\alpha-n+3/2)}$ <p style="text-align: right;"><math>\text{Re } \alpha &gt; -1</math></p> |
| (2) | $\int_{-1}^1 (1-x)^{\alpha} (1+x)^{\beta} T_n(x) dx = \frac{2^{\alpha+\beta+2n+1} (n!)^2 \Gamma(\alpha+1) \Gamma(\beta+1)}{(2n)! \Gamma(\alpha+\beta+2)}$ $\times {}_3F_2(-n, n, \alpha+1; \frac{1}{2}, \alpha+\beta+2; 1) \quad \text{Re } \alpha > -1, \quad \text{Re } \beta > -1$ |
| (3) | $\int_{-1}^1 (x-y)^{-1} (1-x^2)^{-\frac{1}{2}} T_n(x) dx = \pi U_{n-1}(y) \quad -1 < y < 1$                                                                                                                                                                                           |
| (4) | $\int_{-1}^1 \sin(xyz) \cos[(1-x^2)^{\frac{1}{2}} (1-y^2)^{\frac{1}{2}} z] T_{2n+1}(x) dx$ $= (-1)^n \pi T_{2n+1}(y) J_{2n+1}(z)$                                                                                                                                                     |
| (5) | $\int_{-1}^1 \cos(xyz) \cos[(1-x^2)^{\frac{1}{2}} (1-y^2)^{\frac{1}{2}} z] T_{2n}(x) dx$ $= (-1)^n \pi T_{2n}(y) J_{2n}(z)$                                                                                                                                                           |
| (6) | $\int_{-1}^1 [T_n(x)]^2 dx = 1 - (4n^2 - 1)^{-1}$                                                                                                                                                                                                                                     |

**Tchebichef polynomials (cont'd)**  $m, n = 0, 1, 2, \dots$ 

|      |                                                                                                                                                                                                                                                               |                                |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------|
| (7)  | $\int_{-1}^1 (1-x^2)^{-1/2} [T_0(x)]^2 dx = \pi$                                                                                                                                                                                                              |                                |
| (8)  | $\int_{-1}^1 (1-x^2)^{-1/2} [T_n(x)]^2 dx = \frac{1}{2}\pi$                                                                                                                                                                                                   | $n \neq 0$                     |
| (9)  | $\int_{-1}^1 (1-x^2)^{-1/2} T_m(x) T_n(x) dx = 0$                                                                                                                                                                                                             | $m \neq n$                     |
| (10) | $\int_{-1}^1 (1-x)^{-1/2} (1+x)^{m-n-3/2} T_m(x) T_n(x) dx = 0$                                                                                                                                                                                               | $m > n$                        |
| (11) | $\int_{-1}^1 (1-x)^{-1/2} (1+x)^{m+n-3/2} T_m(x) T_n(x) dx$ $= \frac{\pi (2m+2n-2)!}{2^{m+n} (2m-1)! (2n-1)!}$                                                                                                                                                | $m+n \neq 0$                   |
| (12) | $\int_{-1}^1 (1+x)^{-1/2} (1-x)^{\alpha-1} T_m(x) T_n(x) dx$ $= \frac{\pi^{1/2} 2^{\alpha-1/2} \Gamma(\alpha) \Gamma(n-\alpha+1/2)}{\Gamma(1/2-\alpha) \Gamma(\alpha+n+1/2)}$ $\times {}_4F_3(-m, m, \alpha, \alpha+1/2; 1/2, \alpha+n+1/2, \alpha-n+1/2; 1)$ | $\operatorname{Re} \alpha > 0$ |
| (13) | $\int_0^1 x^{-1/2} (1-x^2)^{-1/2} e^{-2\alpha/x} T_n(x) dx = \pi^{1/2} D_{n-1/2}(2\alpha^{1/2}) D_{-n-1/2}(2\alpha^{1/2})$                                                                                                                                    | $\operatorname{Re} \alpha > 0$ |
| (14) | $\int_{-1}^1 (1-x^2)^{-1/2} T_n(1-x^2 y) dx = \frac{1}{2}\pi [P_n(1-y) + P_{n-1}(1-y)]$                                                                                                                                                                       |                                |
| (15) | $\int_0^\infty (1+x^2)^{-n} \operatorname{sech}(\frac{1}{2}\pi x) T_{2n}[(1+x^2)^{-1/2}] dx$ $= (-1)^{n+1} \frac{2\pi^{2n}}{(2n)!} (2^{n-1} - 1) B_{2n}$                                                                                                      |                                |

**Tchebichef polynomials (cont'd)**  $m, n = 0, 1, 2, \dots$ 

|      |                                                                                                                                                                                                                                                                                                                                                                                |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (16) | $\int_0^\infty (1+x^2)^{-\frac{1}{2}n} \operatorname{sech}(\frac{1}{2}\pi x) T_n[(1+x^2)^{-\frac{1}{2}}] dx = 2^{1-n}(1-2^{1-n})\zeta(n)$                                                                                                                                                                                                                                      |
| (17) | $\begin{aligned} \int_0^\infty (1+x^2)^{\frac{1}{2}-n} [\cosh(\frac{1}{2}\pi x)]^{-2} T_{2n-1}[(1+x^2)^{-\frac{1}{2}}] dx \\ = 2(-1)^{n+1} \pi^{2n-1} \frac{2n-1}{(2n)!} B_{2n} \end{aligned}$                                                                                                                                                                                 |
| (18) | $\begin{aligned} \int_0^\infty (1+x^2)^{-\frac{1}{2}n} [\cosh(\frac{1}{2}\pi x)]^{-2} T_n[(1+x^2)^{-\frac{1}{2}}] dx \\ = \pi^{-1} n 2^{1-n} \zeta(n+1) \end{aligned}$                                                                                                                                                                                                         |
| (19) | $\begin{aligned} \int_0^\infty (a^2+x^2)^{-\frac{1}{2}n} \operatorname{sech}(\frac{1}{2}\pi x) T_n[a(a^2+x^2)^{-\frac{1}{2}}] dx \\ = 2^{1-2n} \left[ \zeta\left(n, \frac{a+1}{4}\right) - \zeta\left(n, \frac{a+3}{4}\right) \right] = 2^{1-n} \Phi\left(-1, n, \frac{a+1}{2}\right) \end{aligned}$ <p style="text-align: right;"><math>\operatorname{Re} a &gt; 0</math></p> |
| (20) | $\begin{aligned} \int_0^\infty (a^2+x^2)^{-\frac{1}{2}n} [\cosh(\frac{1}{2}\pi x)]^{-2} T_n[a(a^2+x^2)^{-\frac{1}{2}}] dx \\ = \pi^{-1} n 2^{1-n} \zeta\left(n+1, \frac{a+1}{2}\right) \end{aligned}$ <p style="text-align: right;"><math>\operatorname{Re} a &gt; 0</math></p>                                                                                                |
| (21) | $\int_{-1}^1 (1-x)^{\frac{1}{2}} (1+x)^a U_n(x) dx = \frac{\pi^{1/2} 2^{\alpha+2n+3/2} [(n+1)!]^2 \Gamma(\alpha+\frac{1}{2})\Gamma(\alpha+1)}{(2n+2)! \Gamma(\alpha+n+5/2) \Gamma(\alpha-n+1/2)}$ <p style="text-align: right;"><math>\operatorname{Re} a &gt; -1</math></p>                                                                                                   |
| (22) | $\begin{aligned} \int_{-1}^1 (1-x)^\alpha (1+x)^\beta U_n(x) dx = \frac{2^{\alpha+\beta+2n+2} [(n+1)!]^2 \Gamma(\alpha+1) \Gamma(\beta+1)}{(2n+2)! \Gamma(\alpha+\beta+2)} \\ \times {}_3F_2(-n, n+1, \alpha+1; 3/2, \alpha+\beta+2; 1) \end{aligned}$ <p style="text-align: right;"><math>\operatorname{Re} a &gt; -1, \quad \operatorname{Re} \beta &gt; -1</math></p>       |

**Tchebichef polynomials (cont'd)**  $m, n = 0, 1, 2, \dots$ 

|      |                                                                                                                                 |
|------|---------------------------------------------------------------------------------------------------------------------------------|
| (23) | $\int_{-1}^1 (x-y)^{-1} (1-x^2)^{-1/2} U_n(x) dx = -\pi T_{n+1}(y) \quad -1 < y < 1$                                            |
| (24) | $\int_{-1}^1 \cos(xyz) \sin[(1-x^2)^{1/2} (1-y^2)^{1/2} z] U_{2n}(x) dx$ $= (-1)^n \pi (1-y^2)^{1/2} U_{2n}(y) J_{2n+1}(z)$     |
| (25) | $\int_{-1}^1 \sin(xyz) \sin[(1-x^2)^{1/2} (1-y^2)^{1/2} z] U_{2n+1}(x) dx$ $= (-1)^n \pi (1-y^2)^{1/2} U_{2n+1}(y) J_{2n+2}(z)$ |
| (26) | $\int_{-1}^1 (1-x)^{-1/2} (1+x)^{1/2} [U_n(x)]^2 dx = (n+1) \pi$                                                                |
| (27) | $\int_{-1}^1 (1-x^2)^{1/2} [U_n(x)]^2 dx = \frac{1}{2} \pi$                                                                     |
| (28) | $\int_{-1}^1 (1-x^2)^{1/2} U_m(x) U_n(x) dx = 0 \quad m \neq n$                                                                 |
| (29) | $\int_{-1}^1 (1-x)(1+x)^{1/2} U_m(x) U_n(x) dx$ $= \frac{2^{5/2} (m+1)(n+1)}{(m+n+3/2)(m+n+5/2)[1-4(m-n)^2]}$                   |
| (30) | $\int_{-1}^1 (1-x)^{1/2} (1+x)^{m-n-1/2} U_m(x) U_n(x) dx = 0 \quad m > n$                                                      |
| (31) | $\int_{-1}^1 (1-x)^{1/2} (1+x)^{m+n+3/2} U_m(x) U_n(x) dx$ $\times \frac{\pi(2m+2n+2)!}{2^{m+n+2} (2m+1)! (2n+1)!}$             |



**Tchebichef polynomials (cont'd)**  $m, n = 0, 1, 2, \dots$ 

|      |                                                                                                                                                                                                                                                                                                                                      |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (32) | $\int_{-1}^1 (1+x)^{\frac{1}{2}}(1-x)^{\alpha-1} U_m(x) U_n(x) dx$ $= \frac{\pi^{\frac{1}{2}} 2^{\alpha-\frac{1}{2}} (m+1)(n+1) \Gamma(\alpha) \Gamma(n-\alpha+3/2)}{\Gamma(3/2-\alpha) \Gamma(3/2+\alpha+n)}$ $\times {}_4F_3(-m, m+2, \alpha, \alpha-\frac{1}{2}; 3/2, \alpha+n+3/2, \alpha-n-1/2; 1) \quad \text{Re } \alpha > 0$ |
| (33) | $\int_{-1}^1 (1-x^2)^{-\frac{1}{2}} U_{2n}(xz) dx = \pi P_n(2z^2-1)$                                                                                                                                                                                                                                                                 |
| (34) | $\int_{-1}^1 U_n[x(1-y^2)^{\frac{1}{2}}(1-z^2)^{\frac{1}{2}}+yz] dx = \frac{2}{n+1} U_n(y) U_n(z)$                                                                                                                                                                                                                                   |
| (35) | $\int_0^\infty \frac{x U_{2n-1}[(1+x^2)^{-\frac{1}{2}}]}{(1+x^2)^{n+\frac{1}{2}}(e^{\pi x}+1)} dx = \frac{1}{2(2n-1)} + \frac{(-1)^n \pi^{2n}}{2(2n)!} B_{2n}$                                                                                                                                                                       |
| (36) | $\int_0^\infty \frac{x U_n[(1+x^2)^{-\frac{1}{2}}]}{(1+x^2)^{\frac{1}{2}n+1}(e^{\pi x}+1)} dx = \frac{1}{2n} - 2^{-n-1} \zeta(n+1)$                                                                                                                                                                                                  |
| (37) | $\int_0^\infty \frac{x U_{2n-1}[(1+x^2)^{-\frac{1}{2}}]}{(1+x^2)^{n+\frac{1}{2}}(e^{2\pi x}-1)} dx = \frac{(-1)^{n+1} (2\pi)^{2n}}{4(2n)!} B_{2n} - \frac{1}{4} - \frac{1}{4n-2}$                                                                                                                                                    |
| (38) | $\int_0^\infty \frac{x U_n[(1+x^2)^{-\frac{1}{2}}]}{(1+x^2)^{\frac{1}{2}n+1}(e^{2\pi x}-1)} dx = \frac{1}{2} \zeta(n+1) - \frac{1}{4} - \frac{1}{2n}$                                                                                                                                                                                |
| (39) | $\int_0^\infty \frac{x U_n[a(\alpha^2+x^2)^{-\frac{1}{2}}]}{(\alpha^2+x^2)^{\frac{1}{2}n+1}(e^{\pi x}+1)} dx = \frac{\alpha^{-n}}{2n} - 2^{-n-1} \zeta\left(n+1, \frac{\alpha+1}{2}\right)$ <p style="text-align: right;">Re <math>\alpha &gt; 0</math></p>                                                                          |

**Tchebichef polynomials (cont'd)**  $m, n = 0, 1, 2, \dots$ 

|      |                                                                                                                                                                   |                    |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------|
| (40) | $\int_0^\infty \frac{x U_n [a(a^2 + x^2)^{-1/2}]}{(a^2 + x^2)^{1/2n+1} (e^{2\pi x} - 1)} dx = \frac{1}{2} \zeta(n+1, a) - \frac{a^{-n-1}}{4} - \frac{a^{-n}}{2n}$ | $\text{Re } a > 0$ |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------|

**16.2. Legendre polynomials**

See also under Gegenbauer polynomials, Legendre functions, hypergeometric series.

In this section  $m$  and  $n$  are non-negative integers.

|     |                                                                                                                                         |                                                     |
|-----|-----------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------|
| (1) | $\int_0^1 x^\lambda P_{2m}(x) dx = \frac{(-1)^m (-\frac{1}{2}\lambda)_m}{2(\frac{1}{2} + \frac{1}{2}\lambda)_{m+1}}$                    | $\text{Re } \lambda > -1$                           |
| (2) | $\int_0^1 x^\lambda P_{2m+1}(x) dx = \frac{(-1)^m (\frac{1}{2} - \frac{1}{2}\lambda)_m}{2(1 + \frac{1}{2}\lambda)_{m+1}}$               | $\text{Re } \lambda > -2$                           |
| (3) | $\int_{-1}^1 (1-x)^{-1/2} P_n(x) dx = \frac{2^{3/2}}{2n+1}$                                                                             |                                                     |
| (4) | $\int_{-1}^1 (1-x^2)^{-1/2} P_{2m}(x) dx = \pi \left[ \frac{(\frac{1}{2})_m}{m!} \right]^2$                                             |                                                     |
| (5) | $\int_{-1}^1 x(1-x^2)^{-1/2} P_{2m+1}(x) dx = \pi \frac{(\frac{1}{2})_m (\frac{1}{2})_{m+1}}{m!(m+1)!}$                                 |                                                     |
| (6) | $\int_{-1}^1 (1-x)^{\alpha-1} (1+x)^{\beta-1} P_n(x) dx = \frac{2^{\alpha+\beta-1} \Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha+\beta)}$ | $\text{Re } \alpha > 0, \quad \text{Re } \beta > 0$ |

$\times {}_3F_2(-n, 1+n, \alpha; 1, \alpha+\beta; 1)$

**Legendre polynomials (cont'd)** $m, n = 0, 1, 2, \dots$ 

|      |                                                                                                                                                       |                                      |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------|
| (7)  | $\int_{-1}^1 (z-x)^{-1} P_n(x) dx = 2Q_n(z)$                                                                                                          | $z$ in the cut plane                 |
| (8)  | $\int_{-1}^1 x (z-x)^{-1} P_0(x) dx = 2Q_1(z)$                                                                                                        | $z$ in the cut plane                 |
| (9)  | $\int_{-1}^1 (z-x)^{-1} x^{n+1} P_n(x) dx = 2z^{n+1} Q_n(z) - \frac{2^{n+1} (n!)^2}{(2n+1)!}$                                                         | $z$ in the cut plane                 |
| (10) | $\int_{-1}^1 (z-x)^{-1} x^m P_n(x) dx = 2z^m Q_m(z)$                                                                                                  | $m \leq n, \quad z$ in the cut plane |
| (11) | $\int_{-1}^1 (a^2 + b^2 - 2abx)^{-1/2} \sin[\lambda(a^2 + b^2 - 2abx)^{1/2}] P_n(x) dx$<br>$= \pi(ab)^{-1/2} J_{n+1/2}(a\lambda) J_{n+1/2}(b\lambda)$ | $a, b > 0$                           |
| (12) | $\int_{-1}^1 (a^2 + b^2 - 2abx)^{-1/2} \cos[\lambda(a^2 + b^2 - 2abx)^{1/2}] P_n(x) dx$<br>$= \pi(ab)^{-1/2} J_{n+1/2}(a\lambda) Y_{n+1/2}(b\lambda)$ | $0 \leq a \leq b$                    |
| (13) | $\int_{-1}^1 [P_n(x)]^2 dx = (n + 1/2)^{-1}$                                                                                                          |                                      |
| (14) | $\int_{-1}^1 P_m(x) P_n(x) dx = 0$                                                                                                                    | $m \neq n$                           |
| (15) | $\int_{-1}^1 (1+x)^{m+n} P_m(x) P_n(x) dx = \frac{2^{m+n+1} [(m+n)!]^4}{(m!n!)^2 (2m+2n+1)!}$                                                         |                                      |

The complex  $z$ -plane is cut along the real axis from  $-1$  to  $1$ .

**Legendre polynomials (cont'd)**  $m, n = 0, 1, 2, \dots$ 

|                                                                                                    |                                                                                                                                                                                                                   |                                              |
|----------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------|
| (16)                                                                                               | $\int_{-1}^1 (1+x)^{n-n-1} P_m(x) P_n(x) dx = 0$                                                                                                                                                                  | $m > n$                                      |
| (17)                                                                                               | $\int_{-1}^1 (1-x)^{\alpha-1} P_m(x) P_n(x) dx = \frac{2^\alpha \Gamma(\alpha) \Gamma(n-\alpha+1)}{\Gamma(1-\alpha) \Gamma(n+\alpha+1)}$<br>$\times {}_4F_3(-m, m+1, \alpha, \alpha; 1, \alpha+n+1, \alpha-n; 1)$ | $\operatorname{Re} \alpha > 0$               |
| (18)                                                                                               | $\int_{-1}^1 (z-x)^{-1} P_m(x) P_n(x) dx = 2P_m(z) Q_n(z)$                                                                                                                                                        | $m \leq n, \quad z \text{ in the cut plane}$ |
| (19)                                                                                               | $\int_{-1}^1 (z-x)^{-1} P_n(x) P_{n+1}(x) dx = 2P_{n+1}(z) Q_n(z) - 2(n+1)^{-1}$                                                                                                                                  | $z \text{ in the cut plane}$                 |
| (20)                                                                                               | $\int_{-1}^1 x(z-x)^{-1} [P_n(x)]^2 dx = 2z P_n(z) Q_n(z) - 2(2n+1)^{-1}$                                                                                                                                         | $z \text{ in the cut plane}$                 |
| (21)                                                                                               | $\int_{-1}^1 x(z-x)^{-1} P_m(x) P_n(x) dx = 2z P_m(z) Q_n(z)$                                                                                                                                                     | $m < n, \quad z \text{ in the cut plane}$    |
| For other similar integrals see MacRobert, T.M., 1948: <i>Proc. Glasgow Math. Assoc.</i> 1, 10-12. |                                                                                                                                                                                                                   |                                              |
| (22)                                                                                               | $\int_0^1 x^{2\mu-1} P_n(1-2x^2) dx = \frac{(-1)^n [\Gamma(\mu)]^2}{2\Gamma(\mu+n) \Gamma(\mu-n)}$                                                                                                                | $\operatorname{Re} \mu > 0$                  |
| (23)                                                                                               | $\int_0^1 x(\alpha^2+x^2)^{-\frac{1}{2}} P_n(1-2x^2) dx = \frac{[\alpha+(\alpha^2+1)^{\frac{1}{2}}]^{-2n-1}}{2n+1}$                                                                                               | $\operatorname{Re} \alpha > 0$               |

The complex  $z$ -plane is cut along the real axis from  $-1$  to  $1$ .

**Legendre polynomials (cont'd)**  $k, l, m, n = 0, 1, 2, \dots$

|      |                                                                                                                                                                                          |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (24) | $\int_{-1}^1 (1-x)^{\alpha} (1+x)^{\beta} P_n^m(x) dx$<br>See Shabde, N.G., 1940: <i>Bull. Calcutta Math. Soc.</i> 32, 121-128.                                                          |
| (25) | $\int_{-1}^1 (z-x)^{-1} (1-x^2)^{\frac{1}{2}m} P_n^m(x) dx = (-2)^m (z^2-1)^{\frac{1}{2}m} Q_n^m(z)$<br>$m \leq n, \quad z \text{ in the cut plane}$                                     |
| (26) | $\int_{-1}^1 x^k (z-x)^{-1} (1-x^2)^{\frac{1}{2}m} P_n^m(x) dx = (-2)^m z^k (z^2-1)^{\frac{1}{2}m} Q_n^m(z)$<br>$m \leq n, \quad k = 0, 1, \dots, n-m, \quad z \text{ in the cut plane}$ |
| (27) | $\int_{-1}^1 [P_n^m(x)]^2 dx = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} \quad m \leq n$                                                                                                      |
| (28) | $\int_{-1}^1 P_n^m(x) P_k^m(x) dx = 0 \quad k \neq n$                                                                                                                                    |
| (29) | $\int_{-1}^1 P_n^m(x) Q_k^m(x) dx = (-1)^m \frac{1 - (-1)^{k+n}}{(n-k)(n+k+1)} \frac{(k+m)!}{(k-m)!}$                                                                                    |
| (30) | $\int_{-1}^1 (1-x^2)^{-1} P_n^m(x) P_n^k(x) dx = 0 \quad k \neq m$                                                                                                                       |
| (31) | $\int_{-1}^1 (1-x^2)^{-1} [P_n^m(x)]^2 dx = \frac{(n+m)!}{m(n-m)!}$                                                                                                                      |

The complex  $z$ -plane is cut along the real axis from  $-1$  to  $1$ .



Gegenbauer polynomials (cont'd)  $m, n = 0, 1, 2, \dots$ 

|     |                                                                                                                                                                                                                                                                                                                                                                              |
|-----|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (4) | $\int_{-1}^1 (1-x)^\alpha (1+x)^\beta C_n^\nu(x) dx = \frac{2^{\alpha+\beta+1} \Gamma(\alpha+1) \Gamma(\beta+1) \Gamma(n+2\nu)}{n! \Gamma(2\nu) \Gamma(\alpha+\beta+2)}$ $\times {}_3F_2(-n, n+2\nu, \alpha+1; \nu+\frac{1}{2}, \alpha+\beta+2; 1)$ $\text{Re } \alpha > -1, \quad \text{Re } \beta > -1$                                                                    |
| (5) | $\int_{-1}^1 x^m (z-x)^{-1} (1-x^2)^{\nu-\frac{1}{2}} C_n^\nu(x) dx$ $= \frac{\pi^{1/2} 2^{3/2-\nu}}{\Gamma(\nu)} e^{-(\nu-\frac{1}{2})\pi i} z^m (z^2-1)^{\frac{1}{2}\nu-\frac{1}{4}} Q_{n+\nu-\frac{1}{2}}^{\nu-\frac{1}{2}}(z)$ $m \leq n, \quad \text{Re } \nu > -\frac{1}{2}, \quad z \text{ in the cut plane}$                                                         |
| (6) | $\int_{-1}^1 x^{n+1} (z-x)^{-1} (1-x^2)^{\nu-\frac{1}{2}} C_n^\nu(x) dx$ $= \frac{\pi^{1/2} 2^{3/2-\nu}}{\Gamma(\nu)} e^{-(\nu-\frac{1}{2})\pi i} z^{n+1} (z^2-1)^{\frac{1}{2}\nu-\frac{1}{4}} Q_{n+\nu-\frac{1}{2}}^{\nu-\frac{1}{2}}(z)$ $- \frac{\pi 2^{1-2\nu-n} n!}{\Gamma(\nu) \Gamma(\nu+n+1)} \quad \text{Re } \nu > -\frac{1}{2}, \quad z \text{ in the cut plane}$ |
| (7) | $\int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} e^{i\alpha x} C_n^\nu(x) dx = \frac{\pi 2^{1-\nu} i^n \Gamma(2\nu+n)}{n! \Gamma(\nu)} a^{-\nu} J_{\nu+n}(a)$ $\text{Re } \nu > -\frac{1}{2}$                                                                                                                                                                                          |
| (8) | $\int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} [C_n^\nu(x)]^2 dx = \frac{\pi 2^{1-2\nu} \Gamma(2\nu+n)}{n! (n+\nu) [\Gamma(\nu)]^2}$ $\text{Re } \nu > -\frac{1}{2}$                                                                                                                                                                                                                 |
| (9) | $\int_{-1}^1 (1-x)^{\nu-3/2} (1+x)^{\nu-1/2} [C_n^\nu(x)]^2 dx = \frac{\pi^{\frac{1}{2}} \Gamma(\nu-\frac{1}{2}) \Gamma(2\nu+n)}{n! \Gamma(\nu) \Gamma(2\nu)}$ $\text{Re } \nu > \frac{1}{2}$                                                                                                                                                                                |

The complex  $z$ -plane is cut along the real axis from  $-1$  to  $1$ .

**Gegenbauer polynomials (cont'd)**  $m, n = 0, 1, 2, \dots$ 

|      |                                                                                                                                                                                                                                                                                                                                  |                                                        |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------|
| (10) | $\int_{-1}^1 (1-x)^{\nu-\frac{1}{2}} (1+x)^{2\nu-1} [C_n^\nu(x)]^2 dx$ $= \frac{2^{3\nu-\frac{1}{2}} [\Gamma(2\nu+n)]^2 \Gamma(2n+\nu+\frac{1}{2})}{(n!)^2 \Gamma(2\nu) \Gamma(3\nu+2n+\frac{1}{2})}$                                                                                                                            | $\operatorname{Re} \nu > 0$                            |
| (11) | $\int_{-1}^1 (1-x)^{3\nu+2n-3/2} (1+x)^{\nu-1/2} [C_n^\nu(x)]^2 dx$ $= \frac{\pi^{\frac{1}{2}} [\Gamma(\nu+\frac{1}{2})]^2 \Gamma(\nu+2n+\frac{1}{2}) \Gamma(2\nu+2n) \Gamma(3\nu+2n-\frac{1}{2})}{2^{2\nu+2n} [n! \Gamma(\nu+n+\frac{1}{2}) \Gamma(2\nu)]^2 \Gamma(2\nu+2n+\frac{1}{2})}$                                       | $\operatorname{Re} \nu > 1/6$                          |
| (12) | $\int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} C_m^\nu(x) C_n^\nu(x) dx = 0$                                                                                                                                                                                                                                                             | $m \neq n, \quad \operatorname{Re} \nu > -\frac{1}{2}$ |
| (13) | $\int_{-1}^1 (1-x)^{\nu-1/2} (1+x)^{\nu+m-n-3/2} C_m^\nu(x) C_n^\nu(x) dx$ $= (-1)^m \frac{2^{2-2\nu-m+n} \pi^{3/2} \Gamma(2\nu+n) \Gamma(\nu-\frac{1}{2}+m-n) \Gamma(\frac{1}{2}-\nu+m-n)}{m! (n-m)! [\Gamma(\nu)]^2 \Gamma(\frac{1}{2}+\nu+m) \Gamma(\frac{1}{2}-\nu-n) \Gamma(\frac{1}{2}+m-n)}$                              | $\operatorname{Re} \nu > \frac{1}{2}$                  |
| (14) | $\int_{-1}^1 (1-x)^{2\nu-1} (1+x)^{\nu-\frac{1}{2}} C_m^\nu(x) C_n^\nu(x) dx$ $= \frac{2^{3\nu-\frac{1}{2}} \Gamma(\nu+\frac{1}{2}) \Gamma(2\nu+m) \Gamma(2\nu+n) \Gamma(\nu+\frac{1}{2}+m+n) \Gamma(\frac{1}{2}-\nu+n-m)}{m! n! \Gamma(2\nu) \Gamma(\frac{1}{2}-\nu) \Gamma(\nu+\frac{1}{2}+n-m) \Gamma(3\nu+\frac{1}{2}+m+n)}$ | $\operatorname{Re} \nu > \frac{1}{2}$                  |
| (15) | $\int_{-1}^1 (1-x)^{\nu-1/2} (1+x)^{3\nu+m+n-3/2} C_m^\nu(x) C_n^\nu(x) dx$ $= \frac{2^{4\nu+m+n-1} [\Gamma(\nu+\frac{1}{2}) \Gamma(2\nu+m+n)]^2 \Gamma(\nu+m+n+\frac{1}{2}) \Gamma(3\nu+m+n-\frac{1}{2})}{\Gamma(\nu+m+\frac{1}{2}) \Gamma(\nu+n+\frac{1}{2}) \Gamma(2\nu+m) \Gamma(2\nu+n) \Gamma(4\nu+2m+2n)}$                | $\operatorname{Re} \nu > 1/6$                          |



Gegenbauer polynomials (cont'd)  $m, n = 0, 1, 2, \dots$ 

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (16) | $\int_{-1}^1 (1-x)^a (1+x)^{\nu-\frac{1}{2}} C_m^\mu(x) C_n^\nu(x) dx$ $= \frac{2^{\alpha+\nu+\frac{1}{2}} \Gamma(\alpha+1) \Gamma(\nu+\frac{1}{2}) \Gamma(\nu-\alpha+n-\frac{1}{2}) \Gamma(2\mu+m) \Gamma(2\nu+n)}{m! n! \Gamma(\nu-\alpha-1/2) \Gamma(\nu-\alpha+n+3/2) \Gamma(2\mu) \Gamma(2\nu)}$ $\times {}_4F_3\left(-m, m+2\mu, \alpha+1, \alpha-\nu+\frac{3}{2}; \mu+\frac{1}{2}, \nu+\alpha+n+\frac{3}{2}, \alpha-\nu-n+\frac{3}{2}; 1\right)$ $\text{Re } \alpha > -1, \quad \text{Re } \nu > -\frac{1}{2}$ |
| (17) | $\int_{-1}^1 (z-x)^{-1} (1-x^2)^{\nu-\frac{1}{2}} C_m^\nu(x) C_n^\nu(x) dx$ $= \frac{\pi^{\frac{1}{2}} 2^{\frac{1}{2}-\nu}}{\Gamma(\nu)} e^{-(\nu-\frac{1}{2})\pi i} (z^2-1)^{\frac{1}{2}\nu-\frac{1}{4}} C_m^\nu(z) Q_{n+\nu-\frac{1}{2}}^{\nu-\frac{1}{2}}(z)$ $m \leq n, \quad \text{Re } \nu > -\frac{1}{2}, \quad z \text{ in the cut plane}$                                                                                                                                                                    |
| (18) | $\int_{-1}^1 (1-x^2)^{\nu-\frac{1}{2}} C_m^\nu(x) C_n^\nu(x) C_k^\nu(x) dx$ <p>See Hsü, Hsien-Yü, 1938: <i>Duke Math. J.</i> 4, 374-383.</p>                                                                                                                                                                                                                                                                                                                                                                          |
| (19) | $\int_{-1}^1 (1-x^2)^{\frac{1}{2}\nu-1} C_{2n}^\nu(ax) dx = \frac{\pi^{\frac{1}{2}} \Gamma(\frac{1}{2}\nu)}{\Gamma(\frac{1}{2}\nu+\frac{1}{2})} C_n^{\frac{1}{2}\nu}(2a^2-1) \quad \text{Re } \nu > 0$                                                                                                                                                                                                                                                                                                                |
| (20) | $\int_{-1}^1 (1-x^2)^{\nu-1} C_n^\nu(\cos a \cos \beta + x \sin a \sin \beta) dx$ $= \frac{2^{2\nu-1} n! [\Gamma(\nu)]^2}{\Gamma(2\nu+n)} C_n^\nu(\cos a) C_n^\nu(\cos \beta) \quad \text{Re } \nu > 0$                                                                                                                                                                                                                                                                                                               |
| (21) | $\int_0^1 x^{2\nu} (1-x^2)^{\sigma-1} C_n^\nu(1-x^2 y) dx = \frac{(2\nu)_n \Gamma(\nu+\frac{1}{2}) \Gamma(\sigma)}{2\Gamma(n+\nu+\sigma+\frac{1}{2})} P_n^{(\alpha, \beta)}(1-y)$ $\text{Re } \nu > -\frac{1}{2}, \quad \text{Re } \sigma > 0, \quad \alpha = \nu + \sigma - \frac{1}{2}, \quad \beta = \nu - \sigma - \frac{1}{2}$                                                                                                                                                                                   |

The complex  $z$ -plane is cut along the real axis from  $-1$  to  $1$ .

### 16.4. Jacobi polynomials

See also under hypergeometric series.

In this section  $m$  and  $n$  are non-negative integers.

|     |                                                                                                                                                                                                                                                                                                                                                                                                                                       |
|-----|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $\int_{-1}^1 (1-x)^{\alpha} (1+x)^{\sigma} P_n^{(\alpha, \beta)}(x) dx$ $= \frac{2^{\alpha+\sigma+1} \Gamma(\sigma+1) \Gamma(\alpha+n+1) \Gamma(\sigma-\beta+1)}{\Gamma(\sigma-\beta-n+1) \Gamma(\alpha+\sigma+n+2)}$ <p style="text-align: right;"><math>\operatorname{Re} \alpha &gt; -1, \quad \operatorname{Re} \sigma &gt; -1</math></p>                                                                                         |
| (2) | $\int_{-1}^1 (1-x)^{\rho} (1+x)^{\beta} P_n^{(\alpha, \beta)}(x) dx$ $= \frac{2^{\beta+\rho+1} \Gamma(\rho+1) \Gamma(\beta+n+1) \Gamma(\alpha-\rho+n)}{n! \Gamma(\alpha-\rho) \Gamma(\beta+\rho+n+2)}$ <p style="text-align: right;"><math>\operatorname{Re} \rho &gt; -1, \quad \operatorname{Re} \beta &gt; -1</math></p>                                                                                                           |
| (3) | $\int_{-1}^1 (1-x)^{\rho} (1+x)^{\sigma} P_n^{(\alpha, \beta)}(x) dx = \frac{2^{\rho+\sigma+1} \Gamma(\rho+1) \Gamma(\sigma+1)}{\Gamma(\rho+\sigma+2)}$ $\times {}_3F_2(-n, \alpha+\beta+n+1, \rho+1; \alpha+1, \rho+\sigma+2; 1)$ <p style="text-align: right;"><math>\operatorname{Re} \rho &gt; -1, \quad \operatorname{Re} \sigma &gt; -1</math></p>                                                                              |
| (4) | $\int_{-1}^1 (z-x)^{-1} (1-x)^{\alpha} (1+x)^{\beta} P_n^{(\alpha, \beta)}(x) dx$ $= \frac{2^{\alpha+\beta+n+1} \Gamma(\alpha+n+1) \Gamma(\beta+n+1)}{\Gamma(\alpha+\beta+2n+2) (z-1)^{n+1}}$ $\times {}_2F_1\left(n+1, \alpha+n+1; \alpha+\beta+2n+2; \frac{2}{z-1}\right)$ <p style="text-align: right;"><math>\operatorname{Re} \alpha &gt; -1, \quad \operatorname{Re} \beta &gt; -1, \quad z \text{ in the cut plane}</math></p> |

The complex  $z$ -plane is cut along the real axis from  $-1$  to  $1$ .

**Jacobi polynomials (cont'd)** $m, n = 0, 1, 2, \dots$ 

|      |                                                                                                                                                                                                                                                                                                                                              |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (5)  | $\int_{-1}^1 (1-x)^{\alpha}(1+x)^{\beta} [P_n^{(\alpha, \beta)}(x)]^2 dx = \frac{2^{\alpha+\beta+1} \Gamma(\alpha+n+1) \Gamma(\beta+n+1)}{n! (\alpha+\beta+2n+1) \Gamma(\alpha+\beta+n+1)}$ $\operatorname{Re} \alpha > -1, \quad \operatorname{Re} \beta > -1$                                                                              |
| (6)  | $\int_{-1}^1 (1-x)^{\alpha-1} (1+x)^{\beta} [P_n^{(\alpha, \beta)}(x)]^2 dx = \frac{2^{\alpha+\beta} \Gamma(\alpha+n+1) \Gamma(\beta+n+1)}{n! \alpha \Gamma(\alpha+\beta+n+1)}$ $\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > -1$                                                                                           |
| (7)  | $\int_{-1}^1 (1-x)^{2\alpha} (1+x)^{\beta} [P_n^{(\alpha, \beta)}(x)]^2 dx$ $= \frac{2^{4\alpha+\beta+1} \Gamma(\alpha+\frac{1}{2}) [\Gamma(\alpha+n+1)]^2 \Gamma(\beta+2n+1)}{\pi^{\frac{1}{2}} (n!)^2 \Gamma(\alpha+1) \Gamma(2\alpha+\beta+2n+2)}$ $\operatorname{Re} \alpha > -\frac{1}{2}, \quad \operatorname{Re} \beta > -1$          |
| (8)  | $\int_{-1}^1 (1-x)^{2\alpha+\beta+2n} (1+x)^{\beta} [P_n^{(\alpha, \beta)}(x)]^2 dx$ $= \frac{2^{2\alpha+2\beta+2n+1} \Gamma(\beta+2n+1) [\Gamma(\alpha+\beta+2n+1)]^2 \Gamma(2\alpha+\beta+2n+1)}{[n! \Gamma(\alpha+\beta+n+1)]^2 \Gamma(2\alpha+2\beta+4n+2)}$ $\operatorname{Re} \beta > -1, \quad \operatorname{Re}(2\alpha+\beta) > -1$ |
| (9)  | $\int_{-1}^1 (1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha, \beta)}(x) P_m^{(\alpha, \beta)}(x) dx = 0$ $m \neq n, \quad \operatorname{Re} \alpha > -1, \quad \operatorname{Re} \beta > -1$                                                                                                                                                       |
| (10) | $\int_{-1}^1 (1-x)^{\rho}(1+x)^{\beta} P_n^{(\alpha, \beta)}(x) P_n^{(\rho, \beta)}(x) dx$ $= \frac{2^{\rho+\beta+1} \Gamma(\rho+n+1) \Gamma(\beta+n+1) \Gamma(\alpha+\beta+2n+1)}{n! \Gamma(\rho+\beta+2n+2) \Gamma(\alpha+\beta+n+1)}$ $\operatorname{Re} \rho > -1, \quad \operatorname{Re} \beta > -1$                                   |

## Jacobi polynomials (cont'd)

 $m, n = 0, 1, 2, \dots$ 

|      |                                                                                                                                                                                                                                                                                                                                                                                                        |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (11) | $\int_{-1}^1 (1-x)^{\rho-1} (1+x)^{\beta} P_n^{(\alpha, \beta)}(x) P_n^{(\rho, \beta)}(x) dx$ $= \frac{2^{\rho+\beta} \Gamma(a+n+1) \Gamma(\beta+n+1) \Gamma(\rho)}{n! \Gamma(a+1) \Gamma(\rho+\beta+n+1)}$ $\text{Re } \beta > -1, \quad \text{Re } \rho > 0$                                                                                                                                         |
| (12) | $\int_{-1}^1 (1-x)^a (1+x)^{\sigma} P_n^{(\alpha, \beta)}(x) P_m^{(\alpha, \sigma)}(x) dx$ $= \frac{2^{a+\sigma+1} \Gamma(a+n+1) \Gamma(a+\beta+m+n+1) \Gamma(\sigma+m+1) \Gamma(\sigma-\beta+1)}{m! (n-m)! \Gamma(a+\beta+n+1) \Gamma(a+\sigma+m+n+2) \Gamma(\sigma-\beta+m+1)}$ $\text{Re } a > -1, \quad \text{Re } \sigma > -1$                                                                    |
| (13) | $\int_{-1}^1 (1-x)^a (1+x)^{\beta+\sigma} P_n^{(\alpha, \beta)}(x) P_m^{(\alpha, \sigma)}(x) dx$ $= \frac{2^{a+\beta+\sigma+1} \Gamma(a+m+n+1) \Gamma(\beta+n+1) \Gamma(\beta+\sigma+1) \Gamma(\sigma+m+1)}{m! n! \Gamma(a+\beta+\sigma+m+n+2) \Gamma(\beta-m+n+1) \Gamma(\sigma+m-n+1)}$ $\text{Re } a > -1, \quad \text{Re } (\beta + \sigma) > -1$                                                  |
| (14) | $\int_{-1}^1 (1-x)^a (1+x)^{a+\beta+\sigma+m+n} P_n^{(\alpha, \beta)}(x) P_m^{(\alpha, \sigma)}(x) dx$ $= \frac{2^{2a+\beta+\sigma+m+n+1} \Gamma(a+\beta+\sigma+m+n+1) \Gamma(a+\sigma+m+n+1)}{m! n! \Gamma(a+\beta+n+1) \Gamma(a+\sigma+n+1)}$ $\times \frac{\Gamma(a+m+n+1) \Gamma(a+\beta+m+n+1)}{\Gamma(2a+\beta+\sigma+2m+2n+2)}$ $\text{Re } a > -1, \quad \text{Re } (a + \beta + \sigma) > -1$ |
| (15) | $\int_{-1}^1 (1-x)^a (1+x)^{\sigma+m-n-1} P_n^{(\alpha, \beta)}(x) P_m^{(\alpha, \sigma)}(x) dx$ $= \frac{2^{a+\sigma+m-n} \Gamma(a+n+1) \Gamma(\beta+n+1) \Gamma(\sigma+m-n) \Gamma(\sigma-\beta+m-n)}{n! (n-m)! \Gamma(a+\sigma+m+1) \Gamma(\beta-m+n+1) \Gamma(\sigma-\beta+2m-2n)}$ $\text{Re } a > 0, \quad \text{Re } \sigma > n-m$                                                              |

**Jacobi polynomials (cont'd)** $m, n = 0, 1, 2, \dots$ 

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                      |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (16) | $\int_{-1}^1 (1-x)^{\rho} (1+x)^{\beta} P_n^{(\alpha, \beta)}(x) P_m^{(\rho, \beta)}(x) dx$ $= \frac{2^{\beta+\rho+1} \Gamma(\alpha+\beta+m+n+1) \Gamma(\beta+n+1) \Gamma(\rho+m+1) \Gamma(\rho-\alpha-m+n)}{n! (n-m)! \Gamma(\alpha+\beta+n+1) \Gamma(\beta+\rho+m+n+2) \Gamma(\rho-\alpha)}$ $\text{Re } \beta > -1, \quad \text{Re } \rho > -1$                                                                                                   |
| (17) | $\int_{-1}^1 (1-x)^{\alpha+\rho} (1+x)^{\beta} P_n^{(\alpha, \beta)}(x) P_m^{(\rho, \beta)}(x) dx$ $= \frac{(-1)^{m+n} 2^{\alpha+\beta+\rho+1} \Gamma(\alpha+n+1) \Gamma(\alpha+\rho+1) \Gamma(\beta+m+n+1) \Gamma(\rho+m+1)}{m! n! \Gamma(\alpha-m+n+1) \Gamma(\alpha+\beta+\rho+m+n+2) \Gamma(\rho+m-n+1)}$ $\text{Re } (\alpha+\rho) > -1, \quad \text{Re } \beta > -1$                                                                           |
| (18) | $\int_{-1}^1 (1-x)^{\alpha+\beta+\rho+m+n} (1+x)^{\beta} P_n^{(\alpha, \beta)}(x) P_m^{(\rho, \beta)}(x) dx$ $= \frac{(-1)^{m+n} 2^{\alpha+\beta+\rho+m+n+1} \Gamma(\alpha+\beta+m+n+1)}{m! n! \Gamma(\alpha+\beta+n+1)}$ $\times \frac{\Gamma(\alpha+\beta+\rho+m+n+1) \Gamma(\beta+m+n+1) \Gamma(\beta+\rho+m+n+1)}{\Gamma(\alpha+2\beta+\rho+2m+2n+2) \Gamma(\beta+\rho+m+1)}$ $\text{Re } \beta > -1, \quad \text{Re } (\alpha+\beta+\rho) > -1$ |
| (19) | $\int_{-1}^1 (1-x)^{\rho+m-n-1} (1+x)^{\beta} P_n^{(\alpha, \beta)}(x) P_m^{(\rho, \beta)}(x) dx$ $= \frac{2^{\beta+\rho+m-n} \Gamma(\alpha+n+1) \Gamma(\beta+n+1) \Gamma(\rho+m-n) \Gamma(\alpha-\rho-2m+2n+1)}{n! (n-m)! \Gamma(\alpha-m+n+1) \Gamma(\alpha-\rho-m+n+1) \Gamma(\beta+\rho+m+1)}$ $\text{Re } \beta > -1, \quad \text{Re } \rho > n-m$                                                                                              |



**Hermite polynomials (cont'd)** $m, n = 0, 1, 2, \dots$ 

|      |                                                                                                                                                                                                                                                                                                  |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (6)  | $\int_0^\infty \exp(-\frac{1}{2}x^2) \cos(\beta x) \text{He}_{2n}(x) dx = (-1)^n (\frac{1}{2}\pi)^{\frac{1}{2}} \beta^{2n} \exp(-\frac{1}{2}\beta^2)$                                                                                                                                            |
| (7)  | $\int_0^\infty \exp(-\frac{1}{2}x^2) \sinh(\beta x) \text{He}_{2n+1}(x) dx = (\frac{1}{2}\pi)^{\frac{1}{2}} \beta^{2n+1} \exp(\frac{1}{2}\beta^2)$                                                                                                                                               |
| (8)  | $\int_0^\infty \exp(-\frac{1}{2}x^2) \cosh(\beta x) \text{He}_{2n}(x) dx = (\frac{1}{2}\pi)^{\frac{1}{2}} \beta^{2n} \exp(\frac{1}{2}\beta^2)$                                                                                                                                                   |
| (9)  | $\int_{-\infty}^\infty \exp(-\frac{1}{2}x^2) [\text{He}_n(x)]^2 dx = (2\pi)^{\frac{1}{2}} n!$                                                                                                                                                                                                    |
| (10) | $\int_{-\infty}^\infty e^{-x^2} \text{He}_m(x) \text{He}_n(x) dx = (-1)^{\frac{1}{2}m - \frac{1}{2}n} \Gamma\left(\frac{m+n+1}{2}\right)$<br>$m+n \text{ even}$                                                                                                                                  |
| (11) | $\int_{-\infty}^\infty \exp(-\frac{1}{2}x^2) \text{He}_m(x) \text{He}_n(x) dx = 0$<br>$m \neq n$                                                                                                                                                                                                 |
| (12) | $\int_{-\infty}^\infty \exp(-a^2 x^2) \text{He}_m(x) \text{He}_n(x) dx$<br>$= a^{-m-n-1} (1-2a^2)^{\frac{1}{2}m + \frac{1}{2}n} \Gamma\left(\frac{m+n+1}{2}\right)$<br>$\times {}_2F_1\left(-m, -n; \frac{1-m-n}{2}; \frac{a^2}{2a^2-1}\right) \quad \text{Re } a^2 > 0, \quad m+n \text{ even}$ |
| (13) | $\int_{-\infty}^\infty \exp[-\frac{1}{2}(x-y)^2] \text{He}_m(x) \text{He}_n(x) dx = (2\pi)^{\frac{1}{2}} m! y^{n-m} L_n^{n-m}(-y^2)$<br>$m \leq n$                                                                                                                                               |

**Hermite polynomials (cont'd)**  $k, m, n = 0, 1, 2, \dots$

|      |                                                                                                                                                                                                                                                   |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (14) | $\int_{-\infty}^{\infty} \exp(-x^2) \text{He}_k(x) \text{He}_m(x) \text{He}_n(x) dx$ $= \pi^{-1} \Gamma(s-k) \Gamma(s-m) \Gamma(s-n)$ $k+m+n \text{ even, } 2s = k+m+n+1$                                                                         |
| (15) | $\int_{-\infty}^{\infty} \exp(-\tfrac{1}{2}x^2) \text{He}_k(x) \text{He}_m(x) \text{He}_n(x) dx = \frac{(2\pi)^{\frac{1}{2}} k! m! n!}{(s-k)!(s-m)!(s-n)!}$ $k+m+n = 2s \text{ even}$                                                             |
| (16) | $\int_{-\infty}^{\infty} \exp(-a^2 x^2) \text{He}_m(x) \text{He}_n(x) \text{He}_k(x) \dots dx$ <p>See Busbridge, I.W., 1948: <i>J. London Math. Soc.</i> 23, 135-141.</p>                                                                         |
| (17) | $\int_{-\infty}^{\infty} \exp[-\tfrac{1}{2}(x-y)^2] \text{He}_n(ax) dx = (2\pi)^{\frac{1}{2}} (1-a^2)^{\frac{1}{2}n} \text{He}_n \left[ \frac{ay}{(1-a^2)^{\frac{1}{2}}} \right]$                                                                 |
| (18) | $\int_0^{\infty} \exp(-\tfrac{1}{2}x^2) \sin(\beta x) \text{He}_{2n+1}(ax) dx$ $= (-1)^n (\tfrac{1}{2}\pi)^{\frac{1}{2}} (a^2-1)^{n+\frac{1}{2}} \exp(-\tfrac{1}{2}\beta^2) \text{He}_{2n+1} \left[ \frac{a\beta}{(a^2-1)^{\frac{1}{2}}} \right]$ |
| (19) | $\int_0^{\infty} \exp(-\tfrac{1}{2}x^2) \cos(\beta x) \text{He}_{2n}(ax) dx$ $= (\tfrac{1}{2}\pi)^{\frac{1}{2}} (1-a^2)^n \exp(-\tfrac{1}{2}\beta^2) \text{He}_{2n} \left[ \frac{a\beta}{(a^2-1)^{\frac{1}{2}}} \right]$                          |
| (20) | $\int_{-\infty}^{\infty} \exp(-\tfrac{1}{2}x^2) H_m(ax) H_n(x) dx = 0$ $m < n$                                                                                                                                                                    |



**Hermite polynomials (cont'd)** $m, n = 0, 1, 2, \dots$ 

|      |                                                                                                                                                                                                                                                                                                                                                                   |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (21) | $\int_{-\infty}^{\infty} \exp(-\tfrac{1}{2}x^2) H_{2m+n}(ax) H_n(x) dx = \pi^{\frac{1}{2}} \frac{(2m+n)!}{m! 2^{m-\frac{1}{2}}} (a^2 - 1)^m a^n$                                                                                                                                                                                                                  |
| (22) | $\int_{-\infty}^{\infty} \exp[-\tfrac{1}{2}(a^2 + \beta^2)x^2] \text{He}_m(ax) \text{He}_n(\beta x) dx$ $= (-1)^{\frac{1}{2}m - \frac{1}{2}n} 2^{\frac{1}{2}m + \frac{1}{2}n + \frac{1}{2}} \Gamma\left(\frac{m+n+1}{2}\right) a^n \beta^m (a^2 + \beta^2)^{-m-n-\frac{1}{2}}$ $\text{Re}(a^2 + \beta^2) > 0, \quad m+n \text{ even}$                             |
| (23) | $\int_{-\infty}^{\infty} \exp(-\tfrac{1}{2}x^2) \text{He}_m(ax) \text{He}_n(\beta x) dx$ $= 2^{\frac{1}{2} - \frac{1}{2}m - \frac{1}{2}n} \pi^{\frac{1}{2}} n! (a^2 - 1)^{-\frac{1}{2}m} (\beta^2 - 1)^{-\frac{1}{2}n} (a^2 + \beta^2 - 1)^{\frac{1}{2}m + \frac{1}{2}n} P_{m+n}^n(z)$ $\frac{1}{z^2} = \frac{1}{a^2} + \frac{1}{\beta^2} - \frac{1}{a^2\beta^2}$ |
| (24) | $\int_{-\infty}^{\infty} \exp(-\lambda^2 x^2) \text{He}_m(ax) \text{He}_n(\beta x) dx = 0$ $m+n \text{ odd}$                                                                                                                                                                                                                                                      |
| (25) | $\int_{-\infty}^{\infty} x^\rho \exp(-x^2) \text{He}_{2n}(ax) \text{He}_{2n}(\beta x) dx$ <p>See Buchholz, Herbert, 1953: <i>Die konfluente hypergeometrische Funktion</i>. Springer-Verlag, Berlin, Göttinger, Heidelberg, Sec. 13.</p>                                                                                                                          |
| (26) | $\int_{-\infty}^{\infty} \exp[-\tfrac{1}{2}(x-y)^2] \text{He}_m(ax) \text{He}_n(ax) dx$ $= (2\pi)^{\frac{1}{2}} \sum_{k=0}^{\min(m, n)} k! \binom{m}{k} \binom{n}{k} (1-a^2)^{\frac{1}{2}m + \frac{1}{2}n - k}$ $\times \text{He}_{m+n-2k} \left[ \frac{ay}{(1-a^2)^{\frac{1}{2}}} \right]$                                                                       |

**Hermite polynomials (cont'd)** $k, m, n = 0, 1, 2, \dots$ 

|      |                                                                                                                                                                                                                                                                                                                                         |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (27) | $\int_{-\infty}^{\infty} \exp(-\lambda^2 x^2) \text{He}_m(ax) \text{He}_n(\beta x) \text{He}_k(\gamma x) dx = 0 \quad m+n+k \text{ odd}$                                                                                                                                                                                                |
| (23) | $\int_{-\infty}^{\infty} \exp(-\lambda^2 x^2) \text{He}_k(ax) \text{He}_m(\beta x) \text{He}_n(\gamma x) \dots dx$<br>See Bailey, W.N., 1948: <i>J. London Math. Soc.</i> 23, 291-297.<br>Lord, R.D., 1949: <i>J. London Math. Soc.</i> 24, 101-112.                                                                                    |
| (29) | $\int_0^{\infty} x^{\rho-1} \exp(-\lambda^2 x^2) \text{He}_k(ax) \text{He}_m(\beta x) \text{He}_n(\gamma x) \dots dx$<br>See Appell, Paul and M.J. Kampé de Fériet, 1926: <i>Fonctions hypergéométriques et hypersphériques. Polynômes d'Hermite</i> . Gauthier-Villars, p. 343.<br>Erdélyi, Arthur, 1936: <i>Math. Z.</i> 40, 693-702. |
| (30) | $\int_{-\infty}^{\infty} \exp(-\frac{1}{2} x^2) \text{He}_m(x+y) \text{He}_n(x+z) dx$<br>$= (2\pi)^{\frac{1}{2}} m! z^{n-m} L_m^{n-m}(-yz) \quad m \leq n$                                                                                                                                                                              |
| (31) | $\int_0^{\pi} (\cos x)^n \text{He}_{2n}[\alpha(1 - \sec x)^{\frac{1}{2}}] dx = \frac{(-1)^n \pi (2n)!}{2^n (n!)^2} [\text{He}_n(\alpha)]^2$                                                                                                                                                                                             |

**16.6. Laguerre polynomials**

See also confluent hypergeometric functions

In this section  $m$  and  $n$  are non-negative integers.

|     |                                                                                                                                                                         |
|-----|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $\int_0^{\infty} x^{\beta-1} e^{-x} L_n^{\alpha}(x) dx = \frac{\Gamma(\alpha - \beta + n + 1) \Gamma(\beta)}{n! \Gamma(\alpha - \beta + 1)} \quad \text{Re } \beta > 0$ |
| (2) | $\int_0^{\infty} x^{\alpha} e^{-x} [L_n^{\alpha}(x)]^2 dx = \frac{\Gamma(\alpha + n + 1)}{n!} \quad \text{Re } \alpha > 0$                                              |

**Laguerre polynomials (cont'd)** $m, n = 0, 1, 2, \dots$ 

|     |                                                                                                                                                                                             |                                                                                                                                          |
|-----|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------|
| (3) | $\int_0^\infty x^\alpha e^{-x} L_m^\alpha(x) L_n^\alpha(x) dx = 0$                                                                                                                          | $m \neq n, \quad \operatorname{Re} \alpha > -1$                                                                                          |
| (4) | $\int_0^\infty x^{\alpha+\beta} e^{-x} L_m^\alpha(x) L_n^\beta(x) dx = (-1)^{m+n} \binom{\alpha+m}{n} \binom{\beta+n}{m}$                                                                   | $\operatorname{Re}(\alpha + \beta) > -1$                                                                                                 |
| (5) | $\int_0^1 x^\alpha (1-x)^{\beta-\alpha-1} L_n^\alpha(xy) dx = \frac{\Gamma(\alpha+n+1) \Gamma(\beta-\alpha)}{\Gamma(\beta+n+1)} L_n^\beta(y)$                                               | $\operatorname{Re} \beta > \operatorname{Re} \alpha > -1$                                                                                |
| (6) | $\int_0^\infty x^\nu e^{-x} L_n^\alpha(\lambda x) L_n^\alpha(\mu x) dx$                                                                                                                     | See Buchholz, Herbert, 1953: <i>Die konfluente hypergeometrische Funktion</i> , Springer Verlag, Berlin, Göttingen, Heidelberg, Sec. 12. |
| (7) | $\int_0^1 x^\alpha (1-x)^\beta L_m^\alpha(xy) L_n^\beta[(1-x)y] dx$<br>$= \frac{(m+n)! \Gamma(\alpha+m+1) \Gamma(\beta+n+1)}{m!n! \Gamma(\alpha+\beta+m+n+2)} L_{m+n}^{\alpha+\beta+1}(y)$  | $\operatorname{Re} \alpha > -1, \quad \operatorname{Re} \beta > -1$                                                                      |
| (8) | $\int_{-\infty}^\infty x^{m-n} \exp[-\frac{1}{2}(x-y)^2] L_n^{m-n}(x^2) dx$<br>$= \frac{(2\pi)^{\frac{1}{2}}}{n!} i^{n-m} \operatorname{He}_n(iy) \operatorname{He}_m(iy)$                  |                                                                                                                                          |
| (9) | $\int_0^\infty \exp(-\frac{1}{2}x^2) [L_n^{-\frac{1}{4}}(\frac{1}{2}x^2)]^2 \cos(xy) dx$<br>$= (\frac{1}{2}\pi)^{\frac{1}{2}} \exp(-\frac{1}{2}y^2) [L_n^{-\frac{1}{4}}(\frac{1}{2}y^2)]^2$ |                                                                                                                                          |

**Laguerre polynomials (cont'd)** $m, n = 0, 1, 2, \dots$ 

|      |                                                                                                                                                                                                                                                               |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (10) | $\int_0^\infty x \exp(-\tfrac{1}{2}x^2) [L_n^{\frac{1}{2}}(\tfrac{1}{2}x^2)]^2 \sin(xy) dx$ $= (\tfrac{1}{2}\pi)^{\frac{1}{2}} y \exp(-\tfrac{1}{2}y^2) [L_n^{\frac{1}{2}}(\tfrac{1}{2}y^2)]^2$                                                               |
| (11) | $\int_0^\infty x \exp(-\tfrac{1}{2}x^2) L_n^\alpha(\tfrac{1}{2}x^2) L_n^{\frac{1}{2}-\alpha}(\tfrac{1}{2}x^2) \sin(xy) dx$ $= (\tfrac{1}{2}\pi)^{\frac{1}{2}} y \exp(-\tfrac{1}{2}y^2) L_n^\alpha(\tfrac{1}{2}y^2) L_n^{\frac{1}{2}-\alpha}(\tfrac{1}{2}y^2)$ |
| (12) | $\int_0^\infty \exp(-\tfrac{1}{2}x^2) L_n^\alpha(\tfrac{1}{2}x^2) L_n^{-\frac{1}{2}-\alpha}(\tfrac{1}{2}x^2) \cos(xy) dx$ $= (\tfrac{1}{2}\pi)^{\frac{1}{2}} \exp(-\tfrac{1}{2}y^2) L_n^\alpha(\tfrac{1}{2}y^2) L_n^{-\alpha-\frac{1}{2}}(\tfrac{1}{2}y^2)$   |
| (13) | $\int_0^\infty \exp(-\tfrac{1}{2}x^2) L_n(\tfrac{1}{2}x^2) \text{He}_{2n+1}(\tfrac{1}{2}x) \sin(xy) dx$ $= (\tfrac{1}{2}\pi)^{\frac{1}{2}} \exp(-\tfrac{1}{2}y^2) L_n(\tfrac{1}{2}y^2) \text{He}_{2n+1}(\tfrac{1}{2}y)$                                       |
| (14) | $\int_0^\infty \exp(-\tfrac{1}{2}x^2) L_n(\tfrac{1}{2}x^2) \text{He}_{2n}(\tfrac{1}{2}x) \cos(xy) dx$ $= (\tfrac{1}{2}\pi)^{\frac{1}{2}} \exp(-\tfrac{1}{2}y^2) L_n(\tfrac{1}{2}y^2) \text{He}_{2n}(\tfrac{1}{2}y)$                                           |
| (15) | $\int_0^\infty x^{\rho-1} e^{-x} L_{m_1}^{\alpha_1}(\lambda_1 x) \cdots L_{m_n}^{\alpha_n}(\lambda_n x) dx$ <p>See Erdélyi, Arthur, 1936: <i>Math. Z.</i> 40, 693-702.</p>                                                                                    |

## CHAPTER XVII

### GAMMA FUNCTION, INCOMPLETE GAMMA FUNCTIONS, AND RELATED FUNCTIONS

For these functions see H.T.F. vol. I, Chapter I and vol. II, Chapter IX. The expressions, given below, of incomplete gamma functions and related functions in terms of confluent hypergeometric functions will assist in the evaluation of integrals involving these functions. For this reason, only a small selection of integrals involving incomplete gamma functions and their particular cases is given here.

#### Error functions and Fresnel integrals

$$\begin{aligned}
 \operatorname{Erf}(x) &= \pi^{-\frac{1}{2}} \gamma(\tfrac{1}{2}, x^2) \\
 &= 2\pi^{-\frac{1}{2}} x {}_1F_1(1/2; 3/2; -x^2) \\
 &= 2\pi^{-\frac{1}{2}} x e^{-x^2} {}_1F_1(1; 3/2; x^2) \\
 &= 2\pi^{-\frac{1}{2}} x^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} M_{-\frac{1}{4}, \frac{1}{4}}(x^2) \\
 &= 1 - \operatorname{Erfc}(x)
 \end{aligned}$$

$$\begin{aligned}
 \operatorname{Erfc}(x) &= \pi^{-\frac{1}{2}} \Gamma(\tfrac{1}{2}, x^2) \\
 &= 2^{\frac{1}{2}} \pi^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} D_{-1}(2^{\frac{1}{2}} x) \\
 &= \pi^{-\frac{1}{2}} x^{-1} e^{-x^2} {}_2F_0(1, \tfrac{1}{2}; -x^{-2}) \\
 &= \pi^{-\frac{1}{2}} x^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} W_{-\frac{1}{4}, \frac{1}{4}}(x^2) \\
 &= 1 - \operatorname{Erf}(x)
 \end{aligned}$$

$$\begin{aligned}
 C(x) \pm i S(x) &= e^{\pm \frac{1}{2} \pi i} \operatorname{Erf}(e^{\mp \frac{1}{2} \pi i} x^{\frac{1}{2}}) \\
 &= 2 \pi^{-\frac{1}{2}} x^{\frac{1}{2}} {}_1F_1(1/2; 3/2; \pm ix)
 \end{aligned}$$

### Exponential integral and related functions

$$\begin{aligned}
 -\operatorname{Ei}(-x) &= E_1(x) = \Gamma(0, x) \\
 &= x^{-1} e^{-x} {}_2F_0(1, 1; -x^{-1}) \\
 &= x^{-\frac{1}{2}} e^{-\frac{1}{2}x} W_{-\frac{1}{2}, 0}(x)
 \end{aligned}$$

$$\begin{aligned}
 \overline{\operatorname{Ei}}(x) &= \frac{1}{2} [\operatorname{Ei}(x + i0) + \operatorname{Ei}(x - i0)] \\
 &= x^{-1} e^x {}_2F_0(1, 1; x^{-1})
 \end{aligned}$$

$$\begin{aligned}
 \operatorname{Ci}(x) \pm i \operatorname{si}(x) &= -\operatorname{ci}(x) \pm i \operatorname{si}(x) \\
 &= \operatorname{Ei}(\pm ix) = -\Gamma(0, \mp ix) \\
 &= \mp ix^{-1} e^{\pm ix} {}_2F_0(1, 1; \mp ix^{-1}) \\
 &= -x^{-\frac{1}{2}} e^{\pm \frac{1}{2} \pi i \pm \frac{1}{2} xi} W_{-\frac{1}{2}, 0}(\mp ix)
 \end{aligned}$$

### Incomplete gamma functions

$$\begin{aligned}
 \gamma(a, x) &= a^{-1} x^a {}_1F_1(a; a+1; -x) \\
 &= a^{-1} x^{\frac{1}{2}a-\frac{1}{2}} e^{-\frac{1}{2}x} M_{\frac{1}{2}a-\frac{1}{2}, \frac{1}{2}a}(x) \\
 &= \Gamma(a) - \Gamma(a, x)
 \end{aligned}$$

$$\begin{aligned}
 \Gamma(a, x) &= x^{a-1} e^{-x} {}_2F_0(1, 1-a; -x^{-1}) \\
 &= x^{\frac{1}{2}a-\frac{1}{2}} e^{-\frac{1}{2}x} W_{\frac{1}{2}a-\frac{1}{2}, \frac{1}{2}a}(x) \\
 &= \Gamma(a) - \gamma(a, x)
 \end{aligned}$$

GAMMA FUNCTION, INCOMPLETE GAMMA FUNCTIONS, AND  
RELATED FUNCTIONS

**17.1. The gamma function**

|     |                                                                                                                                                                                                                                                               |
|-----|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $\int_{-\infty}^{\infty} \Gamma(a+x) \Gamma(\beta-x) dx = 0$ $\operatorname{Re}(a+\beta) < 1 \text{ and either } \operatorname{Im} a < 0 < \operatorname{Im} \beta \text{ or } \operatorname{Im} \beta < 0 < \operatorname{Im} a$                             |
| (2) | $\int_{-\infty}^{\infty} \Gamma(a+x) \Gamma(\beta-x) dx = i \pi 2^{1-a-\beta} \Gamma(a+\beta)$ $\operatorname{Re}(a+\beta) < 1, \quad \operatorname{Im} a, \operatorname{Im} \beta < 0$                                                                       |
| (3) | $\int_{-\infty}^{\infty} \Gamma(a+x) \Gamma(\beta-x) dx = -i \pi 2^{1-a-\beta} \Gamma(a+\beta)$ $\operatorname{Re}(a+\beta) < 1, \quad \operatorname{Im} a, \operatorname{Im} \beta > 0$                                                                      |
| (4) | $\int_{-\infty}^{\infty} \frac{\Gamma(a+x)}{\Gamma(\beta+x)} dx = 0$ $\operatorname{Im} a \neq 0, \quad \operatorname{Re}(a-\beta) < -1$                                                                                                                      |
| (5) | $\int_{-\infty}^{\infty} \frac{dx}{\Gamma(a+x) \Gamma(\beta-x)} = \frac{2^{a+\beta-2}}{\Gamma(a+\beta-1)}$ $\operatorname{Re}(a+\beta) > 1$                                                                                                                   |
| (6) | $\int_{-\infty}^{\infty} \frac{\Gamma(a+x)}{\Gamma(\beta+x)} \exp[(2\pi n + \pi - 2\theta)xi] dx = 0$ $\operatorname{Re}(\beta-a) > 0$ $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi, \quad n \text{ integer}, \quad (n + \frac{1}{2}) \operatorname{Im}(a) > 0$ |

**The gamma function (cont'd)**

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (7)  | $\int_{-\infty}^{\infty} \Gamma(a+x) \Gamma(\beta-x) \exp[2(\pi n + \theta) xi] dx$ $= 2\pi i \Gamma(a+\beta) (2 \cos \theta)^{-\alpha-\beta} \exp[(\beta-a)\theta i]$ $\times [\eta_n(\beta) \exp(2n\pi\beta i) - \eta_n(-a) \exp(-2n\pi a i)]$ $\begin{aligned} &\text{Re}(a+\beta) < 1, \quad -\frac{1}{2}\pi < \theta < \frac{1}{2}\pi, \quad n \text{ integer} \\ &\eta_n(\zeta) = 0 \quad \text{if} \quad (\tfrac{1}{2}-n) \text{Im } \zeta > 0 \\ &\eta_n(\zeta) = \text{sgn}(\tfrac{1}{2}-n) \quad \text{if} \quad (\tfrac{1}{2}-n) \text{Im } \zeta < 0 \end{aligned}$ |
| (8)  | $\int_{-\infty}^{\infty} \frac{\Gamma(a+x)}{\Gamma(\beta+x)} \exp[(2\pi n + \pi - 2\theta) xi] dx$ $= 2\pi i \text{sgn}(n + \tfrac{1}{2}) \frac{(2 \cos \theta)^{\beta-\alpha-1}}{\Gamma(\beta-a)} \exp[-(2\pi n + \pi - \theta) ai + (\beta-1)i]$ $\text{Re}(\beta-a) > 0, \quad -\frac{1}{2}\pi < \theta < \frac{1}{2}\pi, \quad n \text{ integer}, \quad (n + \tfrac{1}{2}) \text{Im } a < 0$                                                                                                                                                                                |
| (9)  | $\int_{-\infty}^{\infty} \frac{\sin(cx) dx}{\Gamma(a+x) \Gamma(\beta-x)} = 0 \quad \text{Re}(a+\beta) > 1, \quad c > \pi$                                                                                                                                                                                                                                                                                                                                                                                                                                                       |
| (10) | $\int_{-\infty}^{\infty} \frac{\sin(cx) dx}{\Gamma(a+x) \Gamma(\beta-x)} = \frac{[2 \cos(\tfrac{1}{2}c)]^{\alpha+\beta-2}}{\Gamma(a+\beta-1)} \sin[\tfrac{1}{2}c(\beta-a)]$ $\text{Re}(a+\beta) > 1, \quad 0 < c < \pi$                                                                                                                                                                                                                                                                                                                                                         |
| (11) | $\int_{-\infty}^{\infty} \frac{\sin(2n\pi x)}{\sin(\pi x)} \frac{dx}{\Gamma(a+x) \Gamma(\beta-x)} = 0$ $\text{Re}(a+\beta) > 1, \quad n \text{ integer}$                                                                                                                                                                                                                                                                                                                                                                                                                        |
| (12) | $\int_{-\infty}^{\infty} \frac{\sin[(2n+1)\pi x]}{\sin(\pi x)} \frac{dx}{\Gamma(a+x) \Gamma(\beta-x)} = \frac{2^{\alpha+\beta-2}}{\Gamma(a+\beta-1)}$ $\text{Re}(a+\beta) > 1, \quad n \text{ integer}$                                                                                                                                                                                                                                                                                                                                                                         |



## The gamma function (cont'd)

|      |                                                                                                                                                                                                                                                       |                                                                                                                 |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------|
| (13) | $\int_{-\infty}^{\infty} \frac{\cos(cx) dx}{\Gamma(a+x) \Gamma(\beta-x)} = 0$                                                                                                                                                                         | $\operatorname{Re}(a+\beta) > 1, \quad c > \pi$                                                                 |
| (14) | $\int_{-\infty}^{\infty} \frac{\cos(cx) dx}{\Gamma(a+x) \Gamma(\beta-x)} = \frac{[2 \cos(\frac{1}{2}c)]^{\alpha+\beta-2}}{\Gamma(a+\beta-1)} \cos[\frac{1}{2}c(\beta-a)]$                                                                             | $\operatorname{Re}(a+\beta) > 1, \quad 0 \leq c < \pi$                                                          |
| (15) | $\int_{-\infty}^{\infty} \frac{P(x) e^{icx} dx}{\Gamma(a+x) \Gamma(\beta-x)}, \quad P(x) \text{ polynomial}$                                                                                                                                          | See Ramanujan, Srinivasa, 1920: <i>Quart. J. Math.</i> 48, 294-310.                                             |
| (16) | $\int_{-\infty}^{\infty} \frac{\Phi(x) \exp[(2\pi n + \theta)xi] dx}{\Gamma(a+x) \Gamma(\beta-x)}$<br>$= \frac{[2 \cos(\frac{1}{2}\theta)]^{\alpha+\beta-2}}{\Gamma(a+\beta-1)} \exp[\frac{1}{2}\theta(\beta-a)i] \int_0^1 \Phi(t) \exp(2\pi nti) dt$ | $\operatorname{Re}(a+\beta) > 1, \quad -\pi < \theta < \pi, \quad n \text{ integer}, \quad \Phi(x+1) = \Phi(x)$ |
| (17) | $\int_{-\infty}^{\infty} \frac{\Phi(x) e^{ixc} dx}{\Gamma(a+x) \Gamma(\beta-x)}, \quad \Phi(x) \text{ periodic, period real}$                                                                                                                         | See Ramanujan, Srinivasa, 1920: <i>Quart. J. Math.</i> 48, 294-310.                                             |
| (18) | $\int_{-\infty}^{\infty} \frac{\Gamma(\gamma+x) \Gamma(\delta+x)}{\Gamma(a+x) \Gamma(\beta+x)} dx = 0$                                                                                                                                                | $\operatorname{Re}(a+\beta-\gamma-\delta) > 1, \quad \operatorname{Im} \gamma, \operatorname{Im} \delta > 0$    |

## The gamma function (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (19) | $\int_{-\infty}^{\infty} \frac{\Gamma(\gamma+x) \Gamma(\delta+x)}{\Gamma(a+x) \Gamma(\beta+x)} dx$ $= \frac{\pm 2\pi^2 i \Gamma(a+\beta-\gamma-\delta-1)}{\sin[\pi(\gamma-\delta)] \Gamma(a-\gamma) \Gamma(a-\delta) \Gamma(\beta-\gamma) \Gamma(\beta-\delta)}$ <p style="text-align: right;"> <math>\text{Re}(a+\beta-\gamma-\delta) &gt; 1, \quad \text{Im } \gamma, \text{ Im } \delta &lt; 0</math><br/> <math>\pm \text{ according as } \text{Im } \gamma \gtrless \text{Im } \delta</math> </p> |
| (20) | $\int_{-\infty}^{\infty} \frac{\Gamma(a-\beta-\gamma+x+1) dx}{\Gamma(a+x) \Gamma(\beta-x) \Gamma(\gamma+x)}$ $= \frac{\pi \exp[\pm \frac{1}{2} \pi(\delta-\gamma)i]}{\Gamma(\beta+\gamma-1) \Gamma(\frac{a+\beta}{2}) \Gamma(\frac{\gamma-\delta+1}{2})}$ <p style="text-align: right;"> <math>\text{Re}(\beta+\gamma) &gt; 1, \quad \delta = a - \beta - \gamma + 1, \quad \text{Im } \delta \neq 0</math><br/> <math>\pm \text{ according as } \text{Im } \delta \gtrless 0</math> </p>              |
| (21) | $\int_{-\infty}^{\infty} \frac{dx}{\Gamma(a+x) \Gamma(\beta-x) \Gamma(\gamma+x) \Gamma(\delta-x)}$ $= \frac{\Gamma(a+\beta+\gamma+\delta-3)}{\Gamma(a+\beta-1) \Gamma(\beta+\gamma-1) \Gamma(\gamma+\delta-1) \Gamma(\delta+a-1)}$ <p style="text-align: right;"><math>\text{Re}(a+\beta+\gamma+\delta) &gt; 3</math></p>                                                                                                                                                                              |
| (22) | $\int_{-\infty}^{\infty} \frac{\sin(\pi x) dx}{\Gamma(a+x) \Gamma(\beta-x) \Gamma(\gamma+x) \Gamma(\delta-x)}$ $= \frac{\sin[\frac{1}{2} \pi(\beta-a)]}{2 \Gamma(\frac{a+\beta}{2}) \Gamma(\frac{\gamma+\delta}{2}) \Gamma(a+\delta-1)}$ <p style="text-align: right;"><math>a+\delta = \beta+\gamma, \quad \text{Re}(a+\beta+\gamma+\delta) &gt; 2</math></p>                                                                                                                                         |

## The gamma function (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                  |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (23) | $\int_{-\infty}^{\infty} \frac{\cos(\pi x) dx}{\Gamma(a+x) \Gamma(\beta-x) \Gamma(\gamma+x) \Gamma(\delta-x)}$ $= \frac{\cos[\frac{1}{2}\pi(\beta-a)]}{2\Gamma(\frac{a+\beta}{2}) \Gamma(\frac{\gamma+\delta}{2}) \Gamma(a+\delta-1)}$ $a+\delta=\beta+\gamma, \quad \operatorname{Re}(a+\beta+\gamma+\delta) > 2$                                               |
| (24) | $\int_{-\infty}^{\infty} \frac{\Phi(x) dx}{\Gamma(a+x) \Gamma(\beta-x) \Gamma(\gamma+x) \Gamma(\delta-x)}$ $= \frac{\Gamma(a+\beta+\gamma+\delta-3)}{\Gamma(a+\beta-1) \Gamma(\beta+\gamma-1) \Gamma(\gamma+\delta-1) \Gamma(\delta+a-1)} \int_0^1 \Phi(t) dt$ $\operatorname{Re}(a+\beta+\gamma+\delta) > 3, \quad \Phi(x+1) = \Phi(x)$                         |
| (25) | $\int_{-\infty}^{\infty} \frac{\Phi(x) dx}{\Gamma(a+x) \Gamma(\beta-x) \Gamma(\gamma+x) \Gamma(\delta-x)}$ $= \frac{\int_0^1 \Phi(t) \cos[\frac{1}{2}\pi(2t+a-\beta)] dt}{\Gamma(\frac{a+\beta}{2}) \Gamma(\frac{\gamma+\delta}{2}) \Gamma(a+\delta-1)}$ $a+\delta=\beta+\gamma, \quad \operatorname{Re}(a+\beta+\gamma+\delta) > 2, \quad \Phi(x+1) = -\Phi(x)$ |
| (26) | $\int_{-\infty}^{\infty} [\Gamma(a+x) \Gamma(\beta-x) \Gamma(\gamma+kx) \Gamma(\delta-kx)]^{-1} \exp(\pi cxi) dx = 0$ $\operatorname{Re}(a+\beta+\gamma+\delta) > 2, \quad c, k \text{ real}, \quad  c  >  k  + 1$                                                                                                                                               |

For further similar integrals see Ramanujan, Srinivasa, 1920: *Quart. J. Math.* 48, 294-310.

## The gamma function (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                                                        |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (27) | $\int_0^\infty  \Gamma(a+ix) \Gamma(b+ix) ^2 dx = \frac{1}{2} \pi^{\frac{1}{2}} \Gamma(a) \Gamma(a+\frac{1}{2}) \Gamma(b) \Gamma(b+\frac{1}{2})$ $\times \Gamma(a+b)/\Gamma(a+b+\frac{1}{2}) \quad a > 0, \quad b > 0$                                                                                                                                                                                 |
| (28) | $\int_0^\infty \left  \frac{\Gamma(a+ix)}{\Gamma(b+ix)} \right ^2 dx = \frac{\frac{1}{2} \pi^{\frac{1}{2}} \Gamma(a) \Gamma(a+\frac{1}{2}) \Gamma(b-a-\frac{1}{2})}{\Gamma(b) \Gamma(b-\frac{1}{2}) \Gamma(b-a)}$ $0 < a < b - \frac{1}{2}$                                                                                                                                                            |
| (29) | $(2\pi i)^{-1} \int_{-i\infty}^{i\infty} \Gamma(s-\kappa-\lambda) \Gamma(\lambda+\mu-s+\frac{1}{2}) \Gamma(\lambda-\mu-s+\frac{1}{2}) z^s ds$ $= \Gamma(\frac{1}{2}-\kappa-\mu) \Gamma(\frac{1}{2}-\kappa+\mu) z^\lambda e^{\frac{1}{2}z} W_{\kappa,\mu}(z)$ $\operatorname{Re}(\kappa+\lambda) < 0, \quad \operatorname{Re} \lambda >  \operatorname{Re} \mu  - \frac{1}{2}, \quad  \arg z  < 3\pi/2$ |
| (30) | $\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(\lambda+\mu-s+\frac{1}{2}) \Gamma(\lambda-\mu-s+\frac{1}{2})}{\Gamma(\lambda-\kappa-s+1)} z^s ds = z^\lambda e^{-\frac{1}{2}z} W_{\kappa,\mu}(z)$ $\operatorname{Re} \lambda >  \operatorname{Re} \mu  - \frac{1}{2}, \quad  \arg z  < \frac{1}{2}\pi$                                                                                        |
| (31) | $\frac{1}{(2\pi i)} \int_{-i\infty}^{i\infty} \frac{\Gamma(\kappa-\lambda+s) \Gamma(\lambda+\mu-s+\frac{1}{2})}{\Gamma(\mu-\lambda+s+\frac{1}{2})} z^s ds$ $= \frac{\Gamma(\kappa+\mu+\frac{1}{2})}{\Gamma(2\mu+1)} z^\lambda e^{-\frac{1}{2}z} M_{\kappa,\mu}(z)$ $\operatorname{Re}(\kappa-\lambda) > 0, \quad \operatorname{Re}(\lambda+\mu) > -\frac{1}{2}, \quad  \arg z  < \frac{1}{2}\pi$       |
| (32) | $\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(a+s) \Gamma(\beta+s) \Gamma(\gamma-s) \Gamma(\delta-s) ds$ $= \frac{\Gamma(a+\gamma) \Gamma(a+\delta) \Gamma(\beta+\gamma) \Gamma(\beta+\delta)}{\Gamma(a+\beta+\gamma+\delta)}$ $\operatorname{Re} \alpha, \quad \operatorname{Re} \beta, \quad \operatorname{Re} \gamma, \quad \operatorname{Re} \delta > 0$                                      |

## The gamma function (cont'd)

$$(33) \quad \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left[ \frac{\Gamma(\frac{1}{2}-s)}{\Gamma(s)} \right]^2 z^s ds = z^{\frac{1}{2}} [2\pi^{-1} K_0(4z^{\frac{1}{4}}) - Y_0(4z^{\frac{1}{4}})]$$

$z > 0$

$$(34) \quad \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} z^s ds$$

$$= G_{pq}^{mn} \left( z \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right)$$

$p + q < 2(m + n), \quad |\arg z| < (m + n - \frac{1}{2}p - \frac{1}{2}q)\pi$

$\operatorname{Re} a_j < 1 \quad j = 1, \dots, n, \quad \operatorname{Re} b_j > 0 \quad j = 1, \dots, m$

$$(35) \quad \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} z^s ds$$

$$= G_{pq}^{mn} \left( z \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right)$$

$p + q \leq 2(m + n), \quad |\arg z| \leq (m + n - \frac{1}{2}p - \frac{1}{2}q)\pi$

$\operatorname{Re} a_j < 1 \quad j = 1, \dots, n, \quad \operatorname{Re} b_j > 0 \quad j = 1, \dots, m$

$\operatorname{Re} \left( \sum_{j=1}^p a_j - \sum_{j=1}^q b_j \right) > \frac{1}{2}p - \frac{1}{2}q + 1$

For further integrals of this type see sec. 7.3.

An empty product is interpreted as 1.

**The gamma function (cont'd)**

|      |                                                                                                                                                                                  |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (36) | $\int_0^1 \sin(2\pi nx) \log[\Gamma(a+x)] dx$ $= -(2n\pi)^{-1} [\log a + \cos(2n\pi a) \text{ci}(2n\pi a) - \sin(2n\pi a) \text{si}(2n\pi a)]$ $a > 0, \quad n = 1, 2, 3, \dots$ |
| (37) | $\int_0^1 \cos(2\pi nx) \log[\Gamma(a+x)] dx$ $= -(2n\pi)^{-1} [\sin(2n\pi a) \text{ci}(2n\pi a) + \cos(2n\pi a) \text{si}(2n\pi a)]$ $a > 0, \quad n = 1, 2, 3, \dots$          |
| (38) | $\int_0^1 \exp(2\pi mxi) \log[\Gamma(a+x)] dx$ $= (2n\pi i)^{-1} [\log a - \exp(-2\pi mai) \text{Ei}(2n\pi ia)]$ $a > 0, \quad n = \pm 1, \pm 2, \dots$                          |
| (39) | $\int_0^1 \log[\Gamma(x)] dx = \frac{1}{2} \log(2\pi)$                                                                                                                           |
| (40) | $\int_0^1 \log[\Gamma(a+x)] dx = a \log a - a + \frac{1}{2} \log(2\pi)$ $a \geq 0$                                                                                               |
| (41) | $\int_0^n \log[\Gamma(a+x)] dx = \sum_{k=0}^{n-1} (a+k) \log(a+k) - na$ $+ \frac{1}{2} n \log(2\pi) - \frac{1}{2} n(n-1)$ $a \geq 0, \quad n = 1, 2, 3, \dots$                   |
| (42) | $\int_0^1 \sin(2\pi nx) \log[\Gamma(x)] dx = (2\pi n)^{-1} \log(2\pi \gamma n)$ $n = 1, 2, \dots$                                                                                |

**The gamma function (cont'd)**

|      |                                                                                                                                                                                                                       |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (43) | $\int_0^1 \sin[(2n+1)\pi x] \log[\Gamma(x)] dx$ $= \frac{1}{(2n+1)\pi} \left[ \log\left(\frac{\pi}{2}\right) + 2 \left(1 + \frac{1}{3} + \dots + \frac{1}{2n-1}\right) + \frac{1}{2n+1} \right]$ $n = 0, 1, 2, \dots$ |
| (44) | $\int_0^1 \cos(2\pi n x) \log[\Gamma(x)] dx = \frac{1}{4n}$ $n = 1, 2, 3, \dots$                                                                                                                                      |

**17.2. The  $\psi$  function**

|     |                                                                                                                                                 |
|-----|-------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $\int_0^1 \psi(a+x) dx = \log a$ $a > 0$                                                                                                        |
| (2) | $\int_0^1 e^{2n\pi xi} \psi(a+x) dx = e^{-2n\pi ai} \text{Ei}(2n\pi ai)$ $a > 0, \quad n = \pm 1, \pm 2, \dots$                                 |
| (3) | $\int_0^1 \sin(2n\pi x) \psi(x) dx = -\frac{1}{2}\pi$ $n = 1, 2, \dots$                                                                         |
| (4) | $\int_0^1 \sin(2n\pi x) \psi(a+x) dx = \sin(2n\pi a) \text{ci}(2n\pi a)$ $+ \cos(2n\pi a) \text{si}(2n\pi a)$ $a \geq 0, \quad n = 1, 2, \dots$ |
| (5) | $\int_0^1 \cos(2n\pi x) \psi(a+x) dx = \sin(2n\pi a) \text{si}(2n\pi a)$ $- \cos(2n\pi a) \text{ci}(2n\pi a)$ $a > 0, \quad n = 1, 2, \dots$    |
| (6) | $\int_0^\infty x^{-a} [C + \psi(1+x)] dx = -\pi \csc(\pi a) \zeta(a) \bullet$ $1 < \text{Re } a < 2$                                            |

**The  $\psi$  function (cont'd)**

|      |                                                                                                                                                                                                                     |                                |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------|
| (7)  | $\int_0^\infty x^{-a} [\log x - \psi(1+x)] dx = \pi \csc(\pi a) \zeta(a)$                                                                                                                                           | $0 < \operatorname{Re} a < 1$  |
| (8)  | $\int_0^\infty x^{-a} [\log(1+x) - \psi(1+x)] dx = \pi \csc(\pi a) [\zeta(a) - (a-1)^{-1}]$                                                                                                                         | $0 < \operatorname{Re} a < 1$  |
| (9)  | $\int_0^\infty x^{-a} [(1+x)^{-1} - \psi'(1+x)] dx = -\pi a \csc(\pi a) [\zeta(1+a) - a^{-1}]$                                                                                                                      | $-1 < \operatorname{Re} a < 1$ |
| (10) | $\int_0^\infty x^{-a} [x^{-1} - \psi'(1+x)] dx = -\pi a \csc(\pi a) \zeta(1+a)$                                                                                                                                     | $-2 < \operatorname{Re} a < 0$ |
| (11) | $\int_0^\infty x^{-a} \psi^{(n)}(1+x) dx = (-1)^{n-1} \frac{\pi \Gamma(a+n)}{\Gamma(a) \sin \pi a} \zeta(a+n)$<br>$n = 1, 2, \dots, \quad \psi^{(n)}(z) = \frac{d^n \psi}{dz^n}, \quad 0 < \operatorname{Re} a < 1$ |                                |
| (12) | $\int_0^\infty [\psi(x+1) - \log x] \cos(2\pi xy) dx = \frac{1}{2} [\psi(y+1) - \log y]$                                                                                                                            |                                |

**17.3. Incomplete gamma functions and related functions**

|     |                                                                                                                                                                                                                               |                                                                                         |
|-----|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------|
| (1) | $\int_0^\infty x^{\rho-1} \operatorname{Erfc}(ax) dx = \frac{\Gamma(\frac{1}{2}\rho + \frac{1}{2})}{\pi^{\frac{1}{2}} \rho a^\rho}$                                                                                           | $\operatorname{Re} a > 0, \quad \operatorname{Re} \rho > 0$                             |
| (2) | $\int_0^\infty x^{\nu-1} \exp(\beta^2 x^2) \operatorname{Erfc}(ax) dx = \frac{\Gamma(\frac{\nu+1}{2})}{\pi^{\frac{1}{2}} \nu a^\nu} {}_2F_1\left(\frac{\nu}{2}, \frac{\nu+1}{2}; \frac{\nu}{2}+1; \frac{\beta^2}{a^2}\right)$ | $\operatorname{Re} \nu > 0, \quad 0, \operatorname{Re} \beta^2 < \operatorname{Re} a^2$ |



## Incomplete gamma functions etc. (cont'd)

|     |                                                                                                                                                                                                                                                                                                                           |
|-----|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (3) | $\int_0^\infty x^{\nu-1} \sin(\beta x) \operatorname{Erfc}(ax) dx$ $= \frac{\Gamma(1 + \frac{1}{2}\nu) b}{\pi^{\frac{1}{2}} (\nu + 1) a^{\nu+1}} {}_2F_2\left(\frac{\nu+1}{2}, \frac{\nu}{2} + 1; \frac{3}{2}, \frac{\nu+3}{2}; -\frac{\beta^2}{4a^2}\right)$ $\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1$ |
| (4) | $\int_0^\infty x^{\nu-1} \cos(\beta x) \operatorname{Erfc}(ax) dx$ $= \frac{\Gamma(\frac{1}{2} + \frac{1}{2}\nu)}{\pi^{\frac{1}{2}} \nu a^\nu} {}_2F_2\left(\frac{\nu}{2}, \frac{\nu+1}{2}; \frac{1}{2}, \frac{\nu}{2} + 1; -\frac{\beta^2}{4a^2}\right)$ $\operatorname{Re} \nu > 0, \quad \operatorname{Re} a > 0$      |
| (5) | $\int_0^\infty e^{\beta x} \operatorname{Erfc}(a^{\frac{1}{2}} x^{\frac{1}{2}}) dx = \frac{1}{\beta} \left[ \frac{a^{\frac{1}{2}}}{(a - \beta)^{\frac{1}{2}}} - 1 \right] \quad 0, \operatorname{Re} \beta < \operatorname{Re} a$                                                                                         |
| (6) | $\int_0^\infty \sin(\beta x) \operatorname{Erfc}(a^{\frac{1}{2}} x^{\frac{1}{2}}) dx = \frac{1}{\beta} - \left( \frac{\frac{1}{2}a}{a^2 + \beta^2} \right)^{\frac{1}{2}} [(a^2 + \beta^2)^{\frac{1}{2}} - a]^{-\frac{1}{2}}$ $\operatorname{Re} a >  \operatorname{Im} \beta $                                            |
| (7) | $\int_0^\infty \cos(\beta x) \operatorname{Erfc}(a^{\frac{1}{2}} x^{\frac{1}{2}}) dx = \left( \frac{\frac{1}{2}a}{a^2 + \beta^2} \right)^{\frac{1}{2}} [(a^2 + \beta^2)^{\frac{1}{2}} + a]^{-\frac{1}{2}}$ $\operatorname{Re} a >  \operatorname{Im} \beta $                                                              |
| (8) | $\int_0^\infty \sin(bx) \operatorname{Erfc}(a^{\frac{1}{2}} x^{-\frac{1}{2}}) dx = b^{-1} \exp[-(2ab)^{\frac{1}{2}}] \cos[(2ab)^{\frac{1}{2}}]$ $\operatorname{Re} a > 0, \quad b > 0$                                                                                                                                    |
| (9) | $\int_0^\infty \cos(bx) \operatorname{Erfc}(a^{\frac{1}{2}} x^{-\frac{1}{2}}) dx = -b^{-1} \exp[-(2ab)^{\frac{1}{2}}] \sin[(2ab)^{\frac{1}{2}}]$ $\operatorname{Re} a > 0, \quad b > 0$                                                                                                                                   |

## Incomplete gamma functions etc. (cont'd)

|                                                                                                                                                                                                                                                                                                                                                            |                                                                                                                                                                                                                                                                                                     |
|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (10)                                                                                                                                                                                                                                                                                                                                                       | $\int_0^\infty \cosh(2\nu t) \exp[(a \cosh t)^2] \operatorname{Erfc}(a \cosh t) dt$ $= \frac{1}{2} \sec(\nu\pi) \exp(\frac{1}{2}a^2) K_\nu(a^2) \quad \operatorname{Re} a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                                                            |
| <p>For integrals involving products of error functions and other confluent hypergeometric functions see Bock, Philipp, 1939: <i>Compositio Math.</i> 7, 123-134. Note that the function denoted by <math>\operatorname{Erfc}</math> by Bock is <math>\frac{1}{2}\pi^{\frac{1}{2}} \operatorname{Erfc}</math> in the notation used in the present work.</p> |                                                                                                                                                                                                                                                                                                     |
| (11)                                                                                                                                                                                                                                                                                                                                                       | $\int_0^a e^x \operatorname{Ei}(-x) dx = -\log(a\gamma) + e^a \operatorname{Ei}(-a)$                                                                                                                                                                                                                |
| (12)                                                                                                                                                                                                                                                                                                                                                       | $\int_0^c e^{-\beta x} \operatorname{Ei}(-ax) dx = -\beta^{-1} \{ e^{-\beta c} \operatorname{Ei}(-ac) + \log(1 + \beta/a) - \operatorname{Ei}[-(a + \beta)c] \}$                                                                                                                                    |
| (13)                                                                                                                                                                                                                                                                                                                                                       | $\int_0^\infty x^\nu e^x \operatorname{Ei}(-x) dx = \pi \csc(\pi\nu) \Gamma(\nu + 1) \quad -1 < \operatorname{Re} \nu < 0$                                                                                                                                                                          |
| (14)                                                                                                                                                                                                                                                                                                                                                       | $\int_0^\infty x^{\nu-1} e^{-\beta x} \operatorname{Ei}(-ax) dx = -\frac{\Gamma(\nu)}{\nu(a + \beta)^\nu} {}_2F_1\left(1, \nu; \nu + 1; \frac{\beta}{a + \beta}\right)$ $ \arg a  < \pi, \quad \operatorname{Re}(a + \beta) > 0, \quad \operatorname{Re} \nu > 0$                                   |
| <p>For further integrals involving <math>-\operatorname{Ei}(-x) = E_1(x)</math> see LeCaine, J., 1948: <i>National Research Council of Canada, Division of Atomic Energy</i>, Document No. MT-131(NRC 1553), 45 pp.. and Busbridge, I.W., 1950: <i>Quart. J. Math. Oxford Ser. (2)</i> 1, 176-184.</p>                                                     |                                                                                                                                                                                                                                                                                                     |
| (15)                                                                                                                                                                                                                                                                                                                                                       | $\int_0^\infty x^{\mu-1} e^{-\beta x} \gamma(\nu, ax) dx = \frac{a^\nu \Gamma(\mu + \nu)}{\nu(a + \beta)^{\mu + \nu}} {}_2F_1\left(1, \mu + \nu; \nu + 1; \frac{a}{a + \beta}\right)$ $\operatorname{Re}(a + \beta) > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re}(\mu + \nu) > 0$ |

**Incomplete gamma functions etc. (cont'd)**

|                                                                                                                                                                                                                                                                                                   |                                                                                                                                                                                                                                                                                                     |
|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (16)                                                                                                                                                                                                                                                                                              | $\int_0^\infty x^{\mu-1} e^{-\beta x} \Gamma(\nu, ax) dx = \frac{a^\nu \Gamma(\mu + \nu)}{\mu(a + \beta)^{\mu+\nu}} {}_2F_1\left(1, \mu + \nu; \mu + 1; \frac{\beta}{a + \beta}\right)$ $\operatorname{Re}(a + \beta) > 0, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re}(\mu + \nu) > 0$ |
| (17)                                                                                                                                                                                                                                                                                              | $\int_0^\infty e^{-\beta x} \gamma(\nu, ax^2) dx = 2^{1-\nu} \beta^{-1} \Gamma(2\nu) \exp\left(\frac{\beta^2}{8a}\right) D_{-2\nu}\left[\frac{\beta}{(2a)^{1/2}}\right]$ $\operatorname{Re} a > 0, \quad \operatorname{Re} \beta > 0, \quad \nu \neq 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$ |
| (18)                                                                                                                                                                                                                                                                                              | $\int_0^\infty x^{1-2\nu} \exp(ax^2) \sin(bx) \Gamma(\nu, ax^2) dx$ $= \pi^{1/2} 2^{-\nu} a^{\nu-1} \Gamma\left(\frac{3}{2} - \nu\right) \exp\left(\frac{b^2}{8a}\right) D_{2\nu-2}\left[\frac{b}{(2a)^{1/2}}\right]$ $ \arg a  < 3\pi/2, \quad 0 < \operatorname{Re} \nu < 1$                      |
| <p>For other integrals involving</p> $E_n(x) = x^{n-1} \Gamma(1-n, x)$ <p>see LeCaine, J., 1948: <i>National Research Council of Canada, Division of Atomic Energy</i>, Document No. MT-131 (NRC 1553), 45 pp., and Busbridge, I.W., 1950: <i>Quart. J. Math. Oxford Ser. (2)</i> 1, 176-184.</p> |                                                                                                                                                                                                                                                                                                     |
| (19)                                                                                                                                                                                                                                                                                              | $\int_0^\infty e^{-\beta x} \gamma(\nu, ax^{1/2}) dx = 2^{-1/2} \nu a^\nu \beta^{-1/2} \nu^{-1} \Gamma(\nu) \exp\left(\frac{a^2}{8\beta}\right)$ $\times D_{-\nu}\left[\frac{a}{(2\beta)^{1/2}}\right]$ $\operatorname{Re} \beta > 0, \quad \operatorname{Re} \nu > 0$                              |



## CHAPTER XVIII

### LEGENDRE FUNCTIONS

For the theory of these functions see H.T.F. vol. I, Chapter III and the literature quoted there, especially the books by Hobson, MacRobert, Whittaker and Watson. Numerous expansions of Legendre functions in hypergeometric series are listed in H.T.F. vol. I, p. 124-139, and these may be used to reduce integrals involving Legendre functions to integrals involving hypergeometric series.



# LEGENDRE FUNCTIONS

## 18.1. Legendre functions of variable $\alpha x + \beta$ : finite intervals

|     |                                                                                                                                                                                                                                                                                                                                                                                                                                      |
|-----|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $\int_0^1 x^{\lambda-1} P_\nu(x) dx = \frac{\pi^{\frac{1}{2}} 2^{-\lambda} \Gamma(\lambda)}{\Gamma(\frac{1}{2} + \frac{1}{2}\lambda - \frac{1}{2}\nu) \Gamma(1 + \frac{1}{2}\lambda + \frac{1}{2}\nu)} \quad \text{Re } \lambda > 0$                                                                                                                                                                                                 |
| (2) | $\int_0^1 x^{\lambda-1} P_\nu^m(x) dx = \frac{(-1)^m \pi^{\frac{1}{2}} 2^{-2m-1} \Gamma(\frac{1}{2}\lambda) \Gamma(1+m+\nu)}{\Gamma(\frac{1}{2} + \frac{1}{2}m) \Gamma(1 + \frac{1}{2}\lambda + m) \Gamma(1-m+\nu)}$ $\times {}_3F_2\left(\frac{m+\nu+1}{2}, \frac{m-\nu}{2}, \frac{m}{2}+1; m+1, \frac{\lambda+m}{2}+1; 1\right)$ <p style="text-align: right;">Re <math>\lambda &gt; 0</math>, <math>m = 0, 1, 2, \dots</math></p> |
| (3) | $\int_0^1 x^{\lambda-1} P_\nu^\mu(x) dx = \frac{\pi^{\frac{1}{2}} 2^{2\mu-1} \Gamma(\frac{1}{2}\lambda)}{\Gamma(\frac{1}{2} - \frac{1}{2}\mu) \Gamma(1 + \frac{1}{2}\lambda - \frac{1}{2}\mu)}$ $\times {}_3F_2\left(\frac{\nu-\mu+1}{2}, -\frac{\mu+\nu}{2}, 1-\frac{\mu}{2}; 1-\mu, \frac{\lambda-\mu}{2}+1; 1\right)$ <p style="text-align: right;">Re <math>\lambda &gt; 0</math>, Re <math>\mu &lt; 2</math></p>                |
| (4) | $\int_0^1 x^{\lambda-1} (1-x^2)^{\frac{1}{2}m} P_\nu^m(x) dx$ $= \frac{(-1)^m \pi^{\frac{1}{2}} 2^{-\lambda-m} \Gamma(\lambda) \Gamma(1+m+\nu)}{\Gamma(\frac{1}{2} + \frac{1}{2}\lambda + \frac{1}{2}m - \frac{1}{2}\nu) \Gamma(1 + \frac{1}{2}\lambda + \frac{1}{2}m + \frac{1}{2}\nu) \Gamma(1-m+\nu)}$ <p style="text-align: right;">Re <math>\lambda &gt; 0</math>, <math>m = 0, 1, 2, \dots</math></p>                          |

Variable  $\alpha x + \beta$ : finite intervals (cont'd)

|     |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             |
|-----|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (5) | $\int_0^1 x^{\lambda-1} (1-x^2)^{-\frac{1}{2}\mu} P_\nu^\mu(x) dx$ $= \frac{\pi^{\frac{1}{2}} 2^{\mu-\lambda} \Gamma(\lambda)}{\Gamma(\frac{1}{2} + \frac{1}{2}\lambda - \frac{1}{2}\mu - \frac{1}{2}\nu) \Gamma(1 + \frac{1}{2}\lambda - \frac{1}{2}\mu + \frac{1}{2}\nu)}$ <p style="text-align: right;"><math>\operatorname{Re} \lambda &gt; 0, \quad \operatorname{Re} \mu &lt; 1</math></p>                                                                                                                                            |
| (6) | $\int_0^1 x^{\lambda-1} (1-x^2)^\kappa P_\nu^\mu(x) dx = \frac{2^{\mu-1} \Gamma(1+\kappa-\frac{1}{2}\mu) \Gamma(\frac{1}{2}\lambda)}{\Gamma(1-\mu) \Gamma(1+\kappa+\frac{1}{2}\lambda-\frac{1}{2}\mu)}$ $\times {}_3F_2\left(\frac{\nu-\mu+1}{2}, -\frac{\mu+\nu}{2}, 1+\kappa-\frac{\mu}{2}; 1-\mu, 1+\frac{\lambda-\mu}{2}+\kappa; 1\right)$ <p style="text-align: right;"><math>\operatorname{Re}(\kappa - \frac{1}{2}\mu) &gt; -1, \quad \operatorname{Re} \lambda &gt; 0</math></p>                                                    |
| (7) | $\int_0^1 x^{\lambda-1} (1-x^2)^{-\frac{1}{2}\mu} \sin(\alpha x) P_\nu^\mu(x) dx$ $= \frac{\pi^{\frac{1}{2}} 2^{\mu-\lambda-1} \Gamma(\lambda+1) \alpha}{\Gamma[1 + \frac{1}{2}(\lambda - \mu - \nu)] \Gamma[\frac{1}{2}(3 + \lambda - \mu + \nu)]}$ $\times {}_2F_3\left(\frac{1+\lambda}{2}, 1+\frac{\lambda}{2}; \frac{3}{2}, 1+\frac{\lambda-\mu-\nu}{2}, \frac{3+\lambda-\mu+\nu}{2}; -\frac{\alpha^2}{4}\right)$ <p style="text-align: right;"><math>\operatorname{Re} \lambda &gt; -1, \quad \operatorname{Re} \mu &lt; 1</math></p> |
| (8) | $\int_0^1 x^{\lambda-1} (1-x^2)^{-\frac{1}{2}\mu} \cos(\alpha x) P_\nu^\mu(x) dx$ $= \frac{\pi^{\frac{1}{2}} 2^{\mu-\lambda} \Gamma(\lambda)}{\Gamma[1 + \frac{1}{2}(\lambda - \mu + \nu)] \Gamma[\frac{1}{2}(1 + \lambda - \mu - \nu)]}$ $\times {}_2F_3\left(\frac{\lambda}{2}, \frac{\lambda+1}{2}; \frac{1}{2}, \frac{1+\lambda-\mu-\nu}{2}, 1+\frac{\lambda-\mu+\nu}{2}; -\frac{\alpha^2}{4}\right)$ <p style="text-align: right;"><math>\operatorname{Re} \lambda &gt; 0, \quad \operatorname{Re} \mu &lt; 1</math></p>               |



**Variable  $\alpha x + \beta$ : finite intervals (cont'd)**

|      |                                                                                                                                                                                                                                                                                                                                                                                                             |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (9)  | $\int_0^1 (1-x^2)^{-1} [P_\nu^n(x)]^2 dx = -\frac{n!}{2(n-\nu)\Gamma(1-n+2\nu)}$ $n = 0, 1, 2, \dots, \quad \text{Re } \nu > n$                                                                                                                                                                                                                                                                             |
| (10) | $\int_0^1 P_\lambda(x) P_\nu(x) dx = 2[\pi(\lambda-\nu)(\lambda+\nu+1)]^{-1}$ $\times [A \sin(\tfrac{1}{2}\lambda\pi) \cos(\tfrac{1}{2}\nu\pi) - A^{-1} \cos(\tfrac{1}{2}\lambda\pi) \sin(\tfrac{1}{2}\nu\pi)]$ $A = \frac{\Gamma(\tfrac{1}{2} + \tfrac{1}{2}\nu) \Gamma(1 + \tfrac{1}{2}\lambda)}{\Gamma(\tfrac{1}{2} + \tfrac{1}{2}\lambda) \Gamma(1 + \tfrac{1}{2}\nu)}$                                 |
| (11) | $\int_0^1 P_\nu(x) Q_\lambda(x) dx = [(\lambda-\nu)(\lambda+\nu+1)]^{-1} \{A^{-1} \cos[\tfrac{1}{2}(\nu-\lambda)\pi] - 1\}$ $A = \frac{\Gamma(1 + \tfrac{1}{2}\lambda) \Gamma(\tfrac{1}{2} + \tfrac{1}{2}\nu)}{\Gamma(\tfrac{1}{2} + \tfrac{1}{2}\lambda) \Gamma(1 + \tfrac{1}{2}\nu)}$                                                                                                                     |
| (12) | $\int_0^1 Q_\lambda(x) Q_\nu(x) dx = [(\lambda-\nu)(\lambda+\nu+1)]^{-1} \{\psi(\nu+1) - \psi(\lambda+1)$ $- \tfrac{1}{2}\pi(A - A^{-1}) \sin[\tfrac{1}{2}(\lambda+\nu)\pi] + \tfrac{1}{2}\pi(A + A^{-1}) \sin[\tfrac{1}{2}(\lambda-\nu)\pi]\}$ $A = \frac{\Gamma(1 + \tfrac{1}{2}\lambda) \Gamma(\tfrac{1}{2} + \tfrac{1}{2}\nu)}{\Gamma(\tfrac{1}{2} + \tfrac{1}{2}\lambda) \Gamma(1 + \tfrac{1}{2}\nu)}$ |
| (13) | $\int_0^1 [P_\nu^\mu(x)]^2 dx, \quad \int_0^1 P_\nu^\mu(x) Q_\nu^\mu(x) dx.$ <p>See Barnes, E.W., 1908: <i>Quart. J. Math.</i> 39, 97-204. Note that Barnes' definition of Legendre functions of the second kind differs from the one used in this book.</p>                                                                                                                                                |
| (14) | $\int_0^1 P_\nu^\pi(x) P_\lambda^\pi(x) dx$ <p>See Shabde, N.G., 1937: <i>Bull. Calcutta Math. Soc.</i> 29, 33-40.</p>                                                                                                                                                                                                                                                                                      |

**Variable  $\alpha x + \beta$ : finite intervals (cont'd)**

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (15) | $\int_{-1}^1 (1+x)^{\lambda-1} P_{\nu}(x) dx = \frac{2^{\lambda} [\Gamma(\lambda)]^2}{\Gamma(\lambda + \nu + 1) \Gamma(\lambda - \nu)} \quad \text{Re } \lambda > 0$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |
| (16) | $\int_{-1}^1 (1-x^2)^{\lambda-1} P_{\nu}^{\mu}(x) dx$ $= \frac{\pi 2^{\mu} \Gamma(\lambda + \frac{1}{2}\mu) \Gamma(\lambda - \frac{1}{2}\mu)}{\Gamma(\lambda + \frac{1}{2}\nu + 1) \Gamma(\lambda - \frac{1}{2}\nu) \Gamma(-\frac{1}{2}\mu + \frac{1}{2}\nu + 1) \Gamma(-\frac{1}{2}\mu - \frac{1}{2}\nu + \frac{1}{2})}$ $2 \text{Re}(\lambda) >  \text{Re } \mu $                                                                                                                                                                                                                                                                                                                                                                               |
| (17) | $\int_{-1}^1 (1-x^2)^{-\frac{1}{2}\mu} (z-x)^{-1} P_{\mu+n}^{\mu}(x) dx = 2e^{-i\mu\pi} (z^2-1)^{-\frac{1}{2}\mu} Q_{\mu+n}^{\mu}(z)$ $n = 0, 1, 2, \dots, \quad \text{Re } \mu + n > -1, \quad z \text{ in the cut complex plane}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               |
| (18) | $\int_{-1}^1 (1-x)^{-\frac{1}{2}\mu} (1+x)^{\frac{1}{2}\mu-\frac{1}{2}} (z+x)^{\mu-\frac{1}{2}} P_{\nu}^{\mu}(x) dx$ $= \frac{2e^{-2\mu\pi i} \Gamma(\frac{1}{2} + \mu)}{\pi^{\frac{1}{2}} \Gamma(\mu - \nu) \Gamma(\mu + \nu + 1)} (z-1)^{\mu}$ $\times Q_{\nu}^{\mu} \left[ \left( \frac{1+z}{2} \right)^{\frac{1}{2}} \right] Q_{-\nu-1}^{\mu} \left[ \left( \frac{1+z}{2} \right)^{\frac{1}{2}} \right]$ $-\frac{1}{2} < \text{Re } \mu < 1, \quad z \text{ in the cut complex plane}$                                                                                                                                                                                                                                                        |
| (19) | $\int_{-1}^1 (1-x)^{-\frac{1}{2}\mu-1} (1+x)^{\frac{1}{2}\mu-\frac{1}{2}} (z+x)^{\mu-\frac{1}{2}} [(\nu-\mu) P_{\nu}^{\mu}(x) - (\nu+\mu) P_{\nu-1}^{\mu}(x)] dx$ $= \frac{e^{-2\mu\pi i} \Gamma(\mu + \frac{1}{2}) (z-1)^{\mu}}{(\frac{1}{2}\pi)^{\frac{1}{2}} \Gamma(\mu + \nu) \Gamma(\mu - \nu) (z+1)^{\frac{1}{2}}} \left\{ Q_{\nu}^{\mu} \left[ \left( \frac{1+z}{2} \right)^{\frac{1}{2}} \right] Q_{-\nu}^{\mu} \left[ \left( \frac{1+z}{2} \right)^{\frac{1}{2}} \right] \right.$ $\left. + Q_{\nu-1}^{\mu} \left[ \left( \frac{1+z}{2} \right)^{\frac{1}{2}} \right] Q_{-\nu-1}^{\mu} \left[ \left( \frac{1+z}{2} \right)^{\frac{1}{2}} \right] \right\}$ $-\frac{1}{2} < \text{Re } \mu < 0, \quad z \text{ in the cut complex plane}$ |

The complex  $z$ -plane is cut along the real axis from  $-1$  to  $1$ .

Variable  $\alpha x + \beta$ : finite intervals (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (20) | $\int_{-1}^1 (1-x)^{-\frac{1}{2}\mu} (1+x)^{\frac{1}{2}\mu-\frac{1}{2}} (z+x)^{\mu-3/2} P_{\nu}^{\mu}(x) dx$ $= -\frac{\Gamma(\mu-\frac{1}{2})(z-1)^{\mu-\frac{1}{2}}(z+1)^{-\frac{1}{2}}}{\pi^{\frac{1}{2}} e^{2\mu\pi i} \Gamma(\mu+\nu) \Gamma(\mu-\nu-1)} \left\{ Q_{\nu}^{\mu} \left[ \left( \frac{1+z}{2} \right)^{\frac{1}{2}} \right] Q_{-\nu-1}^{\mu-1} \left[ \left( \frac{1+z}{2} \right)^{\frac{1}{2}} \right] \right.$ $\left. + Q_{\nu}^{\mu-1} \left[ \left( \frac{1+z}{2} \right)^{\frac{1}{2}} \right] Q_{-\nu-1}^{\mu} \left[ \left( \frac{1+z}{2} \right)^{\frac{1}{2}} \right] \right\}$ <p style="text-align: right;"><math>-\frac{1}{2} &lt; \text{Re } \mu &lt; 1, \quad z \text{ in the cut complex plane}</math></p> |
| (21) | $\int_{-1}^1 (1-x)^{-\frac{1}{2}\mu} (1+x)^{\frac{1}{2}\mu+\nu-1} \exp\left(-\frac{1-x}{1+x} y\right) P_{\nu}^{\mu}(x) dx$ $= 2^{\nu} y^{\frac{1}{2}\mu+\nu-\frac{1}{2}} e^{\frac{1}{2}y} W_{\frac{1}{2}\mu-\nu-\frac{1}{2}, \frac{1}{2}\mu}(y)$ <p style="text-align: right;"><math>\text{Re } y &gt; 0</math></p>                                                                                                                                                                                                                                                                                                                                                                                                                           |
| (22) | $\int_{-1}^1 P_{\nu}(x) P_{\lambda}(x) dx$ $= \frac{4 \sin(\nu\pi) \sin(\lambda\pi) [\psi(\nu+1) - \psi(\lambda+1)] + 2\pi \sin[(\lambda-\nu)\pi]}{\pi^2 (\lambda-\nu) (\lambda+\nu+1)}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      |
| (23) | $\int_{-1}^1 [P_{\nu}(x)]^2 dx = \frac{\pi^2 - 2[\sin(\nu\pi)]^2 \psi'(\nu+1)}{\pi^2 (\nu+\frac{1}{2})}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      |
| (24) | $\int_{-1}^1 P_{\nu}(x) P_{\lambda}(x) (1+x)^{\lambda+\nu} dx = \frac{2^{\lambda+\nu+1} [\Gamma(\lambda+\nu+1)]^4}{[\Gamma(\lambda+1)\Gamma(\nu+1)]^2 \Gamma(2\lambda+2\nu+2)}$ <p style="text-align: right;"><math>\text{Re}(\lambda+\nu) &gt; -1</math></p>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 |
| (25) | $\int_{-1}^1 P_{\nu}(x) Q_{\nu}(x) dx = -\frac{\sin(2\pi\nu) \psi'(\nu+1)}{\pi(2\nu+1)}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      |

The complex  $z$ -plane is cut along the real axis from  $-1$  to  $1$ .

**Variable  $\alpha x + \beta$ : finite intervals (cont'd)**

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                               |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (26) | $\int_{-1}^1 P_{\nu}(x) Q_{\lambda}(x) dx = [(\nu - \lambda)(\nu + \lambda + 1)]^{-1} \{ 1 - \cos [(\lambda - \nu)\pi] \\ - 2\pi^{-1} \sin(\nu\pi) \cos(\lambda\pi) [\psi(\nu + 1) - \psi(\lambda + 1)] \}$                                                                                                                                                                                                                                                                   |
| (27) | $\int_{-1}^1 [Q_{\nu}(x)]^2 dx = \frac{\frac{1}{2}\pi^2 - \{1 - [\cos(\nu\pi)]^2\} \psi'(\nu + 1)}{2\nu + 1}$                                                                                                                                                                                                                                                                                                                                                                 |
| (28) | $\int_{-1}^1 Q_{\nu}(x) Q_{\lambda}(x) dx = [(\lambda - \nu)(\lambda + \nu + 1)]^{-1} \{ \frac{1}{2}\pi \sin [(\lambda - \nu)\pi] \\ + [\psi(\nu + 1) - \psi(\lambda + 1)] [1 + \cos(\lambda\pi) \cos(\nu\pi)] \}$                                                                                                                                                                                                                                                            |
| (29) | $\int_{-1}^1 (1 - x^2)^{\frac{1}{2}\mu - M - \frac{1}{2}} \left( \frac{1 - x}{1 + x} \right)^{\frac{1}{2}\nu} P_{\mu}^M(x) P_{\lambda}^M(x) dx$ <p>See Shabde, N.G., 1940: <i>Bull. Calcutta Math. Soc.</i> 32, 121-128.</p>                                                                                                                                                                                                                                                  |
| (30) | $\int_{-1}^1 P_{\lambda}^{\rho}(x) P_{\mu}^{\sigma}(x) P_{\nu}^{\tau}(x) dx$ <p>See Gaunt, J.A., 1929: <i>Philos. Trans. Royal Soc.</i> 228, 151-196.</p>                                                                                                                                                                                                                                                                                                                     |
| (31) | $\int_{-1}^1 (1 - x^2)^{\lambda - 1} (1 - a^2 x^2)^{\frac{1}{2}\mu} P_{\nu}(ax) dx$ $= \frac{\pi 2^{\mu} \Gamma(\lambda)}{\Gamma(\frac{1}{2} + \lambda) \Gamma(\frac{1}{2} - \frac{1}{2}\mu - \frac{1}{2}\nu) \Gamma(1 - \frac{1}{2}\mu + \frac{1}{2}\nu)}$ $\times {}_2F_1 \left( -\frac{\mu + \nu}{2}, \frac{1 - \mu + \nu}{2}; \frac{1}{2} + \lambda; a^2 \right)$ <p style="text-align: right;"><math>\operatorname{Re} \lambda &gt; 0, \quad -1 &lt; a &lt; 1</math></p> |

Variable  $\alpha x + \beta$ : finite intervals (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (32) | $\int_0^1 x^{-\frac{1}{2}\mu-\frac{1}{2}} (1-x)^{-\mu-\frac{1}{2}} (1+\alpha x)^{\frac{1}{2}\mu} P_{\nu}^{\mu}(1+2\alpha x) dx$ $= \pi^{\frac{1}{2}} \Gamma(\frac{1}{2}-\mu) \alpha^{\frac{1}{2}\mu} \{P_{\nu}^{\mu}[(1+\alpha)^{\frac{1}{2}}]\}^2 \quad \operatorname{Re} \mu < \frac{1}{2}, \quad  \arg \alpha  < \pi$                                                                                                                                                                                                                                                              |
| (33) | $\int_0^1 x^{-\mu/2-1/2} (1-x)^{-\mu-3/2} (1+\alpha x)^{\mu/2} P_{\nu}^{\mu}(1+2\alpha x) dx$ $= \pi^{\frac{1}{2}} \Gamma(-\mu-\frac{1}{2})(1+\alpha)^{-\frac{1}{2}} \alpha^{\frac{1}{2}\mu+\frac{1}{2}} P_{\nu}^{\mu+1}[(1+\alpha)^{\frac{1}{2}}] P_{\nu}^{\mu}[(1+\alpha)^{\frac{1}{2}}]$ $\operatorname{Re} \mu < -\frac{1}{2}, \quad  \arg \alpha  < \pi$                                                                                                                                                                                                                         |
| (34) | $\int_0^1 x^{\frac{1}{2}\mu-\frac{1}{2}} (1-x)^{\mu-\frac{1}{2}} (1+\alpha x)^{-\frac{1}{2}\mu} P_{\nu}^{\mu}(1+2\alpha x) dx$ $= \pi^{\frac{1}{2}} \Gamma(\frac{1}{2}+\mu) \alpha^{-\frac{1}{2}\mu} P_{\nu}^{\mu}[(1+\alpha)^{\frac{1}{2}}] P_{\nu}^{-\mu}[(1+\alpha)^{\frac{1}{2}}]$ $\operatorname{Re} \mu > -\frac{1}{2}, \quad  \arg \alpha  < \pi$                                                                                                                                                                                                                              |
| (35) | $\int_0^1 x^{\mu/2-1/2} (1-x)^{\mu-3/2} (1+\alpha x)^{-\mu/2} P_{\nu}^{\mu}(1+2\alpha x) dx$ $= \frac{1}{2} \pi^{\frac{1}{2}} \Gamma(\mu-\frac{1}{2})(1+\alpha)^{-\frac{1}{2}} \alpha^{\frac{1}{2}\mu-\frac{1}{2}} \{P_{\nu}^{\frac{1}{2}-\mu}[(1+\alpha)^{\frac{1}{2}}] P_{\nu}^{\mu}[(1+\alpha)^{\frac{1}{2}}]$ $+ (\mu+\nu)(1-\mu+\nu) P_{\nu}^{-\mu}[(1+\alpha)^{\frac{1}{2}}] P_{\nu}^{\mu}[(1+\alpha)^{\frac{1}{2}}]\}$ $\operatorname{Re} \mu > \frac{1}{2}, \quad  \arg \alpha  < \pi$                                                                                        |
| (36) | $\int_0^1 x^{-\frac{1}{2}\mu-1} (1-x)^{-\mu-\frac{1}{2}} (1+\alpha x)^{\frac{1}{2}\mu-\frac{1}{2}} [(1-\nu-\mu) P_{\nu-1}^{\mu-1}(1+2\alpha x)$ $+ (1+\nu-\mu) P_{\nu}^{\mu-1}(1+2\alpha x)] dx = 2\pi^{\frac{1}{2}} \Gamma(\frac{1}{2}-\mu)(1+\alpha)^{-\frac{1}{2}} \alpha^{\frac{1}{2}\mu+\frac{1}{2}}$ $\times P_{\nu}^{\mu}[(1+\alpha)^{\frac{1}{2}}] P_{\nu-1}^{\mu}[(1+\alpha)^{\frac{1}{2}}] \quad \operatorname{Re} \mu < \frac{1}{2}, \quad  \arg \alpha  < \pi$                                                                                                            |
| (37) | $\int_0^1 x^{-\frac{1}{2}\mu-1} (1-x)^{-\mu-\frac{1}{2}} (1+\alpha x)^{\frac{1}{2}\mu-\frac{1}{2}} [P_{\nu-1}^{\frac{1}{2}-\mu}(1+2\alpha x)$ $- P_{\nu}^{\frac{1}{2}-\mu}(1+2\alpha x)] dx = \pi^{\frac{1}{2}} \Gamma(\frac{1}{2}-\mu)(1+\alpha)^{-\frac{1}{2}} \alpha^{\frac{1}{2}\mu+\frac{1}{2}}$ $\times \{(\mu-\nu) P_{\nu}^{\mu}[(1+\alpha)^{\frac{1}{2}}] P_{\nu-1}^{-\mu}[(1+\alpha)^{\frac{1}{2}}]$ $- (\mu+\nu) P_{\nu-1}^{\mu}[(1+\alpha)^{\frac{1}{2}}] P_{\nu}^{-\mu}[(1+\alpha)^{\frac{1}{2}}]\} \quad \operatorname{Re} \mu < \frac{1}{2}, \quad  \arg \alpha  < \pi$ |

**Variable  $\alpha x + \beta$ : finite intervals (cont'd)**

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                        |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (38) | $\int_0^1 x^{-\frac{1}{2}\mu-\frac{1}{2}} (1-x)^{-\mu-\frac{1}{2}} (1+\alpha x)^{\frac{1}{2}\mu} Q_\nu^\mu(1+2\alpha x) dx$ $= \pi^{\frac{1}{2}} \Gamma(\frac{1}{2}-\mu) \alpha^{\frac{1}{2}\mu} P_\nu^\mu[(1+\alpha)^{\frac{1}{2}}] Q_\nu^\mu[(1+\alpha)^{\frac{1}{2}}]$ $\text{Re } \mu < \frac{1}{2}, \quad  \arg \alpha  < \pi$                                                                                                                    |
| (39) | $\int_0^1 x^{-\mu/2-1/2} (1-x)^{-\mu-3/2} (1+\alpha x)^{\mu/2} Q_\nu^\mu(1+2\alpha x) dx$ $= \frac{1}{2} \pi^{\frac{1}{2}} \Gamma(-\mu-\frac{1}{2}) (1+\alpha)^{-\frac{1}{2}} \alpha^{\frac{1}{2}\mu+\frac{1}{2}}$ $\times \{P_\nu^{\mu+1}[(1+\alpha)^{\frac{1}{2}}] Q_\nu^\mu[(1+\alpha)^{\frac{1}{2}}]$ $+ P_\nu^\mu[(1+\alpha)^{\frac{1}{2}}] Q_\nu^{\mu+1}[(1+\alpha)^{\frac{1}{2}}]\}$ $\text{Re } \mu < -\frac{1}{2}, \quad  \arg \alpha  < \pi$ |

**18.2. Legendre functions of variable  $\alpha x + \beta$ : infinite intervals**

|     |                                                                                                                                                                                                                                                                                                                                                                                                                               |
|-----|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $\int_0^\infty (x^2-1)^{\frac{1}{2}\mu} \sin(\alpha x) P_\nu^\mu(x) dx$ $= \frac{2^\mu \pi^{\frac{1}{2}} \alpha^{-\mu-\frac{1}{2}}}{\Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu) \Gamma(1-\frac{1}{2}\mu+\frac{1}{2}\nu)} S_{\mu+\frac{1}{2}, \nu+\frac{1}{2}}(\alpha)$ $\alpha > 0, \quad \text{Re } \mu < 3/2, \quad \text{Re}(\mu+\nu) < 1$                                                                           |
| (2) | $\int_1^\infty (x^2-1)^{\lambda-1} P_\nu^\mu(x) dx = \frac{2^{\mu-1} \Gamma(\lambda-\frac{1}{2}\mu) \Gamma(1-\lambda+\frac{1}{2}\nu) \Gamma(\frac{1}{2}-\lambda-\frac{1}{2}\nu)}{\Gamma(1-\frac{1}{2}\mu+\frac{1}{2}\nu) \Gamma(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu) \Gamma(1-\lambda-\frac{1}{2}\mu)}$ $\text{Re } \lambda > \text{Re } \mu, \quad \text{Re}(1-2\lambda-\nu) > 0, \quad \text{Re}(2-2\lambda+\nu) > 0$ |
| (3) | $\int_1^\infty x^{-\rho} (x^2-1)^{-\frac{1}{2}\mu} P_\nu^\mu(x) dx = \frac{2^{\rho+\mu-2} \Gamma(\frac{\rho+\mu+\nu}{2}) \Gamma(\frac{\rho+\mu-\nu-1}{2})}{\pi^{\frac{1}{2}} \Gamma(\rho)}$ $\text{Re } \mu < 1, \quad \text{Re}(\rho+\mu+\nu) > 0, \quad \text{Re}(\rho+\mu-\nu) > 1$                                                                                                                                        |

**Variable  $\alpha x + \beta$ : infinite intervals (cont'd)**

|     |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     |
|-----|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (4) | $\int_1^\infty (x-1)^{\lambda-1} (x^2-1)^{\frac{1}{2}\mu} P_\nu^\mu(x) dx$ $= \frac{2^{\lambda+\mu} \Gamma(\lambda) \Gamma(-\lambda-\mu-\nu) \Gamma(1-\lambda-\mu+\nu)}{\Gamma(1-\mu+\nu) \Gamma(-\mu-\nu) \Gamma(1-\lambda-\mu)}$ $\operatorname{Re} \lambda > 0, \quad \operatorname{Re}(\lambda + \mu + \nu) < 0, \quad \operatorname{Re}(\lambda + \mu - \nu) < 1$                                                                                                                                                              |
| (5) | $\int_1^\infty (x-1)^{\lambda-1} (x^2-1)^{-\frac{1}{2}\mu} P_\nu^\mu(x) dx$ $= - \frac{2^{\lambda-\mu} \sin(\nu\pi) \Gamma(\lambda-\mu) \Gamma(-\lambda+\mu-\nu) \Gamma(1-\lambda+\mu+\nu)}{\pi \Gamma(1-\lambda)}$ $\operatorname{Re}(\lambda - \mu) > 0, \quad \operatorname{Re}(\mu - \lambda - \nu) > 0, \quad \operatorname{Re}(\mu - \lambda + \nu) > -1$                                                                                                                                                                     |
| (6) | $\int_1^\infty (x-1)^{-\frac{1}{2}\mu} (x+1)^{\frac{1}{2}\mu-\frac{1}{2}} (z+x)^{\mu-\frac{1}{2}} P_\nu^\mu(x) dx$ $= \pi^{\frac{1}{2}} \frac{\Gamma(-\mu-\nu) \Gamma(1-\mu+\nu)}{\Gamma(\frac{1}{2}-\mu)} (z-1)^\mu \left\{ P_\nu^\mu \left[ \left( \frac{1+z}{2} \right)^{\frac{1}{2}} \right] \right\}^2$ $\operatorname{Re}(\mu + \nu) < 0, \quad \operatorname{Re}(\mu - \nu) < 1, \quad  \arg(z+1)  < \pi$                                                                                                                    |
| (7) | $\int_1^\infty (x-1)^{-\frac{1}{2}\mu} (x+1)^{\frac{1}{2}\mu-\frac{1}{2}} (z+x)^{\mu-3/2} P_\nu^\mu(x) dx$ $= \pi^{\frac{1}{2}} \frac{\Gamma(1-\mu-\nu) \Gamma(2-\mu+\nu)}{\Gamma(3/2-\mu)} (z-1)^{\mu-\frac{1}{2}} (z+1)^{-\frac{1}{2}}$ $\times P_\nu^\mu \left[ \left( \frac{1+z}{2} \right)^{\frac{1}{2}} \right] P_\nu^{\mu-1} \left[ \left( \frac{1+z}{2} \right)^{\frac{1}{2}} \right]$ $\operatorname{Re} \mu < 1, \quad \operatorname{Re}(\mu + \nu) < 1, \quad \operatorname{Re}(\mu - \nu) < 2, \quad  \arg(1+z)  < \pi$ |

**Variable  $\alpha x + \beta$ : infinite intervals (cont'd)**

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (8)  | $\int_1^\infty (x-1)^{-\frac{1}{2}\mu-1} (x+1)^{\frac{1}{2}\mu-\frac{1}{2}} (z+x)^{\mu-\frac{1}{2}} [(\nu-\mu) P_\nu^\mu(x) - (\nu+\mu) P_{\nu-1}^\mu(x)] dx = (2\pi)^{\frac{1}{2}} \frac{\Gamma(1-\mu-\nu)\Gamma(1-\mu+\nu)}{\Gamma(\frac{1}{2}-\mu)} (z-1)^\mu (z+1)^{-\frac{1}{2}}$ $\times P_\nu^\mu \left[ \left( \frac{1+z}{2} \right)^{\frac{1}{2}} \right] P_{\nu-1}^\mu \left[ \left( \frac{1+z}{2} \right)^{\frac{1}{2}} \right]$ $\operatorname{Re} \mu < 0, \quad \operatorname{Re} \mu < 1 -  \operatorname{Re} \nu , \quad  \arg(z+1)  < \pi$                                                                                                                                                                                                                      |
| (9)  | $\int_1^\infty (x-1)^{\lambda-1} (x^2-1)^{\frac{1}{2}\mu} (x+z)^{-\rho} P_\nu^\mu(x) dx$ $= \frac{2^{\lambda+\mu-\rho} \Gamma(\lambda-\rho) \Gamma(\rho-\lambda-\mu-\nu) \Gamma(\rho-\lambda-\mu+\nu+1)}{\Gamma(1-\mu+\nu) \Gamma(-\mu-\nu) \Gamma(1+\rho-\lambda-\mu)}$ $\times {}_3F_2(\rho, \rho-\lambda-\mu-\nu, \rho-\lambda-\mu+\nu+1; \rho-\lambda+1, \rho-\lambda-\mu+1; \frac{1}{2}+\frac{1}{2}z)$ $+ \frac{\Gamma(\rho-\lambda)\Gamma(\lambda)}{\Gamma(\rho)\Gamma(1-\mu)} 2^\mu (z+1)^{\lambda-\rho}$ $\times {}_3F_2(\lambda, -\mu-\nu, 1-\mu+\nu; 1-\mu, 1-\rho+\lambda; \frac{1}{2}+\frac{1}{2}z) \quad \operatorname{Re} \lambda > 0$ $\operatorname{Re}(\rho-\lambda-\mu-\nu) > 0, \quad \operatorname{Re}(\rho-\lambda-\mu+\nu+1) > 0, \quad  \arg(z+1)  < \pi$ |
| (10) | $\int_1^\infty (x-1)^{\lambda-1} (x^2-1)^{-\frac{1}{2}\mu} (x+z)^{-\rho} P_\nu^\mu(x) dx$ $= - \frac{\sin(\nu\pi) \Gamma(\lambda-\mu-\rho) \Gamma(\rho-\lambda+\mu-\nu) \Gamma(\rho-\lambda+\mu+\nu+1)}{2^{\rho-\lambda+\mu} \pi \Gamma(1+\rho-\lambda)}$ $\times {}_3F_2(\rho, \rho-\lambda+\mu-\nu, \rho-\lambda+\mu+\nu+1; 1+\rho-\lambda, 1+\rho-\lambda+\mu; \frac{1}{2}+\frac{1}{2}z)$ $+ \frac{\Gamma(\lambda-\mu) \Gamma(\rho-\lambda+\mu)}{\Gamma(\rho) \Gamma(1-\mu)} (z+1)^{\lambda-\rho-\mu}$ $\times {}_3F_2(\lambda-\mu, -\nu, \nu+1; 1+\lambda-\mu-\rho, 1-\mu; \frac{1}{2}+\frac{1}{2}z) \quad \operatorname{Re}(\lambda-\mu) > 0$ $\operatorname{Re}(\rho-\lambda+\mu-\nu) > 0, \quad \operatorname{Re}(\rho-\lambda+\mu+\nu+1) > 0, \quad  \arg(z+1)  < \pi$   |



**Variable  $\alpha x + \beta$ : infinite intervals (cont'd)**

|      |                                                                                                                                                                                                                                                                                                                                                                                            |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (11) | $\int_1^\infty e^{-\alpha x} (x^2 - 1)^{-\frac{1}{2}\mu} P_\nu^\mu(x) dx = 2^{\frac{1}{2}} \pi^{-\frac{1}{2}} \alpha^{\mu-\frac{1}{2}} K_{\nu+\frac{1}{2}}(\alpha)$ $\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \mu < 1$                                                                                                                                                        |
| (12) | $\int_1^\infty \left( \frac{x+1}{x-1} \right)^{\frac{1}{2}\mu} e^{-\alpha x} P_\nu^\mu(x) dx = \alpha^{-1} W_{\mu, \nu+\frac{1}{2}}(2\alpha)$ $\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \mu < 1$                                                                                                                                                                              |
| (13) | $\int_1^\infty (x-1)^{\lambda-1} (x^2-1)^{\frac{1}{2}\mu} e^{-\alpha x} P_\nu^\mu(x) dx$ $= \frac{\alpha^{-\lambda-\mu} e^{-\alpha}}{\Gamma(1-\mu+\nu) \Gamma(-\mu-\nu)} G_{23}^{31} \left( 2\alpha \left  \begin{matrix} 1+\mu, 1 \\ \lambda+\mu, -\nu, 1+\nu \end{matrix} \right. \right)$ $\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \lambda > 0$                           |
| (14) | $\int_1^\infty (x-1)^{\lambda-1} (x^2-1)^{-\frac{1}{2}\mu} e^{-\alpha x} Q_\nu^\mu(x) dx$ $= \frac{1}{2} e^{\mu\pi i} \alpha^{\mu-\lambda} e^{-\alpha} G_{23}^{22} \left( 2\alpha \left  \begin{matrix} 1-\mu, 1 \\ \lambda-\mu, \nu+1, -\nu \end{matrix} \right. \right)$ $\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re}(\lambda - \mu) > 0$ |
| (15) | $\int_1^\infty (x-1)^{\lambda-1} (x^2-1)^{-\frac{1}{2}\mu} e^{-\alpha x} P_\nu^\mu(x) dx$ $= -\pi^{-1} \sin(\nu\pi) \alpha^{\mu-\lambda} e^{-\alpha} G_{23}^{31} \left( 2\alpha \left  \begin{matrix} 1, 1-\mu \\ \lambda-\mu, 1+\nu, -\nu \end{matrix} \right. \right)$ $\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \lambda > \operatorname{Re} \mu$                           |
| (16) | $\int_1^\infty (\alpha^2 + \beta^2 + 2\alpha\beta x)^{-\frac{1}{2}} \exp[-(\alpha^2 + \beta^2 + 2\alpha\beta x)^{\frac{1}{2}}] P_\nu(x) dx$ $= 2\pi^{-1} (\alpha\beta)^{-\frac{1}{2}} K_{\nu+\frac{1}{2}}(\alpha) K_{\nu+\frac{1}{2}}(\beta)$ $\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \beta > 0$                                                                            |

**Variable  $\alpha x + \beta$ : infinite intervals (cont'd)**

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (17) | $\int_1^\infty (x^2 - 1)^{-\frac{1}{2}\mu} \exp(a^2 x^2) \operatorname{Erfc}(\alpha x) P_\nu^\mu(x) dx$ $= \pi^{-1} 2^{\mu-1} \Gamma\left(\frac{1+\mu+\nu}{2}\right) \Gamma\left(\frac{\mu-\nu}{2}\right) \alpha^{\mu-3/2} \exp\left(\frac{a^2}{2}\right)$ $\times W_{\frac{1}{4}-\frac{1}{2}\mu, \frac{1}{4}+\frac{1}{2}\nu}(a^2)$ $\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \mu < 1, \quad \operatorname{Re}(\mu + \nu) > -1, \quad \operatorname{Re}(\mu - \nu) > 0$ |
| (18) | $\int_1^\infty Q_\nu(x) dx = [\nu(\nu + 1)]^{-1}$ $\operatorname{Re} \nu > 0$                                                                                                                                                                                                                                                                                                                                                                                                        |
| (19) | $\int_1^\infty P_\nu(x) Q_\lambda(x) dx = [(\lambda - \nu)(\lambda + \nu + 1)]^{-1}$ $\operatorname{Re}(\lambda - \nu) > 0, \quad \operatorname{Re}(\lambda + \nu) > -1$                                                                                                                                                                                                                                                                                                             |
| (20) | $\int_1^\infty [Q_\nu(x)]^2 dx = (2\nu + 1)^{-1} \psi'(\nu + 1)$ $\operatorname{Re} \nu > -\frac{1}{2}$                                                                                                                                                                                                                                                                                                                                                                              |
| (21) | $\int_1^\infty Q_\nu(x) Q_\lambda(x) dx = \frac{\psi(\lambda + 1) - \psi(\nu + 1)}{(\lambda - \nu)(\lambda + \nu + 1)}$ $\operatorname{Re}(\lambda + \nu) > -1$                                                                                                                                                                                                                                                                                                                      |
| (22) | $\int_1^\infty [Q_\nu^\mu(x)]^2 dx$ <p>See Barnes, E.W., 1908: <i>Quart. J. Math.</i> 39, 97-204. Note that Barnes' definition of Legendre functions of the second kind differs from the one used in this book.</p>                                                                                                                                                                                                                                                                  |
| (23) | $\int_1^\infty (x^2 - 1)^{\lambda-1} Q_\nu^\mu(x) dx$ $= e^{\mu\pi i} \frac{\Gamma(\frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\mu) \Gamma(1 - \lambda + \frac{1}{2}\nu) \Gamma(\lambda + \frac{1}{2}\mu) \Gamma(\lambda - \frac{1}{2}\mu)}{2^{2\lambda-\mu} \Gamma(1 + \frac{1}{2}\nu - \frac{1}{2}\mu) \Gamma(\frac{1}{2} + \lambda + \frac{1}{2}\nu)}$ $ \operatorname{Re} \mu  < 2 \operatorname{Re} \lambda < \operatorname{Re} \nu + 2$                                          |

Variable  $\alpha x + \beta$ : infinite intervals (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (24) | $\int_1^\infty (x-1)^{\lambda-1} (x^2-1)^{\frac{1}{2}\mu} e^{-\alpha x} Q_\nu^\mu(x) dx$ $= \frac{\Gamma(\nu+\mu+1)}{2\Gamma(\nu-\mu+1)} e^{\mu\pi i} \alpha^{-\lambda-\mu} e^{-\alpha} G_{23}^{22} \left( 2\alpha \left  \begin{matrix} 1+\mu, 1 \\ \lambda+\mu, \nu+1, -\nu \end{matrix} \right. \right)$ $\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re}(\lambda+\mu) > 0$                                                                                                                                                                                                                                |
| (25) | $\int_1^\infty (x^2-1)^{\lambda-1} (a^2 x^2-1)^{\frac{1}{2}\mu} P_\nu^\mu(ax) dx$ $= \frac{\Gamma(\lambda) \Gamma(1-\lambda-\frac{1}{2}\mu+\frac{1}{2}\nu) \Gamma(\frac{1}{2}-\lambda-\frac{1}{2}\mu-\frac{1}{2}\nu)}{\Gamma(1-\frac{1}{2}\mu+\frac{1}{2}\nu) \Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu) \Gamma(1-\lambda-\mu)}$ $\times 2^{\mu-1} \alpha^{\mu-\nu-1} {}_2F_1 \left( \frac{1-\mu+\nu}{2}, 1-\lambda-\frac{\mu-\nu}{2}; 1-\lambda-\mu; 1-\frac{1}{a^2} \right)$ $\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re}(\nu-\mu-2\lambda) > -2, \quad \operatorname{Re}(2\lambda+\mu+\nu) < 1$ |
| (26) | $\int_1^\infty x^{\mu-1} Q_\nu^\mu(ax) dx = e^{\mu\pi i} \Gamma(\mu) \alpha^{-\mu} (a^2-1)^{\frac{1}{2}\mu} Q_\nu^{-\mu}(a)$ $ \arg(a-1)  < \pi, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re}(\nu-\mu) > -1$                                                                                                                                                                                                                                                                                                                                                                                                                                 |
| (27) | $\int_1^\infty (x^2-1)^{\lambda-1} (a^2 x^2-1)^{-\frac{1}{2}\mu} Q_\nu^\mu(ax) dx$ $= \frac{\Gamma(\frac{\mu+\nu+1}{2}) \Gamma(\lambda) \Gamma(1-\lambda+\frac{\mu+\nu}{2})}{\Gamma(\nu+\frac{3}{2})} 2^{\mu-2} e^{\mu\pi i} \alpha^{-\mu-\nu-1}$ $\times {}_2F_1 \left( \frac{\mu+\nu+1}{2}, 1-\lambda+\frac{\mu+\nu}{2}; \nu+\frac{3}{2}; a^{-2} \right)$ $ \arg(a-1)  < \pi, \quad \operatorname{Re} \lambda > 0, \quad \operatorname{Re}(2\lambda-\mu-\nu) < 2$                                                                                                                                                                                      |
| (28) | $\int_1^\infty x^{-\frac{1}{2}\mu-\frac{1}{2}} (x-1)^{-\mu-\frac{1}{2}} (1+\alpha x)^{\frac{1}{2}\mu} Q_\nu^\mu(1+2\alpha x) dx$ $= \pi^{-\frac{1}{2}} e^{-\mu\pi i} \Gamma(\frac{1}{2}-\mu) \alpha^{\frac{1}{2}\mu} \{Q_\nu^\mu[(1+\alpha)^{\frac{1}{2}}]\}^2$ $ \arg \alpha  < \pi, \quad \operatorname{Re} \mu < \frac{1}{2}, \quad \operatorname{Re}(\mu+\nu) > -1$                                                                                                                                                                                                                                                                                  |

**Variable  $\alpha x + \beta$ : infinite intervals (cont'd)**

|      |                                                                                                                                                                                                                                                                                                                                           |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (29) | $\int_1^\infty x^{-\mu/2-1/2} (x-1)^{-\mu-3/2} (1+\alpha x)^{\mu/2} Q_\nu^\mu(1+2\alpha x) dx$ $= -\pi^{-1/2} e^{-\mu\pi i} \Gamma(-\mu-1/2) a^{1/2}\mu^{1/2} (1+a^2)^{-1/2}$ $\times Q_\nu^{\mu+1}[(1+a)^{1/2}] Q_\nu^\mu[(1+a)^{1/2}]$ $ \arg a  < \pi, \quad \operatorname{Re} \mu < -1/2, \quad \operatorname{Re}(\mu + \nu + 2) > 0$ |
| (30) | $\int_1^\infty (x-1)^{\mu-1} P_\nu(\alpha x) Q_\lambda(\alpha x) dx, \quad \int_1^\infty (x-1)^{\mu-1} Q_\nu(\alpha x) Q_\lambda(\alpha x) dx$ <p>See Shabde, N.G., 1937: <i>Bull. Calcutta Math. Soc.</i> 29,33-40.</p>                                                                                                                  |

**18.3. Legendre functions of other variables**

|     |                                                                                                                                                                                                                                                                                                 |
|-----|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $\int_a^\infty P_\nu(2x^2 a^{-2} - 1) \sin(bx) dx$ $= -\frac{\pi a}{4 \cos(\nu\pi)} \left\{ \left[ J_{\nu+1/2}\left(\frac{ab}{2}\right) \right]^2 - \left[ J_{-\nu-1/2}\left(\frac{ab}{2}\right) \right]^2 \right\}$ $a, b > 0, \quad -1 < \operatorname{Re} \nu < 0$                           |
| (2) | $\int_a^\infty P_\nu(2x^2 a^{-2} - 1) \cos(bx) dx$ $= -\frac{1}{4} \pi a [J_{\nu+1/2}(\frac{1}{2}ab) J_{-\nu-1/2}(\frac{1}{2}ab) - Y_{\nu+1/2}(\frac{1}{2}ab) Y_{-\nu-1/2}(\frac{1}{2}ab)]$ $a, b > 0, \quad -1 < \operatorname{Re} \nu < 0$                                                    |
| (3) | $\int_0^\infty (\alpha+x)^{-\mu-\nu-2} P_\mu\left(\frac{\alpha-x}{\alpha+x}\right) P_\nu\left(\frac{\alpha-x}{\alpha+x}\right) dx$ $= \frac{a^{-\mu-\nu-1} [\Gamma(\mu+\nu+1)]^4}{[\Gamma(\mu+1) \Gamma(\nu+1)]^2 \Gamma(2\mu+2\nu+2)}$ $ \arg a  < \pi, \quad \operatorname{Re}(\mu+\nu) > -1$ |

## Other variables (cont'd)

|     |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                |
|-----|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (4) | $\int_0^1 x^{-1} \cos(ax) P_\nu(2x^{-2} - 1) dx$ $= -\frac{1}{2} \pi \csc(\nu\pi) {}_1F_1(\nu+1; 1; ai) {}_1F_1(\nu+1; 1; -ai)$ $a > 0, \quad -1 < \operatorname{Re} \nu < 0$                                                                                                                                                                                                                                                                                                                  |
| (5) | $\int_0^\infty x^{-1} e^{-\beta x} Q_{-\frac{1}{2}}(1+2x^{-2}) dx = \frac{\pi^2}{8} \{ [J_0(\frac{1}{2}\beta)]^2 + [Y_0(\frac{1}{2}\beta)]^2 \}$ $\operatorname{Re} \beta > 0$                                                                                                                                                                                                                                                                                                                 |
| (6) | $\int_0^\infty x^{-1} e^{-\alpha x} Q_\nu(1+2x^{-2}) dx = \frac{1}{2} [\Gamma(\nu+1)]^2 \alpha^{-1}$ $\times W_{-\nu-\frac{1}{2}, 0}(ai) W_{-\nu-\frac{1}{2}, 0}(-ai)$ $\operatorname{Re} \alpha > 0, \quad \operatorname{Re} \nu > -1$                                                                                                                                                                                                                                                        |
| (7) | $\int_0^\infty x^{\lambda-1} (x^2+a^2)^{\frac{1}{2}\nu} e^{-\beta x} P_\nu^\mu \left[ \frac{x}{(x^2+a^2)^{\frac{1}{2}}} \right] dx$ $= \frac{2^{-\nu-2} a^{\lambda+\nu}}{\pi \Gamma(-\mu-\nu)} G_{24}^{32} \left( \frac{a^2 \beta^2}{4} \left  \begin{matrix} 1-\frac{\lambda}{2}, \frac{1-\lambda}{2} \\ 0, \frac{1}{2}, -\frac{\lambda+\mu+\nu}{2}, -\frac{\lambda-\mu+\nu}{2} \end{matrix} \right. \right)$ $a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \lambda > 0$ |
| (8) | $\int_0^\infty x^{\frac{1}{2}} \sin(bx) [P_\nu^{-\frac{1}{2}}(y)]^2 dx = \frac{(\frac{1}{2}\pi)^{-\frac{1}{2}} a^{-1} b^{-\frac{1}{2}}}{\Gamma(5/4+\nu)\Gamma(1/4-\nu)} \left[ K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right) \right]^2$ $\operatorname{Re} \alpha > 0, \quad b > 0, \quad -5/4 < \operatorname{Re} \nu < 1/4$                                                                                                                                                                   |
| (9) | $\int_0^\infty x^{\frac{1}{2}} \sin(bx) P_\nu^{-\frac{1}{2}}(y) Q_\nu^{-\frac{1}{2}}(y) dx$ $= \frac{(\frac{1}{2}\pi)^{\frac{1}{2}} e^{-\frac{1}{2}\pi i} \Gamma(\nu+5/4)}{ab^{\frac{1}{2}} \Gamma(\nu+3/4)} I_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right) K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right)$ $\operatorname{Re} \alpha > 0, \quad b > 0, \quad \operatorname{Re} \nu > -5/4$                                                                                                        |

$$y = (1 + a^2 x^2)^{\frac{1}{2}}$$

## Other variables (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                                                   |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (10) | $\int_0^\infty x^{\frac{1}{2}} y^{-1} \sin(bx) P_\nu^{-\frac{1}{4}}(y) P_{\nu-1}^{-\frac{1}{4}}(y) dx$ $= \frac{(2\pi)^{-\frac{1}{2}} a^{-2} b^{\frac{1}{2}}}{\Gamma(5/4 + \nu) \Gamma(5/4 - \nu)} K_{\nu-\frac{1}{2}}\left(\frac{b}{2a}\right) K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right)$ $\text{Re } a > 0, \quad b > 0, \quad -5/4 < \text{Re } \nu < 5/4$                                  |
| (11) | $\int_0^\infty x^{\frac{1}{2}} y^{-1} \sin(bx) P_\nu^{\frac{1}{4}}(y) P_\nu^{-\frac{1}{4}}(y) dx$ $= \frac{(2\pi)^{-\frac{1}{2}} a^{-2} b^{\frac{1}{2}}}{\Gamma(7/4 + \nu) \Gamma(3/4 - \nu)} \left[ K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right) \right]^2$ $\text{Re } a > 0, \quad b > 0, \quad -7/4 < \text{Re } \nu < 3/4$                                                                   |
| (12) | $\int_0^\infty x^{\frac{1}{2}} \cos(bx) [P_\nu^{\frac{1}{4}}(y)]^2 dx = \frac{a^{-1} (\frac{1}{2}\pi b)^{-\frac{1}{2}}}{\Gamma(\frac{3}{4} + \nu) \Gamma(-\frac{1}{4} - \nu)} \left[ K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right) \right]^2$ $\text{Re } a > 0, \quad b > 0, \quad -\frac{3}{4} < \text{Re } \nu < -\frac{1}{4}$                                                                  |
| (13) | $\int_0^\infty x^{\frac{1}{2}} \cos(bx) P_\nu^{\frac{1}{4}}(y) Q_\nu^{\frac{1}{4}}(y) dx = \frac{(\frac{1}{2}\pi)^{\frac{1}{2}} e^{\frac{1}{4}\pi i} \Gamma(\nu + \frac{3}{4})}{a b^{\frac{1}{2}} \Gamma(\nu + 5/4)}$ $\times I_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right) K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right) \quad \text{Re } a > 0, \quad b > 0, \quad \text{Re } \nu > -\frac{3}{4}$ |
| (14) | $\int_0^\infty x^{\frac{1}{2}} y^{-1} \cos(bx) P_\nu^{-\frac{1}{4}}(y) P_\nu^{\frac{1}{4}}(y) dx = \frac{(2\pi)^{-\frac{1}{2}} a^{-2} b^{\frac{1}{2}}}{\Gamma(5/4 + \nu) \Gamma(1/4 - \nu)}$ $\times \left[ K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right) \right]^2 \quad \text{Re } a > 0, \quad b > 0, \quad -5/4 < \text{Re } \nu < 1/4$                                                        |

$$y = (1 + a^2 x^2)^{\frac{1}{2}}$$

## Other variables (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                                                  |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (15) | $\int_0^\infty x^{\frac{1}{2}} y^{-1} \cos(bx) P_{\nu}^{\frac{1}{2}}(y) P_{\nu-1}^{\frac{1}{2}}(y) dx$ $= \frac{(2\pi)^{-\frac{1}{2}} a^{-2} b^{\frac{1}{2}}}{\Gamma(\frac{3}{4} + \nu) \Gamma(\frac{3}{4} - \nu)} K_{\nu-\frac{1}{2}}\left(\frac{b}{2a}\right) K_{\nu+\frac{1}{2}}\left(\frac{b}{2a}\right)$ $\text{Re } a > 0, \quad b > 0, \quad -\frac{3}{4} < \text{Re } \nu < \frac{3}{4}$ |
| (16) | $\int_0^a \left[ \frac{\sin(a-x)}{\sin x} \right]^{\kappa} P_{\nu}^{-\mu}(\cos x) P_{\nu}^{-\kappa}[\cos(a-x)] \frac{dx}{\sin x}$ $= \frac{2^{\kappa} \Gamma(\mu - \kappa) \Gamma(\kappa + \frac{1}{2})}{\pi^{\frac{1}{2}} \Gamma(\kappa + \mu + 1)} (\sin a)^{\kappa} P_{\nu}^{-\mu}(\cos a)$ $\text{Re } \mu > \text{Re } \kappa > -\frac{1}{2}$                                               |
| (17) | $\int_0^a (\sin x)^{\rho} [\sin(a-x)]^{\sigma} P_{\nu}^{\mu}(\cos x) P_{\kappa}^{\lambda}[\cos(a-x)] dx$ <p>For this integral and several particular cases see Bailey, W.N., 1931: <i>Proc. Cambridge Philos. Soc.</i> 27, 184-189 and 381-386.</p>                                                                                                                                              |
| (18) | $\int_0^\infty \cos(ax) P_{\nu}(\cosh x) dx$ $= -\frac{\sin(\nu\pi)}{4\pi^2} \Gamma\left(\frac{1+\nu+ia}{2}\right) \Gamma\left(\frac{1+\nu-ia}{2}\right) \Gamma\left(-\frac{\nu+ia}{2}\right) \Gamma\left(-\frac{\nu-ia}{2}\right)$ $a > 0, \quad -1 < \text{Re } \nu < 0$                                                                                                                       |
| (19) | $\int_0^\infty P_{-x-\frac{1}{2}}(\cos \theta) dx = \frac{1}{2} \csc(\frac{1}{2}\theta)$ $0 < \theta < \pi$                                                                                                                                                                                                                                                                                      |
| (20) | $\int_{-\infty}^\infty P_x(\cos \theta) dx = \csc(\frac{1}{2}\theta)$ $0 < \theta < \pi$                                                                                                                                                                                                                                                                                                         |

$$y = (1 + a^2 x^2)^{\frac{1}{2}}$$

**Other variables (cont'd)**

|      |                                                                                                                                                                                                                                                                         |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (21) | $\int_0^\infty \cos(bx) P_{-\frac{1}{2}+ix}^\mu(\cosh a) dx$ $= 0 \quad 0 < a < b$ $= \frac{(\frac{1}{2}\pi)^{\frac{1}{2}} (\sinh a)^\mu}{\Gamma(\frac{1}{2}-\mu) (\cosh a - \cosh b)^{\mu+\frac{1}{2}}} \quad 0 < b < a$                                               |
| (22) | $\int_0^\infty x^{-1} \tanh(\pi x) P_{-\frac{1}{2}+ix}(\cosh a) dx = 2 e^{-\frac{1}{2}a} K(e^{-a}) \quad a > 0$                                                                                                                                                         |
| (23) | $\int_0^\infty \frac{x \tanh(\pi x)}{a^2 + x^2} P_{-\frac{1}{2}+ix}(\cosh b) dx = Q_{a-\frac{1}{2}}(\cosh b) \quad \operatorname{Re} a > 0$                                                                                                                             |
| (24) | $\int_0^\infty \cos(bx) \Gamma(\mu + ix) \Gamma(\mu - ix) P_{-\frac{1}{2}+ix}^{\frac{1}{2}-\mu}(\cosh a) dx$ $= \frac{(\frac{1}{2}\pi)^{\frac{1}{2}} \Gamma(\mu) (\sinh a)^{\mu-\frac{1}{2}}}{(\cosh a + \cosh b)^\mu} \quad a, b > 0, \quad \operatorname{Re} \mu > 0$ |



## CHAPTER XIX

### BESSEL FUNCTIONS

For Bessel functions and related functions see H.T.F., vol. II, Chapter VII and the literature quoted there, especially the books by Watson (the standard treatise on the subject), Gray and Mathew, McLachlan (now in a second, revised, edition), and Weyrich. Integrals involving Bessel functions appear in almost every chapter of the present work, and the kernels of the integral transforms listed in Chapters VIII to XII are Bessel functions. The present chapter contains mainly integrals which have not already appeared in the earlier chapters, although some of the integrals already listed have been included for the sake of easy reference.

Bessel functions are particular confluent hypergeometric functions, and the expressions which follow may be used to reduce integrals involving Bessel functions to integrals involving hypergeometric functions.

$$\begin{aligned}
 J_{\nu}(z) &= \frac{(\frac{1}{2}z)^{\nu}}{\Gamma(\nu+1)} {}_0F_1(\nu+1; -\frac{1}{4}z^2) \\
 &= \frac{(\frac{1}{2}z)^{\nu}}{\Gamma(\nu+1)} e^{-iz} {}_1F_1(\nu+\frac{1}{2}; 2\nu+1; 2iz) \\
 &= \frac{z^{-\frac{1}{2}} e^{-\frac{1}{2}(\nu+\frac{1}{2})\pi i}}{2^{2\nu+\frac{1}{2}} \Gamma(\nu+1)} M_{0,\nu}(2iz)
 \end{aligned}$$

$$H_{\nu}^{(1)}(z) = (\frac{1}{2}\pi z)^{-\frac{1}{2}} e^{-\frac{1}{2}(\nu+\frac{1}{2})\pi i} W_{0,\nu}(-2iz)$$

$$H_{\nu}^{(2)}(z) = (\frac{1}{2}\pi z)^{-\frac{1}{2}} e^{\frac{1}{2}(\nu+\frac{1}{2})\pi i} W_{0,\nu}(2iz)$$

$$\begin{aligned}
 I_{\nu}(z) &= \frac{(\frac{1}{2}z)^{\nu}}{\Gamma(\nu+1)} {}_0F_1(\nu+1; \frac{1}{4}z^2) \\
 &= \frac{(\frac{1}{2}z)^{\nu}}{\Gamma(\nu+1)} e^{-z} {}_1F_1(\nu+\frac{1}{2}; 2\nu+1; 2z) \\
 &= \frac{z^{-\frac{1}{2}} 2^{-2\nu-\frac{1}{2}}}{\Gamma(\nu+1)} M_{0,\nu}(2z)
 \end{aligned}$$

$$K_{\nu}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} W_{0,\nu}(2z)$$

$$H_{\nu}(z) = \frac{2(z/2)^{\nu+1}}{\pi^{\frac{1}{2}} \Gamma(\nu+3/2)} {}_1F_2(1; \nu+3/2, 3/2; -z^2/4)$$

$$L_{\nu}(z) = \frac{2(z/2)^{\nu+1}}{\pi^{\frac{1}{2}} \Gamma(\nu+3/2)} {}_1F_2(1; \nu+3/2, 3/2; z^2/4)$$

$$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)} {}_1F_2\left(1; \frac{\mu+\nu+3}{2}, \frac{\mu-\nu+3}{2}; -\frac{z^2}{4}\right)$$

Expressions of various combinations of Bessel functions in terms of Meijer's  $G$ -function are given in the Appendix. Many integrals involving Bessel functions may be obtained by specializing parameters in the small number of known integrals involving the  $G$ -function. Likewise, in the tables which follow, many fairly general integrals involving Bessel functions have been evaluated in terms of confluent hypergeometric functions or  $G$ -functions. For special values of the parameters these expressions simplify considerably. Frequently these particular cases are not given separately, and the user of these tables is expected to perform the necessary operations, the requisite formulas being given in the Appendix.

# BESSEL FUNCTIONS

## 19.1. Bessel functions of argument $x$ . Finite intervals

|     |                                                                                                                                                                                        |                                     |
|-----|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------|
| (1) | $\int_0^a J_\nu(x) dx = 2 \sum_{n=0}^{\infty} J_{\nu+2n+1}(a)$                                                                                                                         | $\operatorname{Re} \nu > -1$        |
| (2) | $\int_0^a x^\nu J_\nu(x) dx = 2^{\nu-1} \pi^{1/2} \Gamma(\nu + 1/2) a [J_\nu(a) \mathbf{H}_{\nu-1}(a) - \mathbf{H}_\nu(a) J_{\nu-1}(a)]$                                               | $\operatorname{Re} \nu > -1/2$      |
| (3) | $\int_0^a x^{\nu+1} J_\nu(x) dx = a^{\nu+1} J_{\nu+1}(a)$                                                                                                                              | $\operatorname{Re} \nu > -1$        |
| (4) | $\int_0^a x^{1-\nu} J_\nu(x) dx = \frac{1}{2^{\nu-1} \Gamma(\nu)} - a^{1-\nu} J_{\nu-1}(a)$                                                                                            |                                     |
| (5) | $\int_0^a x^\mu J_\nu(x) dx = (\mu + \nu - 1) a J_\nu(a) S_{\mu-1, \nu-1}(a) - a J_{\nu-1}(a) S_{\mu, \nu}(a) + 2^\mu \frac{\Gamma(\frac{1+\mu+\nu}{2})}{\Gamma(\frac{1-\mu+\nu}{2})}$ | $\operatorname{Re}(\mu + \nu) > -1$ |
| (6) | $\int_0^a (a-x)^{-1/2} J_\nu(x) dx = \pi^{1/2} (1/2 a)^{1/2} J_{1/2 \nu + 1/4}(1/2 a) J_{1/2 \nu - 1/4}(1/2 a)$                                                                        | $\operatorname{Re} \nu > -1$        |
| (7) | $\int_0^a x^{-\nu} (a^2 - x^2)^{-\nu-1/2} J_\nu(x) dx = \pi^{1/2} 2^{-\nu-1} a^{-2\nu} \Gamma(1/2 - \nu) \times J_\nu(1/2 a) J_{-\nu}(1/2 a)$                                          | $\operatorname{Re} \nu < 1/2$       |

For other similar integrals see sections 8.5 and 13.1.

**Bessel functions of  $x$ ; finite intervals (cont'd)**

|      |                                                                                                                                                |
|------|------------------------------------------------------------------------------------------------------------------------------------------------|
| (8)  | $\int_0^a x^\rho (a^2 - 2abx + b^2)^{-1/2} J_\nu(x) dx$<br>See Bose, S.K., 1946: <i>Bull. Calcutta Math. Soc.</i> 38, 177-180.                 |
| (9)  | $\int_0^a x^\nu \sin x J_\nu(x) dx = \frac{a^{\nu+1}}{2\nu+1} \sin a J_\nu(a) - \cos a J_{\nu+1}(a)$<br>$\text{Re } \nu > -1$                  |
| (10) | $\int_0^a \sin(a-x) J_{2n}(x) dx = a J_{2n+1}(a) + (-1)^n 2n [\cos a$<br>$- J_0(a) - 2 \sum_{m=1}^n (-1)^m J_{2m}(a)]$<br>$n = 0, 1, 2, \dots$ |
| (11) | $\int_0^a \sin(a-x) J_{2n+1}(x) dx = a J_{2n+2}(a)$<br>$+ (-1)^n (2n+1) [\sin a - 2 \sum_{m=0}^n (-1)^m J_{2m+1}(a)]$<br>$n = 0, 1, 2, \dots$  |
| (12) | $\int_0^a \sin(a-x) J_\nu(x) dx = a J_{\nu+1}(a) - 2\nu \sum_{n=0}^{\infty} (-1)^n J_{\nu+2n+2}(a)$<br>$\text{Re } \nu > -1$                   |
| (13) | $\int_0^a x^{-1} \sin(a-x) J_\nu(x) dx = 2\nu^{-1} \sum_{n=0}^{\infty} (-1)^n J_{\nu+2n+1}(a)$<br>$\text{Re } \nu > 0$                         |
| (14) | $\int_0^a x^{-3/2} \sin(a-x) J_\nu(x) dx = (\nu^2 - 1/4)^{-1} a^{1/2} J_\nu(a)$<br>$\text{Re } \nu > 1/2$                                      |
| (15) | $\int_0^a x^\nu \sin(a-x) J_\nu(x) dx = \frac{a^{\nu+1}}{2\nu+1} J_{\nu+1}(a)$<br>$\text{Re } \nu > -1/2$                                      |

**Bessel functions of  $x$ ; finite interval (cont'd)**

|      |                                                                                                                                                                                                                                                                                         |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (16) | $\int_0^a x^\lambda \sin(a-x) J_\nu(x) dx$ $= 2a^{\lambda+1} \sum_{n=0}^{\infty} \frac{(-1)^n (\nu-\lambda)_{2n}}{(\nu+\lambda+1)_{2n+2}} (\nu+2n+1) J_{\nu+2n+1}(a)$ $\text{Re}(\lambda+\nu) > -1$ <p>See also Bailey, W.N., 1930: <i>Proc. London Math. Soc.</i> (2) 31, 200-208.</p> |
| (17) | $\int_0^a (a^2 - x^2)^{-1/2} \sin(\beta x) J_\nu(x) dx$ $= \pi \sum_{n=0}^{\infty} (-1)^n J_{2n+1}(\alpha\beta) J_{1/2\nu+n+1/2}(1/2a) J_{1/2\nu-n-1/2}(a)$ $\text{Re } \nu > -2$                                                                                                       |
| (18) | $\int_0^a x^{\nu+1} \sin[1/2\beta(a^2 - x^2)] J_\nu(x) dx = \beta^{-\nu-1} U_{\nu+2}(a^2\beta, a)$ $\text{Re } \nu > -1$                                                                                                                                                                |
| (19) | $\int_0^a x^{\nu+1} \sin[b(a^2 - x^2)^{1/2}] J_\nu(x) dx = (1/2\pi)^{1/2} a^{\nu+3/2} b(1+b^2)^{-1/2\nu-1/4}$ $\times J_{\nu+3/2}[a(1+b^2)^{1/2}]$ $\text{Re } \mu > -1$                                                                                                                |
| (20) | $\int_0^a x^\nu \cos x J_\nu(x) dx = \frac{a^{\nu+1}}{2\nu+1} [\cos a J_\nu(a) + \sin a J_{\nu+1}(a)]$ $\text{Re } \nu > -1/2$                                                                                                                                                          |
| (21) | $\int_0^a \cos(a-x) J_{2n}(x) dx$ $= a J_{2n}(a) - (-1)^n 2n [\sin a - 2 \sum_{m=0}^{n-1} (-1)^m J_{2m+1}(a)]$ $n = 0, 1, 2, \dots$                                                                                                                                                     |

**Bessel functions of  $x$ ; finite intervals (cont'd)**

|      |                                                                                                                                                                                                                                                                                                                                                                                                            |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (22) | $\int_0^a \cos(a-x) J_{2n+1}(x) dx = a J_{2n+1}(a)$ $+ (-1)^n (2n+1) [\cos a - J_0(a) - 2 \sum_{m=1}^n (-1)^m J_{2m}(a)] \quad n = 0, 1, 2, \dots$                                                                                                                                                                                                                                                         |
| (23) | $\int_0^a \cos(a-x) J_\nu(x) dx = a J_\nu(a) - 2\nu \sum_{n=0}^{\infty} (-1)^n J_{\nu+2n+1}(a)$ <p style="text-align: right;"><math>\operatorname{Re} \nu &gt; -1</math></p>                                                                                                                                                                                                                               |
| (24) | $\int_0^a x^{-1} \cos(a-x) J_\nu(x) dx = \nu^{-1} J_\nu(a) + 2\nu^{-1} \sum_{n=1}^{\infty} (-1)^n J_{\nu+2n}(a)$ <p style="text-align: right;"><math>\operatorname{Re} \nu &gt; 0</math></p>                                                                                                                                                                                                               |
| (25) | $\int_0^a x^\nu \cos(a-x) J_\nu(x) dx = \frac{a^{\nu+1}}{2\nu+1} J_\nu(a)$ <p style="text-align: right;"><math>\operatorname{Re} \nu &gt; -\frac{1}{2}</math></p>                                                                                                                                                                                                                                          |
| (26) | $\int_0^a x^\lambda \cos(a-x) J_\nu(x) dx = \frac{a^{\lambda+1} J_\nu(a)}{\lambda + \nu + 1}$ $+ 2a^{\lambda+1} \sum_{n=1}^{\infty} \frac{(-1)^n (\nu - \lambda)_{2n-1}}{(\nu + \lambda + 1)_{2n+1}} (\nu + 2n) J_{\nu+2n}(a)$ <p style="text-align: right;"><math>\operatorname{Re}(\lambda + \nu) &gt; -1</math></p> <p>See also Bailey, W.N., 1930: <i>Proc. London Math. Soc.</i> (2) 31, 200-208.</p> |
| (27) | $\int_0^a (a^2 - x^2)^{-\frac{1}{2}} \cos(\beta x) J_\nu(x) dx = \frac{1}{2} \pi J_0(a\beta) [J_{\frac{1}{2}\nu}(\frac{1}{2}a)]^2$ $+ \pi \sum_{n=1}^{\infty} (-1)^n J_{2n}(a\beta) J_{\frac{1}{2}\nu+n}(\frac{1}{2}a) J_{\frac{1}{2}\nu-n}(\frac{1}{2}a)$ <p style="text-align: right;"><math>\operatorname{Re} \nu &gt; -1</math></p>                                                                    |
| (28) | $\int_0^a x^{\nu+1} \cos[\frac{1}{2}\beta(a^2 - x^2)] J_\nu(x) dx = \beta^{-\nu-1} U_{\nu+1}(a^2\beta, a)$ <p style="text-align: right;"><math>\operatorname{Re} \nu &gt; -1</math></p>                                                                                                                                                                                                                    |

**Bessel functions of  $x$ ; finite intervals (cont'd)**

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (29) | $\int_0^a x (a^2 - x^2)^{-\frac{1}{2}} \cos [\beta (a^2 - x^2)^{\frac{1}{2}}] J_0(x) dx$ $= (\beta^2 + 1)^{-\frac{1}{2}} \sin [a (\beta^2 + 1)^{\frac{1}{2}}]$                                                                                                                                                                                                                                                                                                                                                                                                            |
| (30) | $\int_0^a (a^2 - x^2)^{-\frac{1}{2}} \cos [\beta (a^2 - x^2)^{\frac{1}{2}}] J_\nu(x) dx$ $= \frac{1}{2} \pi J_{\frac{1}{2}\nu}(\frac{1}{2}ae^u) J_{\frac{1}{2}\nu}(\frac{1}{2}ae^{-u}) \quad \beta = \sinh u, \quad \operatorname{Re} \nu > -1$                                                                                                                                                                                                                                                                                                                           |
| (31) | $\int_0^a x^{\mu-1} P_n(x/a) J_\nu(x) dx, \quad \int_0^a x^{\mu-1} P_n(x/a) J_\mu(x) J_\nu(x) dx$ <p>See Bose, S.K., 1946: <i>Bull. Calcutta Math. Soc.</i> 38, 177-180.</p>                                                                                                                                                                                                                                                                                                                                                                                              |
| (32) | $\int_0^a x^{\mu-1} P_n(2x^2 a^{-2} - 1) J_\nu(x) dx$ $= \frac{2^{-\nu-1} a^{\mu+\nu} [\Gamma(\frac{1}{2}\mu + \frac{1}{2}\nu)]^2}{\Gamma(\nu+1) \Gamma(\frac{1}{2}\mu + \frac{1}{2}\nu + n + 1) \Gamma(\frac{1}{2} + \frac{1}{2}\nu - n)}$ $\times {}_2F_3\left(\frac{\mu+\nu}{2}, \frac{\mu+\nu}{2}; \nu+1, \frac{\mu+\nu}{2} + n + 1, \frac{\mu+\nu}{2} - n; -\frac{a^2}{4}\right)$ <p style="text-align: right;"><math>\operatorname{Re}(\mu + \nu) &gt; 0</math></p> <p>For particular cases see Bose, B.N., 1944: <i>Bull. Calcutta Math. Soc.</i> 36, 125-132.</p> |
| (33) | $\int_0^a x^{\frac{1}{2}-\mu} (a^2 - x^2)^{-\frac{1}{2}\mu} P_\nu^\mu(x/a) J_{\nu+\frac{1}{2}}(x) dx$ $= (\frac{1}{2}\pi)^{\frac{1}{2}} a^{1-\mu} J_{\frac{1}{2}-\mu}(\frac{1}{2}a) J_{\nu+\frac{1}{2}}(\frac{1}{2}a) \quad \operatorname{Re} \mu < 1, \quad \operatorname{Re}(\mu - \nu) < 2$                                                                                                                                                                                                                                                                            |
| (34) | $\int_0^a x^{\frac{1}{2}} (a^2 - x^2)^{-\frac{1}{2}\nu-\frac{1}{4}} P_\mu^{\nu+\frac{1}{2}}(2x^2 a^{-2} - 1) J_\nu(x) dx$ $= \pi^{\frac{1}{2}} 2^{-\nu-1} a J_{\mu+\frac{1}{2}}(\frac{1}{2}a) J_{-\mu-\frac{1}{2}}(\frac{1}{2}a) \quad -1 < \operatorname{Re} \nu < \frac{1}{2}$                                                                                                                                                                                                                                                                                          |

**Bessel functions of  $x$ ; finite intervals (cont'd)**

|      |                                                                                                                                                          |                                                                         |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------|
| (35) | $\int_0^a J_0(x) J_1(x) dx = \frac{1}{2} - \frac{1}{2}[J_0(a)]^2$                                                                                        |                                                                         |
| (36) | $\int_0^a J_n(x) J_{n+1}(x) dx = \frac{1}{2} - \frac{1}{2}[J_0(a)]^2 - \sum_{m=1}^n [J_m(a)]^2$                                                          | $n = 1, 2, 3, \dots$                                                    |
| (37) | $\int_0^a J_\nu(x) J_{\nu+1}(x) dx = \sum_{n=0}^{\infty} [J_{\nu+n+1}(a)]^2$                                                                             | $\text{Re } \nu > -1$                                                   |
| (38) | $\int_0^a x^{\rho-1} (a^2 - x^2)^{\sigma-1} J_\mu(x) J_\nu(x) dx$                                                                                        | See Bailey, W.N., 1938: <i>Quart. J. Math. Oxford Ser. 9</i> , 141-147. |
| (39) | $\int_0^a x P_n(1 - 2x^2 a^{-2}) [J_0(x)]^2 dx = \frac{a^2}{2(2n+1)} \{[J_n(a)]^2 + [J_{n+1}(a)]^2\}$                                                    | $n = 0, 1, 2, \dots$                                                    |
| (40) | $\int_0^a x^{2\nu+1} P_n(2x^2 a^{-2} - 1) [J_\nu(x)]^2 dx$                                                                                               | See Bose, B.N., 1944: <i>Bull. Calcutta Math. Soc.</i> 36, 125-132.     |
| (41) | $\int_a^b \frac{dx}{x [J_\nu(x)]^2} = \frac{\pi}{2} \left[ \frac{Y_\nu(b)}{J_\nu(b)} - \frac{Y_\nu(a)}{J_\nu(a)} \right]$                                |                                                                         |
| (42) | $\int_a^b \frac{dx}{x J_\nu(x) J_{-\nu}(x)} = \frac{\pi}{2 \sin(\nu\pi)} \log \left[ \frac{J_{-\nu}(a) J_\nu(b)}{J_\nu(a) J_{-\nu}(b)} \right]$          |                                                                         |
| (43) | $\int_0^a x^\nu Y_\nu(x) dx = 2^{\nu-1} \pi^{\frac{1}{2}} \Gamma(\nu + \frac{1}{2}) a [Y_\nu(a) \mathbf{H}_{\nu-1}(a) - \mathbf{H}_\nu(a) Y_{\nu-1}(a)]$ | $\text{Re } \nu > -\frac{1}{2}$                                         |



**Bessel functions of  $x$ ; finite intervals (cont'd)**

|      |                                                                                                                                                                                                                                                                                                                                                                                                                 |                              |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------|
| (44) | $\int_0^a x^{\nu+1} Y_\nu(x) dx = a^{\nu+1} Y_{\nu+1}(a) + 2^{\nu+1} \Gamma(\nu+1)$                                                                                                                                                                                                                                                                                                                             | $\operatorname{Re} \nu > -1$ |
| (45) | $\int_0^a x^{1-\nu} Y_\nu(x) dx = \frac{\operatorname{ctn}(\nu\pi)}{2^{\nu-1} \Gamma(\nu)} - a^{1-\nu} Y_{\nu-1}(a)$                                                                                                                                                                                                                                                                                            | $\operatorname{Re} \nu < 1$  |
| (46) | $\int_a^b Y_\nu(x) dx = 2 \sum_{n=0}^{\infty} [Y_{\nu+2n+1}(b) - Y_{\nu+2n+1}(a)]$                                                                                                                                                                                                                                                                                                                              |                              |
| (47) | $\int_0^a (a^2 - x^2)^{-\frac{1}{2}} \cos[\beta(a^2 - x^2)^{\frac{1}{2}}] Y_{2\nu}(x) dx$ $= \frac{1}{4} \pi [J_\nu(u) Y_\nu(v) + J_\nu(v) Y_\nu(u)] - \frac{1}{4} \pi \tan(\nu\pi) [J_\nu(u) J_\nu(v) + Y_\nu(u) Y_\nu(v)]$ $u = \frac{1}{2} a [(\beta^2 + 1)^{\frac{1}{2}} + \beta], \quad v = \frac{1}{2} a [(\beta^2 + 1)^{\frac{1}{2}} - \beta], \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$ |                              |
| (48) | $\int_0^a x P_n(1 - 2x^2 a^{-2}) J_0(x) Y_0(x) dx$ $= \frac{a^2}{2(2n+1)} [J_n(a) Y_n(a) + J_{n+1}(a) Y_{n+1}(a)]$                                                                                                                                                                                                                                                                                              | $n = 0, 1, 2, \dots$         |
| (49) | $\int_a^b \frac{dx}{x [Y_\nu(x)]^2} = \frac{\pi}{2} \left[ \frac{J_\nu(a)}{Y_\nu(a)} - \frac{J_\nu(b)}{Y_\nu(b)} \right]$                                                                                                                                                                                                                                                                                       |                              |
| (50) | $\int_a^b \frac{dx}{x J_\nu(x) Y_\nu(x)} = \frac{\pi}{2} \log \left[ \frac{J_\nu(a) Y_\nu(b)}{J_\nu(b) Y_\nu(a)} \right]$                                                                                                                                                                                                                                                                                       |                              |

**19.2. Bessel functions of argument  $x$ . Infinite intervals**

|     |                                 |                              |
|-----|---------------------------------|------------------------------|
| (1) | $\int_0^\infty J_\nu(x) dx = 1$ | $\operatorname{Re} \nu > -1$ |
|-----|---------------------------------|------------------------------|

**Bessel functions of  $x$ ; infinite intervals (cont'd)**

|     |                                                                                                                                                                                                                                                                                                                                        |
|-----|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (2) | $\int_0^\infty \frac{J_\nu(x)}{x^2 + a^2} dx = \frac{\pi [\mathbf{J}_\nu(a) - J_\nu(a)]}{a \sin(\nu\pi)} \quad \text{Re } a > 0, \quad \text{Re } \nu > -1$                                                                                                                                                                            |
| (3) | $\int_0^\infty x^{-1} (a^2 + x^2)^{-\frac{1}{2}} [J_0(x) - 1] dx = a^{-1} [\text{Ei}(-a) - \log(\gamma a)] \quad \text{Re } a > 0$                                                                                                                                                                                                     |
| (4) | $\int_0^\infty x [(x^2 + a^2)(x^2 + \beta^2)]^{-\frac{1}{2}} [(x^2 + a^2)^{\frac{1}{2}} + (x^2 + \beta^2)^{\frac{1}{2}}]^{-2\nu} J_0(x) dx$ $= (a^2 - \beta^2)^{-\nu} I_\nu\left(\frac{a - \beta}{2}\right) K_\nu\left(\frac{a + \beta}{2}\right)$ $\text{Re } a > 0, \quad \text{Re } \beta > 0, \quad \text{Re } \nu > -\frac{3}{4}$ |
| (5) | $\int_0^\infty x^3 (x^4 - a^4)^{-1} J_0(x) dx = \frac{1}{2} K_0(a) - \frac{1}{4} \pi Y_0(a) \quad a > 0$                                                                                                                                                                                                                               |
| (6) | $\int_0^\infty x^\nu (e^{ax} - 1)^{-1} J_\nu(x) dx = 2^\nu \pi^{-\frac{1}{2}} \Gamma(\nu + \frac{1}{2}) \sum_{n=1}^\infty (1 + a^2 n^2)^{-\nu - \frac{1}{2}}$ $\text{Re } a > 0, \quad \text{Re } \nu > 0$                                                                                                                             |
| (7) | $\int_0^\infty x^{-1} \exp\left(-\frac{2a}{x}\right) J_\nu(x) dx = 2 J_\nu(2a^{\frac{1}{2}}) K_\nu(2a^{\frac{1}{2}}) \quad \text{Re } a > 0$                                                                                                                                                                                           |
| (8) | $\int_0^\infty \frac{x^\nu}{x + \beta} \sin(x + \beta) J_\nu(x) dx = \frac{1}{2} \pi \sec(\nu\pi) \beta^\nu J_{-\nu}(\beta)$ $ \arg \beta  < \pi, \quad  \text{Re } \nu  < \frac{1}{2}$                                                                                                                                                |
| (9) | $\int_0^\infty \frac{x^\nu}{x + \beta} \cos(x + \beta) J_\nu(x) dx = -\frac{1}{2} \pi \sec(\nu\pi) \beta^\nu Y_{-\nu}(\beta)$ $ \arg \beta  < \pi, \quad  \text{Re } \nu  < \frac{1}{2}$                                                                                                                                               |

**Bessel functions of  $x$ ; infinite intervals (cont'd)**

|      |                                                                                                                                                                                                                                                 |                                |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------|
| (10) | $\int_0^\infty x^{1/4} \sin(2ax^{1/2}) J_{-1/4}(x) dx = \pi^{1/2} a^{3/2} J_{3/4}(a^2)$                                                                                                                                                         | $a > 0$                        |
| (11) | $\int_0^\infty x^{1/4} \sin(2ax^{1/2}) J_{1/4}(x) dx = \pi^{1/2} a^{3/2} J_{-1/4}(a^2)$                                                                                                                                                         | $a > 0$                        |
| (12) | $\int_0^\infty x^{1/4} \cos(2ax^{1/2}) J_{1/4}(x) dx = \pi^{1/2} a^{3/2} J_{-3/4}(a^2)$                                                                                                                                                         | $a > 0$                        |
| (13) | $\int_0^\infty x^{1/4} \cos(2ax^{1/2}) J_{-1/4}(x) dx = \pi^{1/2} a^{3/2} J_{1/4}(a^2)$                                                                                                                                                         | $a > 0$                        |
| (14) | $\int_0^\infty x^{-1/4} e^{-\alpha x} \sin(2\beta x^{1/2}) J_{-1/4}(x) dx$ $= \pi^{1/2} \left( \frac{\beta}{\alpha^2 + 1} \right)^{1/2} \exp\left(-\frac{\alpha\beta^2}{\alpha^2 + 1}\right) J_{-1/4}\left(\frac{\beta^2}{\alpha^2 + 1}\right)$ | $\operatorname{Re} \alpha > 0$ |
| (15) | $\int_0^\infty x^{-1/4} e^{-\alpha x} \sin(2\beta x^{1/2}) J_{1/4}(x) dx$ $= \pi^{1/2} \left( \frac{\beta}{\alpha^2 + 1} \right)^{1/2} \exp\left(-\frac{\alpha\beta^2}{\alpha^2 + 1}\right) J_{1/4}\left(\frac{\beta^2}{\alpha^2 + 1}\right)$   | $\operatorname{Re} \alpha > 0$ |
| (16) | $\int_0^\infty x^{-1/2} \sin x \sin(4ax^{1/2}) J_0(x) dx = (1/2)\pi^{1/2} \cos(a^2 + 1/4\pi) J_0(a^2)$                                                                                                                                          | $a > 0$                        |
| (17) | $\int_0^\infty x^{-1/3} \sin x \sin(4ax^{1/2}) J_{1/3}(x) dx$ $= -2^{-5/2} \pi^{1/2} a^{1/3} [\sin(a^2 + \pi/12) J_{1/3}(a^2)$ $+ \cos(a^2 + \pi/12) Y_{1/3}(a^2)]$                                                                             | $a > 0$                        |
| (18) | $\int_0^\infty x^{-1/2} \sin x \cos(4ax^{1/2}) J_0(x) dx$ $= -2^{-3/2} \pi^{1/2} [\cos(a^2 - 1/4\pi) J_0(a^2) - \sin(a^2 - 1/4\pi) Y_0(a^2)]$                                                                                                   | $a > 0$                        |

**Bessel functions of  $x$ ; infinite intervals (cont'd)**

|      |                                                                                                                                                                                                                                                           |                                                                 |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------|
| (19) | $\int_0^\infty x^{-1/3} \sin x \cos(4ax^{1/2}) J_{-1/3}(x) dx$ $= -2^{-3/2} \pi^{1/2} a^{1/3} \sin(a^2 - \pi/12) J_{-1/3}(a^2)$                                                                                                                           | $a > 0$                                                         |
| (20) | $\int_0^\infty x^{-1/2} \cos x \sin(4ax^{1/2}) J_0(x) dx$ $= (\frac{1}{2}\pi)^{1/2} \cos(a^2 - \frac{1}{4}\pi) J_0(a^2)$                                                                                                                                  | $a > 0$                                                         |
| (21) | $\int_0^\infty x^{-1/3} \cos x \sin(4ax^{1/2}) J_{1/3}(x) dx$ $= 2^{-5/2} \pi^{1/2} a^{1/3} [\cos(a^2 + \pi/12) J_{1/3}(a^2)$ $- \sin(a^2 + \pi/12) Y_{1/3}(a^2)]$                                                                                        | $a > 0$                                                         |
| (22) | $\int_0^\infty x^{-1/2} \cos x \cos(4ax^{1/2}) J_0(x) dx$ $= -2^{-3/2} \pi^{1/2} [\sin(a^2 - \frac{1}{4}\pi) J_0(a^2) + \cos(a^2 - \frac{1}{4}\pi) Y_0(a^2)]$                                                                                             | $a > 0$                                                         |
| (23) | $\int_0^\infty x^{-1/3} \cos x \cos(4ax^{1/2}) J_{-1/3}(x) dx$ $= 2^{-3/2} \pi^{1/2} a^{1/3} \cos(a^2 - \pi/12) J_{-1/3}(a^2)$                                                                                                                            | $a > 0$                                                         |
| (24) | $\int_0^\infty x^{-\rho} J_\mu(x) J_\nu(x) dx$ $= \frac{2^{-\rho} \Gamma(\rho) \Gamma[\frac{1}{2}(\mu + \nu - \rho + 1)]}{\Gamma[\frac{1}{2}(\rho + \mu + \nu + 1)] \Gamma[\frac{1}{2}(\rho - \mu + \nu + 1)] \Gamma[\frac{1}{2}(\rho + \mu - \nu + 1)]}$ | $0 < \operatorname{Re} \rho < \operatorname{Re}(\mu + \nu) + 1$ |
| (25) | $\int_0^\infty x^{1-2\nu} [J_\nu(x)]^4 dx = \frac{\Gamma(\nu) \Gamma(2\nu)}{2\pi [\Gamma(\nu + \frac{1}{2})]^2 \Gamma(3\nu)}$                                                                                                                             | $\operatorname{Re} \nu > 0$                                     |
| (26) | $\int_0^\infty \frac{x}{x^2 + a^2} [J_\nu(x)]^2 dx = I_\nu(a) K_\nu(a)$                                                                                                                                                                                   | $\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -1$     |

Bessel functions of  $x$ ; infinite intervals (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                      |                                                                         |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------|
| (27) | $\int_0^\infty (x^2 + a^2)^{-\frac{1}{2}} J_\mu(x) J_\nu(x) dx$                                                                                                                                                                                                                                                                                      |                                                                         |
|      | See Bouwkamp, C.S., 1950: <i>Nederl. Akad. Wetensch., Proc.</i> 53, 654-661.                                                                                                                                                                                                                                                                         |                                                                         |
| (28) | $\int_0^\infty x^{\rho-1} (x^2 + a^2)^{-\sigma} J_\mu(x) J_\nu(x) dx$                                                                                                                                                                                                                                                                                |                                                                         |
|      | See Watson, G.N., 1922: <i>A treatise on the theory of Bessel functions</i> , Cambridge, p. 436.                                                                                                                                                                                                                                                     |                                                                         |
| (29) | $\int_0^\infty x^{\mu+\nu} e^{-\alpha x} J_\mu(x) J_\nu(x) dx$                                                                                                                                                                                                                                                                                       |                                                                         |
|      | See Watson, G.N., 1922: <i>A treatise on the theory of Bessel functions</i> , Cambridge, p. 390.                                                                                                                                                                                                                                                     |                                                                         |
| (30) | $\int_0^\infty \sin(2ax) [J_\nu(x)]^2 dx$<br>$= \frac{1}{2} P_{\nu-\frac{1}{2}}(1-2a^2)$<br>$= \pi^{-1} \cos(\nu\pi) Q_{\nu-\frac{1}{2}}(2a^2-1)$                                                                                                                                                                                                    | $0 < a < 1$<br>$a > 1$<br>$\text{Re } \nu > -1$                         |
| (31) | $\int_0^\infty \sin(2ax) [x^\nu J_\nu(x)]^2 dx$<br>$= \frac{a^{-2\nu} \Gamma(\frac{1}{2} + \nu)}{2\pi^{\frac{1}{2}} \Gamma(1-\nu)} {}_2F_1(\frac{1}{2} + \nu, \frac{1}{2}; 1-\nu; a^2)$<br>$= \frac{a^{-4\nu-1} \Gamma(\frac{1}{2} + \nu)}{2\Gamma(1+\nu) \Gamma(\frac{1}{2} - 2\nu)} {}_2F_1(\frac{1}{2} + \nu, \frac{1}{2} + 2\nu; 1+\nu; a^{-2})$ | $0 < a < 1$<br>$a > 1$<br>$-\frac{1}{2} < \text{Re } \nu < \frac{1}{2}$ |

**Bessel functions of  $x$ ; infinite intervals (cont'd)**

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (32) | $\int_0^\infty \cos(2ax) [J_\nu(x)]^2 dx$ $= \pi^{-1} Q_{\nu-\frac{1}{2}}(1-2a^2) \quad 0 < a < 1$ $= -\pi^{-1} \sin(\nu\pi) Q_{\nu-\frac{1}{2}}(2a^2-1) \quad a > 1$ $\text{Re } \nu > -\frac{1}{2}$                                                                                                                                                                                                                                                                                                                                                         |
| (33) | $\int_0^\infty \cos(2ax) [x^\nu J_\nu(x)]^2 dx$ $= \frac{a^{-2\nu} \Gamma(\nu)}{2\pi^{\frac{1}{2}} \Gamma(\frac{1}{2}-\nu)} {}_2F_1(\nu+\frac{1}{2}, \frac{1}{2}; 1-\nu; a^2)$ $+ \frac{\Gamma(-\nu) \Gamma(\frac{1}{2}+2\nu)}{2\pi \Gamma(\frac{1}{2}-\nu)} {}_2F_1(\frac{1}{2}+\nu, \frac{1}{2}+2\nu; 1+\nu; a^2) \quad 0 < a < 1$ $= -\frac{\sin(\nu\pi) a^{-4\nu-1} \Gamma(\frac{1}{2}+2\nu)}{\Gamma(1+\nu) \Gamma(\frac{1}{2}-\nu)} {}_2F_1(\frac{1}{2}+\nu, \frac{1}{2}+2\nu; 1+\nu; a^{-2}) \quad a > 1$ $-\frac{1}{4} < \text{Re } \nu < \frac{1}{2}$ |
| (34) | $\int_a^\infty (x^2-a^2)^{-\frac{1}{2}} J_\nu(x) dx = -\frac{1}{2}\pi J_{\frac{1}{2}\nu}(\frac{1}{2}a) Y_{\frac{1}{2}\nu}(\frac{1}{2}a) \quad \text{Re } \nu > -1$                                                                                                                                                                                                                                                                                                                                                                                            |
| (35) | $\int_a^\infty x^{-1} (x^2-a^2)^{-\frac{1}{2}} J_0(x) dx = -\text{si}(a)$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     |
| (36) | $\int_a^\infty x^{\frac{1}{2}} P_{\nu-\frac{1}{2}}(x/a) J_\nu(x) dx = -(\frac{1}{2}a)^{-\frac{1}{2}} [\cos(\frac{1}{2}a) Y_\nu(\frac{1}{2}a)$ $+ \sin(\frac{1}{2}a) J_\nu(\frac{1}{2}a)] \quad -\frac{1}{2} < \text{Re } \nu < \frac{1}{2}$                                                                                                                                                                                                                                                                                                                   |
| (37) | $\int_a^\infty x^{\frac{1}{2}-\mu} (x^2-a^2)^{-\frac{1}{2}\mu} P_{\nu-\frac{1}{2}}^\mu(x/a) J_\nu(x) dx$ $= -2^{-3/2} \pi^{1/2} a^{1-\mu} [J_{\mu-\frac{1}{2}}(\frac{1}{2}a) Y_\nu(\frac{1}{2}a) + Y_{\mu-\frac{1}{2}}(\frac{1}{2}a) J_\nu(\frac{1}{2}a)]$ $-\frac{1}{4} < \text{Re } \mu < 1, \quad  \text{Re } \nu  < \frac{1}{2} + 2\text{Re } \mu$                                                                                                                                                                                                        |

Bessel functions of  $x$ ; infinite intervals (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                               |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (38) | $\int_a^\infty x^\nu (x^2 - a^2)^{\frac{1}{2}\lambda - \frac{1}{2}} P_\lambda^{\lambda-1}(x/a) J_\nu(x) dx$ $= \frac{2^{\lambda+\nu} a^\nu \Gamma(\frac{1}{2} + \nu)}{\pi^{\frac{1}{2}} \Gamma(1 - \lambda)} S_{\lambda-\nu, \lambda+\nu}(a)$ $\text{Re } \nu < 5/2, \quad \text{Re}(2\lambda + \nu) < 3/2$                                                                   |
| (39) | $\int_a^\infty x^{\frac{1}{2}} (x^2 - a^2)^{\frac{1}{2}\nu - \frac{1}{4}} P_{\frac{\mu}{2}}^{\frac{1}{2}-\nu}(2x^2 a^{-2} - 1) J_\nu(x) dx$ $= -2^{\nu-2} \pi^{\frac{1}{2}} a \sec(\mu\pi) \{ [J_{\mu+\frac{1}{2}}(\frac{1}{2}a)]^2 - [J_{-\mu-\frac{1}{2}}(\frac{1}{2}a)]^2 \}$ $\text{Re } \nu > -1/2, \quad \text{Re } \nu - 3/2 < 2\text{Re } \mu < 1/2 - \text{Re } \nu$ |
| (40) | $\int_a^\infty x^{1-2\nu} (x^2 - a^2)^{\nu-3/2} [J_\nu(x)]^2 dx = \frac{\Gamma(\nu - \frac{1}{2})}{2\pi^{\frac{1}{2}} a^{\nu+1}} \mathbf{H}_\nu(2a) \quad \text{Re } \nu > \frac{1}{2}$                                                                                                                                                                                       |
| (41) | $\int_a^\infty x^{2\nu+1} (a^2 - x^2)^{-\nu-3/2} \{ [J_\nu(x)]^2 + [J_{-\nu}(x)]^2 \} dx$ $= \pi^{-\frac{1}{2}} a^{\nu-1} \Gamma(-\nu - \frac{1}{2}) \sin(\nu\pi) J_\nu(2a) \quad \text{Re } \nu < -\frac{1}{2}$                                                                                                                                                              |
| (42) | $\int_{-\infty}^\infty \frac{\sin[a(x + \beta)]}{x + \beta} J_0(x) dx$ $= 2 \int_0^a (1 - u^2)^{-\frac{1}{2}} \cos(\beta u) du \quad 0 \leq a \leq 1$ $= \pi J_0(\beta) \quad 1 \leq a < \infty$                                                                                                                                                                              |
| (43) | $\int_{-\infty}^\infty \frac{ x }{x + \beta} \sin[a(x + \beta)] J_0(x) dx = 0 \quad 0 \leq a < 1$                                                                                                                                                                                                                                                                             |
| (44) | $\int_{-\infty}^\infty \frac{\sin[a(x + \beta)]}{x^\nu (x + \beta)} J_{\nu+2n}(x) dx = \pi \beta^{-\nu} J_{\nu+2n}(\beta) \quad 1 \leq a < \infty$ $n = 0, 1, 2, \dots, \quad \text{Re } \nu > -3/2$                                                                                                                                                                          |

Bessel functions of  $x$ ; infinite intervals (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                                                  |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (45) | $\int_{-\infty}^{\infty} \frac{\sin [a(x + \beta)]}{x + \beta} [J_{n+\frac{1}{2}}(x)]^2 dx = \pi [J_{n+\frac{1}{2}}(\beta)]^2$ $2 \leq a < \infty$ $n = 0, 1, 2, \dots$                                                                                                                                                                                                                          |
| (46) | $\int_{-\infty}^{\infty} \frac{\sin [a(x + \beta)]}{x + \beta} J_{n+\frac{1}{2}}(x) J_{-n-\frac{1}{2}}(x) dx = \pi J_{n+\frac{1}{2}}(\beta) J_{-n-\frac{1}{2}}(\beta)$ $n = 0, 1, 2, \dots, \quad 2 \leq a < \infty$                                                                                                                                                                             |
| (47) | $\int_{-\infty}^{\infty} \frac{\sin [a(x + \beta)]}{x^{2\nu}(x + \beta)} [J_{\nu+n}(x)]^2 dx = \pi \beta^{-2\nu} [J_{\nu+n}(\beta)]^2$ $2 \leq a < \infty$ $n = 0, 1, 2, \dots, \quad \operatorname{Re} \nu > -1$                                                                                                                                                                                |
| (48) | $\int_0^{\infty} Y_{\nu}(x) dx = -\tan(\frac{1}{2}\nu\pi)$ $-1 < \operatorname{Re} \nu < 1$                                                                                                                                                                                                                                                                                                      |
| (49) | $\int_0^{\infty} x^{\rho}(x^2 + a^2)^{-\mu} Y_{\nu}(x) dx$ $= \frac{a^{\rho-2\mu}}{\Gamma(\mu)} G_{24}^{31} \left( \frac{a^2}{4} \left  \begin{array}{c} 1 - \frac{\rho}{2}, -\frac{\nu}{2} \\ \mu - \frac{\rho}{2}, \frac{1+\nu}{2}, \frac{1-\nu}{2}, -\frac{\nu}{2} \end{array} \right. \right)$ $ \operatorname{Re} \nu  - 1 < \operatorname{Re} \rho < 2\operatorname{Re} \mu + \frac{1}{2}$ |
| (50) | $\int_0^{\infty} x^{-1} \exp\left(-\frac{2a}{x}\right) Y_{\nu}(x) dx = 2 Y_{\nu}(2a^{\frac{1}{2}}) K_{\nu}(2a^{\frac{1}{2}})$ $\operatorname{Re} a > 0$                                                                                                                                                                                                                                          |
| (51) | $\int_0^{\infty} x^{-1} \sin\left(\frac{a}{2x}\right) [\sin x J_0(x) + \cos x Y_0(x)] dx$ $= \pi J_0(a^{\frac{1}{2}}) Y_0(a^{\frac{1}{2}})$ $a > 0$                                                                                                                                                                                                                                              |



**Bessel functions of  $x$ ; infinite intervals (cont'd)**

|      |                                                                                                                                                                                                             |                                                                       |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------|
| (52) | $\int_0^\infty x^{-1} \cos\left(\frac{a}{2x}\right) [\sin x Y_0(x) - \cos x J_0(x)] dx$ $= \pi J_0(a^{1/2}) Y_0(a^{1/2})$                                                                                   | $a > 0$                                                               |
| (53) | $\int_0^\infty x^{1/4} \sin(2ax^{1/2}) Y_{1/4}(x) dx = -\pi^{1/2} a^{3/2} \mathbf{H}_{-1/4}(a^2)$                                                                                                           | $a > 0$                                                               |
| (54) | $\int_0^\infty x^{1/4} \cos(2ax^{1/2}) Y_{1/4}(x) dx = -\pi^{1/2} a^{3/2} \mathbf{H}_{-1/4}(a^2)$                                                                                                           | $a > 0$                                                               |
| (55) | $\int_0^\infty x^{-1/2} \sin x \cos(4ax^{1/2}) Y_0(x) dx$ $= 2^{-3/2} \pi^{1/2} [3 \sin(a^2 - \frac{1}{4}\pi) J_0(a^2) - \cos(a^2 - \frac{1}{4}\pi) Y_0(a^2)]$                                              | $a > 0$                                                               |
| (56) | $\int_0^\infty x^{-1/2} \cos x \cos(4ax^{1/2}) Y_0(x) dx$ $= -2^{-3/2} \pi^{1/2} [3 \cos(a^2 - \frac{1}{4}\pi) J_0(a^2) + \sin(a^2 - \frac{1}{4}\pi) Y_0(a^2)]$                                             | $a > 0$                                                               |
| (57) | $\int_0^\infty J_{\nu+n}(x) Y_{\nu-n}(x) dx = \frac{(-1)^{n+1}}{2}$                                                                                                                                         | $\operatorname{Re} \nu > -\frac{1}{2}, \quad n = 0, 1, 2, \dots$      |
| (58) | $\int_0^\infty e^{-2ax} J_0(x) Y_0(x) dx = \frac{K[a(a^2 + 1)^{-1/2}]}{\pi(a^2 + 1)^{1/2}}$                                                                                                                 | $\operatorname{Re} a > 0$                                             |
| (59) | $\int_0^\infty x^{2\nu+1} \exp(-ax^2) J_\nu(x) Y_\nu(x) dx$ $= -\frac{1}{2} \pi^{-1/2} a^{-3\nu/2-1/2} \exp\left(-\frac{1}{2a}\right) \mathcal{W}_{\frac{1}{2}\nu, \frac{1}{2}\nu}\left(\frac{1}{a}\right)$ | $\operatorname{Re} a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$ |

**Bessel functions of  $x$ ; infinite intervals (cont'd)**

|      |                                                                                                                                                                                                                                                                                                |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (60) | $\int_0^\infty \sin(2ax) J_0(x) Y_0(x) dx$ $= 0 \quad 0 < a < 1$ $= -\frac{K[(1-a^{-2})^{1/2}]}{\pi a} \quad a > 1$                                                                                                                                                                            |
| (61) | $\int_0^\infty \cos(2ax) J_0(x) Y_0(x) dx$ $= -\pi^{-1} K(a) \quad 0 < a < 1$ $= -(\pi a)^{-1} K(a^{-1}) \quad a > 1$                                                                                                                                                                          |
| (62) | $\int_0^\infty \cos(2ax) [Y_0(x)]^2 dx$ $= \pi^{-1} K[(1-a^2)^{1/2}] \quad 0 < a < 1$ $= 2(\pi a)^{-1} K[(1-a^{-2})^{1/2}] \quad a > 1$                                                                                                                                                        |
| (63) | $\int_0^\infty x^{1-2\nu} \sin(2ax) J_\nu(x) Y_\nu(x) dx$ $= -\frac{\Gamma(3/2-\nu) a}{2 \Gamma(2\nu-\frac{1}{2}) \Gamma(2-\nu)} {}_2F_1\left(\frac{3}{2}-\nu, \frac{3}{2}-2\nu; 2-\nu; a^2\right)$ $0 < \operatorname{Re} \nu < 3/2, \quad 0 < a < 1$                                         |
| (64) | $\int_0^\infty x^{1-2\nu} \sin(2ax) \{[J_\nu(x)]^2 - [Y_\nu(x)]^2\} dx$ $= \frac{\sin(2\nu\pi) \Gamma(3/2-\nu) \Gamma(3/2-2\nu) a}{\pi \Gamma(2-\nu)} {}_2F_1\left(\frac{3}{2}-\nu, \frac{3}{2}-2\nu; 2-\nu; a^2\right)$ $0 < \operatorname{Re} \nu < \frac{3}{4}, \quad 0 < a < 1$            |
| (65) | $\int_0^\infty x^{2-2\nu} \sin(2ax) [J_\nu(x) J_{\nu-1}(x) - Y_\nu(x) Y_{\nu-1}(x)] dx$ $= -\frac{\sin(2\nu\pi) \Gamma(3/2-\nu) \Gamma(5/2-2\nu) a}{\pi \Gamma(2-\nu)} {}_2F_1\left(\frac{3}{2}-\nu, \frac{5}{2}-2\nu; 2-\nu; a^2\right)$ $1/2 < \operatorname{Re} \nu < 5/4, \quad 0 < a < 1$ |

Bessel functions of  $x$ ; infinite intervals (cont'd)

|      |                                                                                                                                                                                                                                                                              |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (66) | $\int_0^\infty x^{2-2\nu} \sin(2ax) [J_\nu(x) Y_{\nu-1}(x) + Y_\nu(x) J_{\nu-1}(x)] dx$ $= -\frac{\Gamma(3/2-\nu) a}{\Gamma(2\nu-3/2) \Gamma(2-\nu)} {}_2F_1\left(\frac{3}{2}-\nu, \frac{5}{2}-2\nu; 2-\nu; a^2\right)$ $1/2 < \operatorname{Re} \nu < 5/2, \quad 0 < a < 1$ |
| (67) | $\int_a^\infty x^{1/2-\mu} (a^2-x^2)^{-1/2} P_{\nu-1/2}^\mu(x/a) Y_\nu(x) dx$ $= 2^{-3/2} \pi^{1/2} a^{1-\mu} [J_\nu(1/2 a) J_{\mu-1/2}(1/2 a) - Y_\nu(1/2 a) Y_{\mu-1/2}(1/2 a)]$ $-1/4 < \operatorname{Re} \mu < 1, \quad \operatorname{Re}(2\mu-\nu) > -1/2$              |

19.3. Bessel functions of arguments  $ax + \beta$ ,  $x^2$ ,  $x^{-1}$ 

|     |                                                                                                                                                                                                                                                                                                                                                                             |
|-----|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $\int_0^\infty x^{\rho-1} J_\mu(ax) J_\nu(bx) dx = \frac{2^{\rho-1} a^\mu b^{-\mu-\rho} \Gamma(1/2\mu + 1/2\nu + 1/2\rho)}{\Gamma(\mu+1) \Gamma(1-1/2\mu + 1/2\nu - 1/2\rho)}$ $\times {}_2F_1\left(\frac{\mu+\nu+\rho}{2}, \frac{\mu-\nu+\rho}{2}; \mu+1; \frac{a^2}{b^2}\right)$ $\operatorname{Re}(\mu+\nu+\rho) > 0, \quad \operatorname{Re} \rho < 2, \quad 0 < a < b$ |
| (2) | $\int_0^\infty x^{1/2} (x^2 + \lambda^2)^{-1/2} J_\mu(ax) J_\nu(bx) dx$ <p>See Bouwkamp, 'C.S., 1950: <i>Nederl. Akad. Wetensch., Proc.</i> 53, 654-661.</p>                                                                                                                                                                                                                |
| (3) | $\int_0^\infty x^{1+\nu} [J_\nu(ax)]^2 J_\nu(2bx) dx$ $= -\frac{\Gamma(1/2+\nu)}{2\pi^{3/2}} \sin(\nu\pi) a^{2\nu} b^{-\nu-1} (b^2 - a^2)^{-\nu-1/2} \quad 0 < a < b$ $= \frac{\Gamma(1/2+\nu)}{2\pi^{3/2}} a^{2\nu} b^{-\nu-1} (a^2 - b^2)^{-\nu-1/2} \quad 0 < b < a$ $-1/2 < \operatorname{Re} \nu < 1/2$                                                                |

**Bessel functions of  $\alpha x + \beta$ ,  $x^2$ ,  $x^{-1}$  (cont'd)**

|     |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |
|-----|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (4) | $\int_0^\infty x^{1+\nu} J_\nu(ax) J_{-\nu}(ax) J_\nu(2bx) dx$ $= 0 \quad 0 < a < b$ $= \frac{a^{2\nu} b^{-\nu-1}}{2\pi^{1/2} \Gamma(1/2 - \nu)} (a^2 - b^2)^{-\nu-1/2} \quad 0 < b < a$ $-1 < \operatorname{Re} \nu < 1/2$                                                                                                                                                                                                                                                                                                                        |
| (5) | $\int_0^\infty x J_{1/2\nu-1/4}(ax) J_{1/2\nu+1/4}(ax) J_\nu(2bx) dx$ $= 0 \quad 0 < a < b$ $= 2^{-3/2} \pi^{-1} a^{-1/2} b^{-1} (a-b)^{-1/2} \quad 0 < b < a$ $\operatorname{Re} \nu > -1$                                                                                                                                                                                                                                                                                                                                                        |
| (6) | $\int_0^\infty x^{1+\nu} J_\mu(ax) J_{-\mu}(ax) J_\nu(2bx) dx$ $= 0 \quad 0 < a < b$ $= \frac{(a^2 - b^2)^{-1/2\nu-1/4}}{2\pi^{1/2} ab^{1/2}} P_{\mu-1/2}^{\nu+1/2}(2b^2 a^{-2} - 1) \quad 0 < b < a$ $-1 < \operatorname{Re} \nu < 1/2$                                                                                                                                                                                                                                                                                                           |
| (7) | $\int_0^\infty x^{\rho-1} J_\lambda(ax) J_\mu(ax) J_\nu(2bx) dx$ $= \frac{a^{\lambda+\mu} b^{-\lambda-\mu-\rho} \Gamma(1/2\lambda + 1/2\mu + 1/2\nu + 1/2\rho)}{2^{\lambda+\mu} \Gamma(\lambda+1) \Gamma(\mu+1) \Gamma(1-1/2\lambda-1/2\mu+1/2\nu-1/2\rho)}$ $\times {}_4F_3\left(\frac{\lambda+\mu+1}{2}, \frac{\lambda+\mu}{2}+1, \frac{\lambda+\mu+\nu+\rho}{2}, \frac{\lambda+\mu-\nu+\rho}{2}; \right.$ $\left. \lambda+1, \mu+1, \lambda+\mu+1; \frac{a^2}{b^2}\right)$ $\operatorname{Re}(\lambda + \mu + \nu + \rho) > 0, \quad 0 < a < b$ |

**Bessel functions of  $ax + \beta$ ,  $x^2$ ,  $x^{-1}$ , (cont'd)**

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (8)  | $\int_0^\infty x^{1-\nu} J_\nu(ax) J_\nu(bx) J_\nu(cx) dx$ $= \frac{2^{\nu-1} \Delta^{2\nu-1}}{\pi^{\frac{1}{2}} (abc)^\nu \Gamma(\nu + \frac{1}{2})} \quad \begin{array}{l} \text{if } a, b, c \text{ are sides of a triangle} \\ \text{of area } \Delta \end{array}$ $= 0 \quad \begin{array}{l} \text{if } a, b, c \text{ are not sides of a triangle} \\ a, b, c > 0, \quad \text{Re } \nu > -\frac{1}{2} \end{array}$                                                                                                                                                                                                                                                                                                                                                                                   |
| (9)  | $\int_0^\infty x^{\rho-1} J_\lambda(ax) J_\mu(bx) J_\nu(cx) dx$ $= \frac{2^{\rho-1} a^\lambda b^\mu c^{-\lambda-\mu-\rho} \Gamma(\frac{1}{2}\lambda + \frac{1}{2}\mu + \frac{1}{2}\nu + \frac{1}{2}\rho)}{\Gamma(\lambda + 1) \Gamma(\mu + 1) \Gamma(1 - \frac{1}{2}\lambda - \frac{1}{2}\mu + \frac{1}{2}\nu - \frac{1}{2}\rho)}$ $\times F_4\left(\frac{\lambda + \mu - \nu + \rho}{2}, \frac{\lambda + \mu + \nu + \rho}{2}; \lambda + 1, \mu + 1; \frac{a^2}{c^2}, \frac{b^2}{c^2}\right)$ $\text{Re}(\lambda + \mu + \nu + \rho) > 0, \quad \text{Re } \rho < 5/2, \quad a, b, c > 0, \quad c > a + b$ <p>For particular cases see Watson, G.N., 1922: <i>A treatise on the theory of Bessel functions</i>. Cambridge, Sec.13.46; Bailey, W.N., 1936: <i>Proc. London Math. Soc.</i> (2) 40, 37-48.</p> |
| (10) | $\int_0^\infty x^{1-2\nu} [J_\nu(ax) J_\nu(bx)]^2 dx$ $= \frac{a^{2\nu-1} b^{-1} \Gamma(\nu)}{2\pi \Gamma(\nu + \frac{1}{2}) \Gamma(2\nu + \frac{1}{2})} {}_2F_1\left(\nu, \frac{1}{2} - \nu, 2\nu + \frac{1}{2}; \frac{a^2}{b^2}\right)$ $\text{Re } \nu > 0, \quad 0 < a < b$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |
| (11) | $\int_0^\infty x^{1-\nu} [J_\nu(x) Y_{-\nu}(x) + Y_\nu(x) J_{-\nu}(x)] J_\nu(2ax) dx$ $= 0 \quad 0 < a < 1$ $= \frac{a^{\nu-1} (a^2 - 1)^{\nu-\frac{1}{2}}}{\pi^{\frac{1}{2}} \Gamma(\nu + \frac{1}{2})} \quad a > 1$ $-\frac{1}{2} < \text{Re } \nu < 1$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |

**Bessel functions of  $ax + \beta$ ,  $x^2$ ,  $x^{-1}$  (cont'd)**

|      |                                                                                                                                                                                                                                                                                                                                                                                          |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (12) | $\int_0^\infty x^{\mu+1} [J_\nu(x) Y_\mu(x) + J_\mu(x) Y_\nu(x)] J_\nu(2ax) dx$ $= 0 \quad 0 < a < 1$ $= -\frac{a^{-\mu-1} (a^2 - 1)^{-\frac{1}{2}\mu - \frac{1}{4}}}{\pi^{\frac{1}{2}} 2^{\mu+\frac{1}{2}}} P_{\nu-\frac{1}{2}}^{\mu+\frac{1}{2}}(a) \quad a > 1$ $-1 < \operatorname{Re} \mu < \frac{1}{2}, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re}(\mu + \nu) > -1$ |
| (13) | $\int_0^\infty x^{1+\mu} Y_\mu(ax) J_\nu(bx) J_\nu(cx) dx = 0$ $0 < b < c, \quad 0 < a < c - b$                                                                                                                                                                                                                                                                                          |
| (14) | $\int_0^\infty x J_\nu^2(ax) J_\nu(bx) Y_\nu(bx) dx$ $= 0 \quad 0 < a < b$ $= -(2\pi ab)^{-1} \quad 0 < b < a$ $\operatorname{Re} \nu > -\frac{1}{2}$                                                                                                                                                                                                                                    |
| (15) | $\int_0^\infty x^{2\nu+1} J_\nu(ax) Y_\nu(ax) J_\nu(bx) Y_\nu(bx) dx$ $= \frac{a^{2\nu} \Gamma(3\nu+1)}{2\pi b^{4\nu+2} \Gamma(\frac{1}{2}-\nu) \Gamma(2\nu+3/2)} {}_2F_1\left(\nu+\frac{1}{2}, 3\nu+1; 2\nu+\frac{3}{2}; \frac{a^2}{b^2}\right)$ $0 < a < b, \quad -1/3 < \operatorname{Re} \nu < 1/2$                                                                                  |
| (16) | $\int_0^\infty \frac{J_\nu(ax) Y_\nu(bx) - J_\nu(bx) Y_\nu(ax)}{x \{ [J_\nu(bx)]^2 + [Y_\nu(bx)]^2 \}} dx = -\frac{\pi}{2} \left(\frac{b}{a}\right)^\nu \quad 0 < b < a$                                                                                                                                                                                                                 |
|      | <p>For other similar integrals see sec. 6.8.</p>                                                                                                                                                                                                                                                                                                                                         |

**Bessel functions of  $\alpha x + \beta$ ,  $x^2$ ,  $x^{-1}$  (cont'd)**

|      |                                                                                                                                                                                                                                                                                                                |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (17) | $\int_0^\infty \frac{[J_\nu(ax) Y_\nu(bx) - J_\nu(bx) Y_\nu(ax)]}{[J_\nu(bx)]^2 + [Y_\nu(bx)]^2}$ $\times [J_{\nu+1}(cx) Y_\nu(bx) - J_\nu(bx) Y_{\nu+1}(cx)] dx$ $= -\frac{b^{2\nu}}{a^\nu c^{\nu+1}} \quad 0 < b < c < a$ $= \frac{a^\nu}{c^{\nu+1}} - \frac{b^{2\nu}}{a^\nu c^{\nu+1}} \quad 0 < b < a < c$ |
| (18) | $\int_0^\infty \frac{[J_\nu(ax) Y_\nu(bx) - J_\nu(bx) Y_\nu(ax)]}{[J_\nu(bx)]^2 + [Y_\nu(bx)]^2}$ $\times [J_{\nu+1}(ax) Y_\nu(bx) - J_\nu(bx) Y_{\nu+1}(ax)] dx$ $= \frac{1}{2a} - \frac{b^{2\nu}}{a^{2\nu+1}} \quad 0 < b < a$                                                                               |
| (19) | $\int_0^\infty \frac{J_0(ax) Y_0(bx) - J_0(bx) Y_0(ax)}{[J_0(bx)]^2 + [Y_0(bx)]^2} \frac{x dx}{\lambda^2 + x^2} = -\frac{\pi H_\nu^{(2)}(a\lambda)}{2 H_\nu^{(2)}(b\lambda)}$ <p align="right"><math>\operatorname{Re} \lambda &gt; 0, \quad 0 &lt; b &lt; a</math></p>                                        |
| (20) | $\int_0^\infty \frac{J_0(ax) Y_0(bx) - J_0(bx) Y_0(ax)}{[J_0(bx)]^2 + [Y_0(bx)]^2} \frac{x dx}{c^2 - x^2}$ $= \frac{\pi}{2} \frac{J_0(ac) J_0(bc) + Y_0(ac) Y_0(bc)}{[J_0(bc)]^2 + [Y_0(bc)]^2} \quad 0 < b < a, \quad c > 0$                                                                                  |
| (21) | $\int_0^a J_\nu(x) J_{-\nu}(a-x) dx = \sin a \quad -1 < \operatorname{Re} \nu < 1$                                                                                                                                                                                                                             |
| (22) | $\int_0^a J_\nu(x) J_{1-\nu}(a-x) dx = J_0(a) - \cos a \quad -1 < \operatorname{Re} \nu < 2$                                                                                                                                                                                                                   |

**Bessel functions of  $ax + \beta$ ,  $x^2$ ,  $x^{-2}$ (cont'd)**

|      |                                                                                                                                                                                                                                                                                                                                                   |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (23) | $\int_0^a J_\mu(x) J_\nu(a-x) dx = 2 \sum_{m=0}^{\infty} (-1)^m J_{\mu+\nu+2m+1}(a)$ <p align="right"><math>\operatorname{Re} \mu &gt; -1, \quad \operatorname{Re} \nu &gt; -1</math></p>                                                                                                                                                         |
| (24) | $\int_0^a x^{-1} J_\mu(x) J_\nu(a-x) dx = \mu^{-1} J_{\mu+\nu}(a)$ <p align="right"><math>\operatorname{Re} \mu &gt; 0, \quad \operatorname{Re} \nu &gt; -1</math></p>                                                                                                                                                                            |
| (25) | $\int_0^a x^{\lambda-1} J_\mu(x) J_\nu(a-x) dx = 2^\lambda \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(\lambda+\mu+m) (\lambda)_m}{m! \Gamma(\mu+m+1)}$ $\times J_{\lambda+\mu+\nu+2m}(a)$ <p align="right"><math>\operatorname{Re}(\lambda+\mu) &gt; 0, \quad \operatorname{Re} \nu &gt; -1</math></p>                                               |
| (26) | $\int_0^a x^{-1}(a-x)^{-1} J_\mu(x) J_\nu(a-x) dx = (\mu^{-1} + \nu^{-1}) a^{-1} J_{\mu+\nu}(a)$ <p align="right"><math>\operatorname{Re} \mu &gt; 0, \quad \operatorname{Re} \nu &gt; 0</math></p>                                                                                                                                               |
| (27) | $\int_0^a x^{\lambda-1}(a-x)^{-1} J_\mu(x) J_\nu(a-x) dx$ $= \frac{2^\lambda}{\nu a} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(\lambda+\mu+m) (\lambda)_m}{m! \Gamma(\mu+m+1)} (\lambda+\mu+\nu+2m)$ $\times J_{\lambda+\mu+\nu+2m}(a)$ <p align="right"><math>\operatorname{Re}(\lambda+\mu) &gt; 0, \quad \operatorname{Re} \nu &gt; 0</math></p> |
| (28) | $\int_0^a x^\mu(a-x)^\nu J_\mu(x) J_\nu(a-x) dx$ $= \frac{\Gamma(\mu+\frac{1}{2}) \Gamma(\nu+\frac{1}{2})}{2^{\frac{1}{2}} \pi^{\frac{1}{2}} \Gamma(\mu+\nu+1)} a^{\mu+\nu+\frac{1}{2}} J_{\mu+\nu+\frac{1}{2}}(a)$ <p align="right"><math>\operatorname{Re} \mu &gt; -\frac{1}{2}, \quad \operatorname{Re} \nu &gt; -\frac{1}{2}</math></p>      |
| (29) | $\int_0^a x^\mu(a-x)^{\nu+1} J_\mu(x) J_\nu(a-x) dx$ $= \frac{\Gamma(\mu+1/2) \Gamma(\nu+3/2)}{2^{\frac{1}{2}} \pi^{\frac{1}{2}} \Gamma(\mu+\nu+2)} a^{\mu+\nu+3/2} J_{\mu+\nu+1/2}(a)$ <p align="right"><math>\operatorname{Re} \nu &gt; -1, \quad \operatorname{Re} \mu &gt; -\frac{1}{2}</math></p>                                            |



**Bessel functions of  $ax + \beta, x^2, x^{-1}$  (cont'd)**

|      |                                                                                                                                                                                                                                                                                                                                                                 |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (30) | $\int_0^a x^\mu (a-x)^{-\mu-1} J_\mu(x) J_\nu(a-x) dx = \frac{2^\mu \Gamma(\mu + \frac{1}{2}) \Gamma(\nu - \mu)}{\pi^{\frac{1}{2}} \Gamma(\mu + \nu + 1)} a^\mu J_\nu(a)$ <p align="right"><math>\operatorname{Re} \nu &gt; \operatorname{Re} \mu &gt; -\frac{1}{2}</math></p>                                                                                  |
| (31) | $\int_0^a x^{\rho-1} (a-x)^{\sigma-1} J_\mu(x) J_\nu(a-x) dx$ $\int_0^a x^{\rho-1} (a-x)^{\sigma-1} J_\lambda(bx) J_\mu(cx) J_\nu(a-x) dx$ <p>For these integrals and several particular cases see Bailey, W.N., 1930: <i>Proc. London Math. Soc.</i> (2) 30, 422-424 and 31, 200-208; Rutgers, J.G., 1931: <i>Nederl. Akad. Wetensch. Proc.</i> 44, 75-85.</p> |
| (32) | $\int_{-\infty}^{\infty} \frac{J_\mu[a(x+y)]}{(x+y)^\mu} \frac{J_\nu[a(x+z)]}{(x+z)^\nu} dx$ $= \frac{(2\pi/a)^{\frac{1}{2}} \Gamma(\mu + \nu)}{\Gamma(\mu + \frac{1}{2}) \Gamma(\nu + \frac{1}{2})} \frac{J_{\mu+\nu-\frac{1}{2}}[a(y-z)]}{(y-z)^{\mu+\nu-\frac{1}{2}}}$ <p align="right"><math>a &gt; 0, \operatorname{Re}(\mu + \nu) &gt; 0</math></p>       |
| (33) | $\int_0^\infty x^2 J_{2\nu}(2ax) J_{\nu-\frac{1}{2}}(x^2) dx = \frac{1}{2} a J_{\nu+\frac{1}{2}}(a^2) \quad a > 0, \quad \operatorname{Re} \nu > -\frac{1}{2}$                                                                                                                                                                                                  |
| (34) | $\int_0^\infty x^3 J_{2\nu}(2ax) J_{\nu-\frac{1}{2}}(x^2) dx = \frac{1}{2} \csc(\nu\pi) \mathbf{H}_{-\nu-\frac{1}{2}}(a^2)$ $- \frac{1}{2} \operatorname{ctn}(\nu\pi) J_{\nu+\frac{1}{2}}(a^2) - \frac{1}{2} Y_{\nu+\frac{1}{2}}(a^2) \quad a > 0, \quad \operatorname{Re} \nu > -\frac{3}{4}$                                                                  |
| (35) | $\int_0^\infty x^2 J_{2\nu}(2ax) J_{\nu+\frac{1}{2}}(x^2) dx = \frac{1}{2} a J_{\nu-\frac{1}{2}}(a^2) \quad a > 0, \quad \operatorname{Re} \nu > -2$                                                                                                                                                                                                            |
| (36) | $\int_0^\infty x^2 J_{2\nu}(2ax) Y_{\nu+\frac{1}{2}}(x^2) dx = -\frac{1}{2} a \mathbf{H}_{\nu-\frac{1}{2}}(a^2) \quad a > 0, \quad \operatorname{Re} \nu > -2$                                                                                                                                                                                                  |

**Bessel functions of  $\alpha x + \beta$ ,  $x^2$ ,  $x^{-1}$  (cont'd)**

|      |                                                                                                                                                                                                                                                                         |                                |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------|
| (37) | $\int_0^\infty x J_0(4ax) [J_0(x^2)]^2 dx = -\frac{1}{4} J_0(a^2) Y_0(a^2)$                                                                                                                                                                                             | $a > 0$                        |
| (38) | $\int_0^\infty x J_0(4ax) J_0(x^2) Y_0(x^2) dx = -\frac{1}{4} [J_0(a^2)]^2$                                                                                                                                                                                             | $a > 0$                        |
| (39) | $\int_0^\infty x^{(\nu+2)/3} \sin(x^2) J_{(\nu+1/2)/3}(x^2) J_\nu(4ax) dx$ $= -\frac{a^{(\nu-1)/3}}{8} \left[ J_{(\nu+1/2)/3}(a^2) \sin\left(a^2 + \frac{\nu-1}{6}\pi\right) \right.$ $\left. + Y_{(\nu+1/2)/3}(a^2) \cos\left(a^2 + \frac{\nu-1}{4}\pi\right) \right]$ | $-5/2 < \text{Re } \nu < -1/2$ |
| (40) | $\int_0^\infty x^{(\nu+2)/3} \cos(x^2) J_{(\nu+1/2)/3}(x^2) J_\nu(2ax) dx$ $= \frac{a^{(\nu-1)/3}}{8} \left[ J_{(\nu+1/2)/3}(a^2) \cos\left(a^2 + \frac{\nu-1}{6}\pi\right) \right.$ $\left. + Y_{(\nu+1/2)/3}(a^2) \sin\left(a^2 + \frac{\nu-1}{6}\pi\right) \right]$  | $-1 < \text{Re } \nu < 11/2$   |
| (41) | $\int_0^\infty x \sin(ax^2) J_\nu(bx^2) J_{2\nu}(2cx) dx$ $= \frac{1}{2} h \sin(ac^2 h) J_\nu(bc^2 h)$ $= \frac{1}{2} k \cos(ac^2 k) J_\nu(bc^2 k)$ $\text{Re } \nu > -1, \quad h = (b^2 - a^2)^{-1/2}, \quad k = (a^2 - b^2)^{-1/2}$                                   | $0 < a < b$<br>$0 < b < a$     |
| (42) | $\int_0^\infty x \cos(ax^2) J_\nu(bx^2) J_{2\nu}(2cx) dx$ $= \frac{1}{2} h \cos(ac^2 h) J_\nu(bc^2 h)$ $= \frac{1}{2} k \sin(ac^2 k) J_\nu(bc^2 k)$ $\text{Re } \nu > -\frac{1}{2}, \quad h = (b^2 - a^2)^{-1/2}, \quad k = (a^2 - b^2)^{-1/2}$                         | $0 < a < b$<br>$0 < b < a$     |

Bessel functions of  $ax + \beta$ ,  $x^2$ ,  $x^{-1}$  (cont'd)

|      |                                                                                                                                                                                                                                                                      |                                                                                                   |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------|
| (43) | $\int_0^a \frac{x^2}{a^2 - x^2} J_{\frac{1}{4}}(x) J_{-\frac{1}{4}}(x) J_{2\nu}(2a^2 - 2x^2) dx$ $= \frac{a}{4\nu} J_{\nu+\frac{1}{4}}(a^2) J_{\nu-\frac{1}{4}}(a^2)$                                                                                                | $\operatorname{Re} \nu > 0$                                                                       |
| (44) | $\int_0^\infty J_\nu\left(\frac{x}{a}\right) J_{\nu+1}\left(\frac{b}{x}\right) \frac{dx}{x} = \left(\frac{a}{b}\right)^{\frac{1}{2}} J_{2\nu+1}\left(\frac{2b^{\frac{1}{2}}}{a^{\frac{1}{2}}}\right)$                                                                | $a, b > 0, \quad \operatorname{Re} \nu > -1$                                                      |
| (45) | $\int_0^\infty x^{\rho-1} J_\mu(ax) J_\nu(bx^{-1}) dx$ $= 2^{\rho-1} a^{-\rho} G_{04}^{20} \left( \frac{a^2 b^2}{16} \middle  \frac{\nu}{2}, \frac{\rho+\mu}{2}, \frac{\rho-\mu}{2}, -\frac{\nu}{2} \right)$                                                         | $a, b > 0, \quad \operatorname{Re}(\rho - \nu) < 3/2, \quad \operatorname{Re}(\rho + \mu) > -3/2$ |
| (46) | $\int_0^\infty J_\nu\left(\frac{a}{x}\right) Y_\nu\left(\frac{x}{b}\right) dx = b \left[ \frac{2}{\pi} K_{2\nu}\left(\frac{2a^{\frac{1}{2}}}{b^{\frac{1}{2}}}\right) + Y_{2\nu}\left(\frac{2a^{\frac{1}{2}}}{b^{\frac{1}{2}}}\right) \right]$                        | $a, b > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                              |
| (47) | $\int_0^\infty J_\nu\left(\frac{a}{x}\right) Y_\nu\left(\frac{x}{b}\right) \frac{dx}{x^2} = -\frac{1}{a} \left[ \frac{2}{\pi} K_{2\nu}\left(\frac{2a^{\frac{1}{2}}}{b^{\frac{1}{2}}}\right) - Y_{2\nu}\left(\frac{2a^{\frac{1}{2}}}{b^{\frac{1}{2}}}\right) \right]$ | $a, b > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                              |
| (48) | $\int_0^\infty Y_\nu\left(\frac{a}{x}\right) Y_\nu\left(\frac{x}{b}\right) dx = -b J_{2\nu}\left(\frac{2a^{\frac{1}{2}}}{b^{\frac{1}{2}}}\right)$                                                                                                                    | $a, b > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                              |

## 19.4. Bessel functions of other arguments

|     |                                                                                                                                                                                                                                                                                                                                                                                                                                                 |
|-----|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $\int_0^\infty x^{\nu+1} y^\mu J_\nu(ax) J_\mu(by) dx$ $= a^\nu b^\mu (\beta/h)^{\mu+\nu+1} [\sin(\nu\pi) Y_{\mu+\nu+1}(\beta h) - \cos(\nu\pi) J_{\mu+\nu+1}(\beta h)]$ $= -2\pi^{-1} a^\nu b^\mu (\beta/k)^{\mu+\nu+1} \sin(\mu\pi) K_{\mu+\nu+1}(\beta k)$ $a, b > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re}(\mu + \nu) < 0, \quad \operatorname{Re} \nu > -1$                                                           |
| (2) | $\int_0^\infty x^{\nu+1} y^\mu J_\nu(ax) Y_\mu(by) dx$ $= -a^\nu b^\mu (\beta/h)^{\mu+\nu+1} [\sin(\nu\pi) J_{\mu+\nu+1}(\beta h) + \cos(\nu\pi) Y_{\mu+\nu+1}(\beta h)]$ $= -2\pi^{-1} a^\nu b^\mu (\beta/k)^{\mu+\nu+1} \cos(\mu\pi) K_{\mu+\nu+1}(\beta k)$ $a, b > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re}(\mu + \nu) < 0, \quad \operatorname{Re} \nu > -1$                                                          |
| (3) | $\int_0^\infty x^{\nu+1} y^{-\mu} J_\nu(ax) Y_\mu(by) dx$ $= a^\nu b^{-\mu} (\beta/h)^{\nu-\mu+1} Y_{\mu-\nu-1}(\beta h)$ $= -2\pi^{-1} a^\nu b^{-\mu} (\beta/k)^{\nu-\mu+1} K_{\mu-\nu-1}(\beta k)$ $a, b > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re} \mu > \operatorname{Re} \nu > -1$                                                                                                                                    |
| (4) | $\int_0^\infty x^{\rho-1} y^{-\mu} (x^2 + \lambda^2)^{-1} \left[ \cos\left(\frac{\rho-\nu}{2}\pi\right) J_\nu(ax) + \sin\left(\frac{\rho-\nu}{2}\pi\right) Y_\nu(ax) \right] J_\mu(by) dx$ $= -\lambda^{\rho-2} \frac{J_\mu[b(\beta^2 - \lambda^2)^{\frac{1}{2}}]}{(\beta^2 - \lambda^2)^{\frac{1}{2}\mu}} K_\nu(a\lambda)$ $ \operatorname{Re} \nu  < \operatorname{Re} \rho < \operatorname{Re} \mu + 4, \quad \operatorname{Re} \lambda > 0$ |

$$y = (x^2 + \beta^2)^{\frac{1}{2}}, \quad h = (b^2 - a^2)^{\frac{1}{2}}, \quad k = (a^2 - b^2)^{\frac{1}{2}}$$

## Bessel functions of other arguments (cont'd)

|      |                                                                                                                                                                                                                                                                                                 |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (5)  | $\int_{-\infty}^{\infty} \frac{\sin [a(x + \lambda)]}{x + \lambda} \frac{J_{\nu}(by)}{y^{\nu}} dx = \pi \frac{J_{\nu}[b(\beta^2 + \lambda^2)^{1/2}]}{(\beta^2 + \lambda^2)^{1/2\nu}}$ $0 < b < a, \quad \operatorname{Re} \nu > -3/2$                                                           |
| (6)  | $\int_{-\infty}^{\infty} \frac{\sin [a(x - z)]}{x - z} \frac{J_{\nu}[a(x^2 - 2bx \cos \theta + b^2)^{1/2}]}{(x^2 - 2bx \cos \theta + b^2)^{1/2\nu}} dx$ $= \pi \frac{J_{\nu}[a(z^2 - 2bz \cos \theta + b^2)^{1/2}]}{(z^2 - 2bz \cos \theta + b^2)^{1/2\nu}} \quad \operatorname{Re} \nu > -1/2$ |
| (7)  | $\int_0^{\infty} x^{\rho-1} \left( \frac{a+bx}{ax+b} \right)^{\nu} J_{2\nu}[x^{-1/2}(a+bx)^{1/2}(ax+b)^{1/2}] dx$ $= -\pi [J_{\nu+\rho}(a) Y_{\nu-\rho}(b) + J_{\nu-\rho}(b) Y_{\nu+\rho}(a)]$ $a, b > 0, \quad -3/4 < \operatorname{Re} \rho < 3/4$                                            |
| (8)  | $\int_0^{\infty} x^{\rho-1} \left( \frac{a+bx}{ax+b} \right)^{\nu} Y_{2\nu}[x^{-1/2}(a+bx)^{1/2}(ax+b)^{1/2}] dx$ $= \pi [J_{\nu+\rho}(a) J_{\nu-\rho}(b) - Y_{\nu+\rho}(a) Y_{\nu-\rho}(b)]$ $a, b > 0, \quad -3/4 < \operatorname{Re} \nu < 3/4$                                              |
| (9)  | $\int_0^{\infty} x^{\rho-1} \left( \frac{a+bx}{ax+b} \right)^{\nu} H_{\nu}^{(2)}[x^{-1/2}(a+bx)^{1/2}(ax+b)^{1/2}] dx$ $= -i\pi H_{\nu+\rho}^{(2)}(a) H_{\nu-\rho}^{(2)}(b) \quad a, b > 0, \quad -3/4 < \operatorname{Re} \nu < 3/4$                                                           |
| (10) | $\int_0^{\infty} \cosh x \cos(2a \sinh x) J_{\nu}(be^x) J_{\nu}(be^{-x}) dx$ $= 1/2 (b^2 - a^2)^{-1/2} J_{2\nu}[2(b^2 - a^2)^{1/2}] \quad 0 < a < b$ $= 0 \quad 0 < b < a$ $\operatorname{Re} \nu > -1$                                                                                         |

$$\gamma = (x^2 + \beta^2)^{1/2}, \quad h = (b^2 - a^2)^{1/2}, \quad k = (a^2 - b^2)^{1/2}$$

**Bessel functions of other arguments (cont'd)**

|      |                                                                                                                                                                                                                                                                                                                                                                                |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (11) | $\int_0^\infty \cosh x \cos(2a \sinh x) Y_\nu(b e^x) Y_\nu(b e^{-x}) dx$ $= -\frac{1}{2}(b^2 - a^2)^{-\frac{1}{2}} J_{2\nu}[2(b^2 - a^2)^{\frac{1}{2}}] \quad 0 < a < b$ $= 2\pi^{-1} \cos(\nu\pi) (a^2 - b^2)^{-\frac{1}{2}} K_{2\nu}[2(a^2 - b^2)^{\frac{1}{2}}] \quad 0 < b < a$ $-1 < \operatorname{Re} \nu < 1$                                                           |
| (12) | $\int_0^\infty \cosh x \sin(2a \sinh x) [J_\nu(b e^x) Y_\nu(b e^{-x}) - Y_\nu(b e^x) J_\nu(b e^{-x})] dx$ $= 0 \quad 0 < a < b$ $= -2\pi^{-1} \cos(\nu\pi) (a^2 - b^2)^{-\frac{1}{2}} K_{2\nu}[2(a^2 - b^2)^{\frac{1}{2}}] \quad 0 < b < a$ $-\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                                                                               |
| (13) | $\int_0^\pi \sin(2\mu x) J_{2\nu}(2a \sin x) dx = \pi \sin(\mu\pi) J_{\nu-\mu}(a) J_{\nu+\mu}(a)$ $\operatorname{Re} \nu > -1$                                                                                                                                                                                                                                                 |
| (14) | $\int_0^\pi \cos(2\mu x) J_{2\nu}(2a \sin x) dx = \pi \cos(\mu\pi) J_{\nu-\mu}(a) J_{\nu+\mu}(a)$ $\operatorname{Re} \nu > -\frac{1}{2}$                                                                                                                                                                                                                                       |
| (15) | $\int_0^{\frac{1}{2}\pi} \cos(2nx) J_0(2a \sin x) dx = \frac{1}{2}\pi [J_n(a)]^2 \quad n = 0, 1, 2, \dots$                                                                                                                                                                                                                                                                     |
| (16) | $\int_0^{\frac{1}{2}\pi} \cos(2nx) Y_0(2a \sin x) dx = \frac{1}{2}\pi J_n(a) Y_n(a)$ $n = 0, 1, 2, \dots$                                                                                                                                                                                                                                                                      |
| (17) | $\int_0^\pi [\tan(\frac{1}{2}x)]^{-2\kappa} e^{-\beta \cos x} J_{2\nu}(a \sin x) dx$ $= \frac{\Gamma(\frac{1}{2} + \kappa + \nu) \Gamma(\frac{1}{2} - \kappa + \nu)}{a [\Gamma(2\nu + 1)]^2} M_{\kappa, \nu}[\beta + (\beta^2 - a^2)^{\frac{1}{2}}] M_{\kappa, \nu}[\beta - (\beta^2 - a^2)^{\frac{1}{2}}]$ $\operatorname{Re} \nu + \frac{1}{2} >  \operatorname{Re} \kappa $ |

## Bessel functions of other arguments (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                            |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (18) | $\int_0^{\frac{1}{2}\pi} \cos(2\beta \cos x) J_{2\nu}(2a \sin x) dx$ $= \frac{1}{2}\pi J_\nu[(\beta^2 + a^2)^{\frac{1}{2}} + \beta] J_\nu[(\beta^2 + a^2)^{\frac{1}{2}} - \beta] \quad \text{Re } \nu > -\frac{1}{2}$                                                                                                                      |
| (19) | $\int_0^{\frac{1}{2}\pi} (\sin x)^{\nu+1} \cos(\beta \cos x) J_\nu(a \sin x) dx$ $= 2^{-\frac{1}{2}} \pi^{\frac{1}{2}} a^\nu (a^2 + \beta^2)^{-\frac{1}{2}\nu - \frac{1}{4}} J_{\nu+\frac{1}{2}}[(a^2 + \beta^2)^{\frac{1}{2}}] \quad \text{Re } \nu > -1$                                                                                 |
| (20) | $\int_0^{\frac{1}{2}\pi} \sin(2x) P_n(\cos 2x) J_0(a \sin x) dx = a^{-1} J_{2n+1}(a)$ $n = 0, 1, 2, \dots$                                                                                                                                                                                                                                 |
| (21) | $\int_0^{\frac{1}{2}\pi} (\sin x)^{\nu+1} \cos(a \cos \theta \cos x) C_{2n}^{\nu+\frac{1}{2}}(\cos x) J_\nu(a \sin \theta \sin x) dx$ $= (-1)^n 2^{-\frac{1}{2}} \pi^{\frac{1}{2}} (\sin \theta)^\nu a^{-\frac{1}{2}} C_{2n}^{\nu+\frac{1}{2}}(\cos \theta) J_{\nu+2n+\frac{1}{2}}(a)$ $n = 0, 1, 2, \dots, \quad \text{Re } \nu > -1$     |
| (22) | $\int_0^{\frac{1}{2}\pi} (\sin x)^{\nu+1} \sin(a \cos \theta \cos x) C_{2n+1}^{\nu+\frac{1}{2}}(\cos x) J_\nu(a \sin \theta \sin x) dx$ $= (-1)^n 2^{-\frac{1}{2}} \pi^{\frac{1}{2}} (\sin \theta)^\nu a^{-\frac{1}{2}} C_{2n+1}^{\nu+\frac{1}{2}}(\cos \theta) J_{\nu+2n+\frac{3}{2}}(a)$ $n = 0, 1, 2, \dots, \quad \text{Re } \nu > -1$ |
| (23) | $\int_0^{\frac{1}{2}\pi} \cos(2\mu x) J_{2\nu}(2a \cos x) dx = \frac{1}{2}\pi J_{\nu+\mu}(a) J_{\nu-\mu}(a) \quad \text{Re } \nu > -\frac{1}{2}$                                                                                                                                                                                           |
| (24) | $\int_0^{\frac{1}{2}\pi} \cos(2\mu x) Y_{2\nu}(2a \cos x) dx = \frac{1}{2}\pi \cotn(2\nu\pi) J_{\nu+\mu}(a) J_{\nu-\mu}(a)$ $- \frac{1}{2}\pi \csc(2\nu\pi) J_{\mu-\nu}(a) J_{-\mu-\nu}(a) \quad -\frac{1}{2} < \text{Re } \nu < \frac{1}{2}$                                                                                              |

## Bessel functions of other arguments (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                             |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (25) | $\int_0^{\frac{1}{2}\pi} (\sin x)^{\mu+1} (\cos x)^{\nu+1} J_{\mu}(a \sin x) J_{\nu}(\beta \cos x) dx$ $= a^{\mu} \beta^{\nu} (a^2 + \beta^2)^{-\frac{1}{2}(\mu+\nu+1)} J_{\mu+\nu+1}[(a^2 + \beta^2)^{\frac{1}{2}}]$ $\text{Re } \mu > -1, \quad \text{Re } \nu > -1$                                                                                      |
| (26) | $\int_0^{\frac{1}{2}\pi} (\sin x)^{\rho} (\cos x)^{\sigma} J_{\mu}(a \sin x) J_{\nu}(\beta \sin x) dx$ $\int_0^{\frac{1}{2}\pi} (\sin x)^{\rho} (\cos x)^{\sigma} J_{\mu}(a \sin x) J_{\nu}(\beta \cos x) dx$ <p>See Bailey, W.N., 1938: <i>Quart. J. Math. Oxford Ser. 9</i>, 141-147.</p>                                                                 |
| (27) | $\int_0^{\pi} (\sin x)^{2\nu} \frac{J_{\nu}(\omega)}{\omega^{\nu}} dx = 2^{\nu} \pi^{\frac{1}{2}} \Gamma(\nu + \frac{1}{2}) \frac{J_{\nu}(a)}{a^{\nu}} \frac{J_{\nu}(\beta)}{\beta^{\nu}}$ $\omega = (a^2 + \beta^2 - 2a\beta \cos x)^{\frac{1}{2}}, \quad \text{Re } \nu > -\frac{1}{2}$                                                                   |
| (28) | $\int_0^{\pi} (\sin x)^{2\nu} \frac{Y_{\nu}(\omega)}{\omega^{\nu}} dx = 2^{\nu} \pi^{\frac{1}{2}} \Gamma(\nu + \frac{1}{2}) \frac{J_{\nu}(a)}{a^{\nu}} \frac{Y_{\nu}(\beta)}{\beta^{\nu}}$ $\omega = (a^2 + \beta^2 - 2a\beta \cos x)^{\frac{1}{2}}, \quad  a  <  \beta , \quad \text{Re } \nu > -\frac{1}{2}$                                              |
| (29) | $\int_0^{\pi} (\sin x)^{2\nu} C_n^{\nu}(\cos x) \frac{J_{\nu}(\omega)}{\omega^{\nu}} dx = \frac{\pi \Gamma(2\nu + n)}{2^{\nu-1} n! \Gamma(\nu)} \frac{J_{\nu+n}(a)}{a^{\nu}} \frac{J_{\nu+n}(\beta)}{\beta^{\nu}}$ $n = 0, 1, 2, \dots, \quad \omega = (a^2 + \beta^2 - 2a\beta \cos x)^{\frac{1}{2}}, \quad \text{Re } \nu > -\frac{1}{2}$                 |
| (30) | $\int_0^{\pi} (\sin x)^{2\nu} C_n^{\nu}(\cos x) \frac{Y_{\nu}(\omega)}{\omega^{\nu}} dx = \frac{\pi \Gamma(2\nu + n)}{2^{\nu-1} n! \Gamma(\nu)} \frac{J_{\nu+n}(a)}{a^{\nu}} \frac{Y_{\nu+n}(\beta)}{\beta^{\nu}}$ $n = 0, 1, 2, \dots, \quad \omega = (a^2 + \beta^2 - 2a\beta \cos x)^{\frac{1}{2}}, \quad  a  <  \beta $ $\text{Re } \nu > -\frac{1}{2}$ |



## Bessel functions of other arguments (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (31) | $\int_0^\infty e^{-2\mu x} Y_{2\nu}(2a \sinh x) dx = \operatorname{ctn}(2\nu\pi) I_{\mu+\nu}(a) K_{\mu-\nu}(a)$ $- \operatorname{csc}(2\nu\pi) I_{\mu-\nu}(a) K_{\mu+\nu}(a)$ $a > 0, \quad \operatorname{Re} \mu > -3/2, \quad -1/2 < \operatorname{Re} \nu < 1/2$                                                                                                                                                                                                                                                                          |
| (32) | $\int_0^\infty [\operatorname{ctnh}(\tfrac{1}{2}x)]^{-2\kappa} \{ \sin[(\mu - \kappa)\pi] J_{2\mu}(a \sinh x)$ $+ \cos[(\mu - \kappa)\pi] Y_{2\mu}(a \sinh x) \} dx = -a^{-1} W_{\kappa, \mu}(a) W_{-\kappa, \mu}(a)$ $a > 0, \quad \operatorname{Re} \kappa >  \operatorname{Re} \mu  - \tfrac{1}{2}$                                                                                                                                                                                                                                       |
| (33) | $\int_0^\infty \sinh x [\tanh(\tfrac{1}{2}x)]^\nu e^{-\beta \cosh x} J_\nu(a \sinh x) dx$ $= (a^2 + \beta^2)^{-\frac{1}{2}} \left[ \frac{(a^2 + \beta^2)^{\frac{1}{2}} + \beta}{(a^2 + \beta^2)^{\frac{1}{2}} - \beta} \right]^{-\frac{1}{2}\nu} \exp[-(a^2 + \beta^2)^{\frac{1}{2}}]$ $\operatorname{Re} \beta >  \operatorname{Re} a , \quad \operatorname{Re} \nu > -1$                                                                                                                                                                   |
| (34) | $\int_0^\infty [\operatorname{ctnh}(\tfrac{1}{2}x)]^{2\kappa} e^{-\beta \cosh x} J_{2\mu}(a \sinh x) dx$ $= \frac{\Gamma(\frac{1}{2} - \kappa + \mu)}{a \Gamma(2\mu + 1)} M_{-\kappa, \mu}[(a^2 + \beta^2)^{\frac{1}{2}} - \beta] W_{\kappa, \mu}[(a^2 + \beta^2)^{\frac{1}{2}} + \beta]$ $\operatorname{Re} \beta >  \operatorname{Re} a , \quad \operatorname{Re}(\mu - \kappa) > -\tfrac{1}{2}$                                                                                                                                           |
| (35) | $\int_0^\infty [\operatorname{ctnh}(\tfrac{1}{2}x)]^{2\kappa} e^{-\beta \cosh x} Y_{2\mu}(a \sinh x) dx$ $= -a^{-1} \sec[(\mu + \kappa)\pi] W_{\kappa, \mu}(h) W_{-\kappa, \mu}(k)$ $- \frac{\tan[(\mu + \kappa)\pi] \Gamma(\frac{1}{2} - \kappa + \mu)}{a \Gamma(2\mu + 1)} W_{\kappa, \mu}(h) M_{-\kappa, \mu}(k)$ $h = (a^2 + \beta^2)^{\frac{1}{2}} + \beta, \quad k = (a^2 + \beta^2)^{\frac{1}{2}} - \beta$ $\operatorname{Re} \beta >  \operatorname{Re} a , \quad \operatorname{Re} \kappa < \tfrac{1}{2} -  \operatorname{Re} \mu $ |

**Bessel functions of other arguments (cont'd)**

|      |                                                                                                                                                                                                                                                                                                                                                                                                       |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (36) | $\int_0^\infty (\sinh x)^{\mu+1} (\cosh x)^{\nu+1} J_\mu(a \sinh x) H_\nu^{(2)}(b \cosh x) dx$ $= -e^{-\mu\pi i} a^\mu b^\nu h^{-\mu-\nu-1} H_{\mu+\nu+1}^{(2)}(h) \quad 0 < a < b$ $= 2i\pi^{-1} e^{\nu\pi i} a^\mu b^\nu k^{-\mu-\nu-1} K_{\mu+\nu+1}(k) \quad 0 < b < a$ $h = (b^2 - a^2)^{1/2}, \quad k = (a^2 - b^2)^{1/2}$ $\operatorname{Re} \mu > -1, \quad \operatorname{Re}(\mu + \nu) < 0$ |
| (37) | $\int_0^\infty (\sinh x)^{\mu+1} (\cosh x)^{1-\nu} J_\mu(a \sinh x) H_\nu^{(2)}(b \cosh x) dx$ $= a^\mu b^{-\nu} h^{\nu-\mu-1} H_{\nu-\mu-1}^{(2)}(h) \quad 0 < a < b$ $= 2i\pi^{-1} a^\mu b^{-\nu} k^{\nu-\mu-1} K_{\nu-\mu-1}(k) \quad 0 < b < a$ $h = (b^2 - a^2)^{1/2}, \quad k = (a^2 - b^2)^{1/2}, \quad \operatorname{Re} \nu > \operatorname{Re} \mu > -1$                                    |
| (38) | $\int_{-\infty}^\infty \operatorname{sech} x e^{2\kappa x - \beta \tanh x} J_{2\mu}(a \operatorname{sech} x) dx$ $= \frac{\Gamma(\frac{1}{2} + \kappa + \mu) \Gamma(\frac{1}{2} - \kappa + \mu)}{a[\Gamma(2\mu + 1)]^2} M_{\kappa, \mu}(h) M_{\kappa, \mu}(k)$ $h + k = 2\beta, \quad hk = a^2, \quad \operatorname{Re} \mu >  \operatorname{Re} \kappa  - \frac{1}{2}$                               |

**19.5. Modified Bessel functions of argument  $x$** 

For integrals involving  $\operatorname{ber}_\nu x$ ,  $\operatorname{bei}_\nu x$ ,  $\operatorname{ker}_\nu x$ ,  $\operatorname{kei}_\nu x$  and similar functions see McLachlan, N.W., 1954: *Bessel functions for engineers*. Oxford, Second edition.

|     |                                                                                                                                                                                         |
|-----|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $\int_0^a I_\nu(x) dx = 2 \sum_{n=0}^{\infty} (-1)^n I_{\nu+2n+1}(a) \quad \operatorname{Re} \nu > -1$                                                                                  |
| (2) | $\int_0^a x^\nu I_\nu(x) dx = 2^{\nu-1} \pi^{1/2} \Gamma(\nu + \frac{1}{2}) a [I_\nu(a) \mathbf{L}_{\nu-1}(a) - \mathbf{L}_\nu(a) I_{\nu-1}(a)]$ $\operatorname{Re} \nu > -\frac{1}{2}$ |

Modified functions of  $x$  (cont'd)

|      |                                                                                                                                                                                                                                                                                             |                                                                             |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------|
| (3)  | $\int_0^a x^{\nu+1} I_\nu(x) dx = a^{\nu+1} I_{\nu+1}(a)$                                                                                                                                                                                                                                   | $\operatorname{Re} \nu > -1$                                                |
| (4)  | $\int_0^a x^{1-\nu} I_\nu(x) dx = a^{1-\nu} I_{\nu-1}(a) - \frac{2^{1-\nu}}{\Gamma(\nu)}$                                                                                                                                                                                                   |                                                                             |
| (5)  | $\int_0^a x^\nu (a^2 - x^2)^{\nu-\frac{1}{2}} I_\nu(x) dx = 2^{-\nu-1} \pi^{\frac{1}{2}} a^{2\nu} \Gamma(\nu + \frac{1}{2}) [I_\nu(\frac{1}{2}a)]^2$                                                                                                                                        |                                                                             |
| (6)  | $\int_0^a x^{\nu+1} (a^2 - x^2)^{\sigma-1} I_\nu(x) dx = 2^{\sigma-1} a^{\nu+\sigma} \Gamma(\sigma) I_{\nu+\sigma}(a)$                                                                                                                                                                      | $\operatorname{Re} \nu > -1, \quad \operatorname{Re} \sigma > 0$            |
| (7)  | $\int_0^a x^{\rho-1} (a^2 - x^2)^{\sigma-1} I_\nu(x) dx$ $= \frac{\Gamma(\frac{\nu+\rho}{2}) \Gamma(\sigma) a^{\nu+\rho+2\sigma-2}}{2^{\nu+1} \Gamma(\nu+1) \Gamma(\frac{\nu+\rho}{2} + \sigma)} {}_1F_2\left(\frac{\nu+\rho}{2}; \nu+1, \frac{\nu+\rho}{2} + \sigma; \frac{a^2}{4}\right)$ | $\operatorname{Re}(\rho + \nu) > 0, \quad \operatorname{Re}(\sigma) > 0$    |
| (8)  | $\int_0^a x^{n+1} e^{-x^2} I_n(2ax) dx = \frac{1}{4} a^n [e^{a^2} - e^{-a^2} \sum_{r=-n}^n I_r(2a^2)]$                                                                                                                                                                                      | $n = 0, 1, 2, \dots$                                                        |
| (9)  | $\int_0^a x^{\nu+1} y^{-1} \cos y I_\nu(x) dx = \frac{\pi^{\frac{1}{2}} a^{2\nu+1}}{2^{\nu+1} \Gamma(\nu + 3/2)}$                                                                                                                                                                           | $y = (a^2 - x^2)^{\frac{1}{2}}, \quad \operatorname{Re} \nu > -1$           |
| (10) | $\int_0^a y^{-1} \cosh(y \sinh t) I_{2\nu}(x) dx = \frac{1}{2} \pi I_\nu(ae^t) I_\nu(ae^{-t})$                                                                                                                                                                                              | $y = (a^2 - x^2)^{\frac{1}{2}}, \quad \operatorname{Re} \nu > -\frac{1}{2}$ |

Modified functions of  $x$  (cont'd)

|      |                                                                                                                                                                                                        |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (11) | $\int_0^a e^{-x} P_n(1 - 2xa^{-1}) I_0(x) dx = \frac{ae^{-a}}{2n+1} [I_n(a) + I_{n+1}(a)]$                                                                                                             |
| (12) | $\int_0^a x^\mu e^{-x} P_\nu(1 - 2x/a) I_\mu(x) dx$ <p>See Bose, B.N., 1948: <i>Bull. Calcutta Math. Soc.</i> 40, 8-14.</p>                                                                            |
| (13) | $\int_0^\infty x^{-1/3} e^{-x} \sin(4ax^{1/2}) I_{1/3}(x) dx = (2\pi)^{-1/2} a^{1/3} e^{-a^2} K_{1/3}(a^2)$ $a > 0$                                                                                    |
| (14) | $\int_0^\infty x^{-\nu} e^{-x} \sin(4ax^{1/2}) I_\nu(x) dx = (2^{3/2} a)^{\nu-1} e^{-a^2}$ $\times W_{1/2-3\nu/2, 1/2-\nu/2}(2a^2)$ $a > 0, \quad \operatorname{Re} \nu > 0$                           |
| (15) | $\int_0^\infty x^{-1/2} e^{-x} \cos(4ax^{1/2}) I_0(x) dx = (2\pi)^{-1/2} e^{-a^2} K_0(a^2)$ $a > 0$                                                                                                    |
| (16) | $\int_0^\infty x^{-\nu-1/2} e^{-x} \cos(4ax^{1/2}) I_\nu(x) dx$ $= 2^{3\nu/2-1} a^{\nu-1} e^{-a^2} W_{-3\nu/2, \nu/2}(2a^2)$ $a > 0, \quad \operatorname{Re} \nu > -1/2$                               |
| (17) | $\int_a^\infty (x^2 - a^2)^{-1/2} T_n(ax^{-1}) K_{2\mu}(x) dx = \frac{\pi}{2a} W_{\frac{1}{2}n, \mu}(a) W_{-\frac{1}{2}n, \mu}(a)$ $n = 0, 1, 2, \dots$                                                |
| (18) | $\int_0^\infty (1+x/a)^\mu e^{-x} P_\nu^{-2\mu}(1+2x/a) I_\mu(x) dx = 0$ $-1/2 < \operatorname{Re} \mu < 0, \quad -1/2 + \operatorname{Re} \mu < \operatorname{Re} \nu < -1/2 - \operatorname{Re} \mu$ |

Modified functions of  $x$  (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                              |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (19) | $\int_0^\infty (x+a)^{-\mu} e^{-x} P_\nu^{-2\mu}(1+2x/a) I_\mu(x) dx$ $= \frac{2^{\mu-1} \Gamma(\mu+\nu+\frac{1}{2}) \Gamma(\mu-\nu-\frac{1}{2})}{\pi^{\frac{1}{2}} \Gamma(2\mu+\nu+1) \Gamma(2\mu-\nu)} e^a W_{\frac{1}{2}-\mu, \frac{1}{2}+\nu}(2a)$ $ \arg a  < \pi, \quad \operatorname{Re} \mu >  \operatorname{Re} \nu + \frac{1}{2} $ |
| (20) | $\int_a^\infty x^{1-n} e^{-x^2} I_n(2ax) dx = \frac{1}{4} a^{-n} [e^{a^2} - e^{-a^2} \sum_{r=1-n}^n I_r(2a^2)]$ $n = 1, 2, \dots$                                                                                                                                                                                                            |
| (21) | $\int_0^a x^\nu K_\nu(x) dx = 2^{\nu-1} \pi^{\frac{1}{2}} \Gamma(\nu+\frac{1}{2}) a [K_\nu(a) \mathbf{L}_{\nu-1}(a) + \mathbf{L}_\nu(a) K_{\nu-1}(a)]$ $\operatorname{Re} \nu > -\frac{1}{2}$                                                                                                                                                |
| (22) | $\int_0^a x^{\nu+1} K_\nu(x) dx = 2^\nu \Gamma(\nu+1) - a^{\nu+1} K_{\nu+1}(a)$ $\operatorname{Re} \nu > -1$                                                                                                                                                                                                                                 |
| (23) | $\int_0^a x^{1-\nu} K_\nu(x) dx = 2^{-\nu} \Gamma(1-\nu) - a^{1-\nu} K_{\nu-1}(a)$ $\operatorname{Re} \nu < 1$                                                                                                                                                                                                                               |
| (24) | $\int_0^a x^\mu (a^2 - x^2)^{\mu-\frac{1}{2}} K_\mu(x) dx = 2^{\mu-1} \pi^{\frac{1}{2}} a^{2\mu} \Gamma(\mu+\frac{1}{2})$ $\times I_\mu(\frac{1}{2}a) K_\mu(\frac{1}{2}a)$ $\operatorname{Re} \mu > -\frac{1}{2}$                                                                                                                            |
| (25) | $\int_0^a y^{-1} \cosh(y \sinh t) K_{2\nu}(x) dx = \frac{1}{4} \pi^2 \csc(\nu\pi) [I_{-\nu}(ae^t) I_{-\nu}(ae^{-t})$ $- I_\nu(ae^t) I_\nu(ae^{-t})]$ $y = (a^2 - x^2)^{\frac{1}{2}}, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{2}$                                                                                               |
| (26) | $\int_0^a x J_\nu(\lambda x) K_\nu(\kappa x) dx = (\kappa^2 + \lambda^2)^{-1} [(\lambda/\kappa)^\nu + \lambda a J_{\nu+1}(\lambda a) K_\nu(\kappa a)$ $- \kappa a J_\nu(\lambda a) K_{\nu+1}(\kappa a)]$ $\operatorname{Re} \nu > -1$                                                                                                        |

Modified functions of  $x$  (cont'd)

|      |                                                                                                                                                                                                                                                     |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (27) | $\int_0^a x^{2\nu+1} P_\mu(1-2x^2 a^{-2}) I_\nu(x) K_\nu(x) dx$<br>See Bose, B.N., 1948: <i>Bull. Calcutta Math. Soc.</i> 40, 8-14.                                                                                                                 |
| (28) | $\int_0^a x^{2\nu+1} P_n[(1-x^2 a^{-2})^{1/2}] I_\nu(x) K_\nu(x) dx$<br>See Bose, B.N., 1944: <i>Bull. Calcutta Math. Soc.</i> 36, 125-132.                                                                                                         |
| (29) | $\int_0^\infty x^{-1/2} (x+a)^{-1} e^{-x} K_\nu(x) dx = \frac{\pi e^a}{a^{1/2} \cos(\nu\pi)} K_\nu(a)$<br>$ \arg a  < \pi, \quad  \operatorname{Re} \nu  < 1/2$                                                                                     |
| (30) | $\int_0^\infty x^{-1/2} \exp(-2ax^{1/2}) K_{1/4}(x) dx = \left(\frac{a\pi}{2}\right)^{1/2} K_{1/4}(a^2)$<br>$+ 1/2 \pi^{3/2} a^{1/2} [\mathbf{L}_{-1/4}(a^2) - \mathbf{L}_{1/4}(a^2)]$                                                              |
| (31) | $\int_0^\infty x^{-1/2} y^{-1} e^{-y} K_\nu(x) dx = \frac{\pi^{3/2} a^{-1/2} \sec(\nu\pi)}{\Gamma(3/4 + 1/2\nu) \Gamma(3/4 - 1/2\nu)} K_\nu(a)$<br>$y = (x^2 + a^2)^{1/2}, \quad \operatorname{Re} a > 0, \quad -1/2 < \operatorname{Re} \nu < 1/2$ |
| (32) | $\int_0^\infty x^{-1/2} y^{-1} e^{-y} K_\nu(x) dx = \pi a^{-1/2} \sec(\nu\pi) P_{\nu-1/2}(-\cos\phi) K_\nu(a)$<br>$y = (x^2 + a^2 - 2ax \cos\phi)^{1/2}, \quad  \arg a  +  \operatorname{Re} \phi  < \pi, \quad  \operatorname{Re} \nu  < 1/2$      |
| (33) | $\int_0^\infty x^{-3/2} (1+a^2/x)^{-1/2} \exp[-(\beta+x)(1+a^2/x)^{1/2}] K_\nu(x) dx$<br>$= 4 a^{-1} K_\nu(\beta) K_{2\nu}(2a\beta^{1/2}) \quad \operatorname{Re} a > 0, \quad \operatorname{Re}(a\beta) > 0$                                       |
| (34) | $\int_0^\infty x \sin\left(\frac{a}{2x}\right) K_0(x) dx = 1/2 \pi a J_1(a^{1/2}) K_1(a^{1/2}) \quad a > 0$                                                                                                                                         |

Modified functions of  $x$  (cont'd)

|      |                                                                                                                                                                                                                                                                                 |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (35) | $\int_0^\infty x \cos\left(\frac{a}{2x}\right) K_0(x) dx = -\frac{1}{2} \pi a Y_1(a^{1/2}) K_1(a^{1/2}) \quad a > 0$                                                                                                                                                            |
| (36) | $\int_0^\infty x^{-1/3} e^x \sin(4ax^{1/2}) K_{1/3}(x) dx = (\frac{1}{2}\pi)^{1/2} a^{1/3} e^{a^2} K_{1/3}(a^2) \quad a > 0$                                                                                                                                                    |
| (37) | $\int_0^\infty x^{-1/3} e^{-x} \sin(4ax^{1/2}) K_{1/3}(x) dx = 2^{-1/2} \pi^{3/2} a^{1/3} e^{-a^2} I_{1/3}(a^2)$                                                                                                                                                                |
| (38) | $\int_0^\infty x^{-\nu} e^x \sin(4ax^{1/2}) K_\nu(x) dx = (2^{3/2} a)^{\nu-1} \pi \frac{\Gamma(3/2 - 2\nu)}{\Gamma(1/2 + \nu)} \times e^{a^2} W_{3\nu/2-1/2, 1/2-\nu/2}(2a^2) \quad a > 0, \quad 0 < \operatorname{Re} \nu < \frac{3}{4}$                                       |
| (39) | $\int_0^\infty x^{\rho-3/2} e^{-x} \sin(4ax^{1/2}) K_\nu(x) dx = \frac{\pi^{1/2} a \Gamma(\rho + \nu) \Gamma(\rho - \nu)}{2^{\rho-2} \Gamma(\rho + 1/2)} \times {}_2F_2(\rho + \nu, \rho - \nu; 3/2, \rho + 1/2; -2a^2) \quad \operatorname{Re} \rho >  \operatorname{Re} \nu $ |
| (40) | $\int_0^\infty x^{-1/2} e^x \cos(4ax^{1/2}) K_0(x) dx = (\frac{1}{2}\pi)^{1/2} e^{a^2} K_0(a^2) \quad a > 0$                                                                                                                                                                    |
| (41) | $\int_0^\infty x^{-1/2} e^{-x} \cos(4ax^{1/2}) K_0(x) dx = 2^{-1/2} \pi^{3/2} e^{-a^2} I_0(a^2)$                                                                                                                                                                                |
| (42) | $\int_0^\infty x^{-\nu-1/2} e^x \cos(4ax^{1/2}) K_\nu(x) dx = 2^{3\nu/2-1} \pi a^{\nu-1} \frac{\Gamma(\frac{1}{2} - 2\nu)}{\Gamma(\frac{1}{2} + \nu)} e^{a^2} W_{3\nu/2, -\nu/2}(2a^2) \quad a > 0, \quad -\frac{1}{2} < \operatorname{Re} \nu < \frac{1}{4}$                   |

Modified functions of  $x$  (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                               |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (43) | $\int_0^\infty x^{\rho-1} e^{-x} \cos(4ax^{1/2}) K_\nu(x) dx = \frac{\pi^{1/2} \Gamma(\rho + \nu) \Gamma(\rho - \nu)}{2^\rho \Gamma(\rho + 1/2)}$ $\times {}_2F_2(\rho + \nu, \rho - \nu; 1/2, \rho + 1/2; -2a^2) \quad \operatorname{Re} \rho >  \operatorname{Re} \nu $                                                                                     |
| (44) | $\int_0^\infty x^{n+2\nu-1/2} \exp[-(1+a)x] L_n^{2\nu}(ax) K_\nu(x) dx$ $= \frac{\pi^{1/2} \Gamma(n + \nu + 1/2) \Gamma(n + 3\nu + 1/2)}{2^{n+2\nu+1/2} n! \Gamma(2\nu + 1)}$ $\times {}_2F_1(n + \nu + 1/2, n + 3\nu + 1/2; 2\nu + 1; -1/2 a)$ $\operatorname{Re} a > -2, \quad \operatorname{Re}(n + \nu) > -1/2, \quad \operatorname{Re}(n + 3\nu) > -1/2$ |
| (45) | $\int_a^\infty x^{-\nu} (x^2 - a^2)^{1/4 - 1/2 \nu} P_\mu^{\nu-1/2}(2a^2 x^{-2} - 1) K_\nu(x) dx$ $= \pi^{1/2} 2^{-\nu} a^{-1/2 - \nu} W_{\mu+1/2, \nu-1/2}(a) W_{-\mu-1/2, \nu-1/2}(a) \quad \operatorname{Re} \nu < 3/2$                                                                                                                                    |
| (46) | $\int_0^\infty x^{-1} \exp\left(\frac{a^2}{2x} - x\right) \operatorname{Erfc}\left[\frac{a}{(2x)^{1/2}}\right] K_\nu(x) dx$ $= 1/4 \pi^{5/2} \sec(\nu\pi) \{[J_\nu(a)]^2 + [Y_\nu(a)]^2\}$ $\operatorname{Re} a > 0, \quad -1/2 < \operatorname{Re} \nu < 1/2$                                                                                                |
| (47) | $\int_0^\infty J_\mu(x) K_\nu(x) x^{\mu-\nu+1} dx = 1/2 \Gamma(\mu - \nu + 1)$ $\operatorname{Re} \mu > -1, \quad \operatorname{Re}(\mu - \nu) > -1$                                                                                                                                                                                                          |
| (48) | $\int_0^\infty e^{-2ax} I_0(x) K_0(x) dx$ $= 1/2 K[(1 - a^2)^{1/2}] \quad 0 < a < 1$ $= (2a)^{-1} K[(1 - a^{-2})^{1/2}] \quad 1 < a < \infty$                                                                                                                                                                                                                 |



Modified functions of  $x$  (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (49) | $\int_0^{\infty} x \exp\left(-\frac{x^2}{2a}\right) [I_{\nu}(x) + I_{-\nu}(x)] K_{\nu}(x) dx = a e^a K_{\nu}(a)$ <p style="text-align: right;"><math>\operatorname{Re} a &gt; 0, \quad -1 &lt; \operatorname{Re} \nu &lt; 1</math></p>                                                                                                                                                                                                                                                                                                                                                                                                                                       |
| (50) | $\int_0^{\infty} x^{\rho-1} \operatorname{si}r(2ax) K_{\mu}(x) K_{\nu}(x) dx = \frac{2^{\rho-1} a}{\Gamma(\rho+1)} \Gamma\left(\frac{\rho+\mu+\nu+1}{2}\right)$ $\times \Gamma\left(\frac{\rho+\mu-\nu+1}{2}\right) \Gamma\left(\frac{\rho-\mu+\nu+1}{2}\right) \Gamma\left(\frac{\rho-\mu-\nu+1}{2}\right)$ $\times {}_4F_3\left(\frac{\rho+\mu+\nu+1}{2}, \frac{\rho+\mu-\nu+1}{2}, \frac{\rho-\mu+\nu+1}{2}, \frac{\rho-\mu-\nu+1}{2}; \frac{3}{2}, \frac{\rho+1}{2}, \frac{\rho}{2}+1; -a^2\right)$ <p style="text-align: right;"><math> \operatorname{Re} a  &lt; 1, \quad \operatorname{Re} \rho &gt;  \operatorname{Re} \mu  +  \operatorname{Re} \nu  - 1</math></p> |
| (51) | $\int_0^{\infty} x^{\rho-1} \cos(2ax) K_{\mu}(x) K_{\nu}(x) dx = \frac{2^{\rho-3}}{\Gamma(\rho)} \Gamma\left(\frac{\rho+\mu+\nu}{2}\right)$ $\times \Gamma\left(\frac{\rho+\mu-\nu}{2}\right) \Gamma\left(\frac{\rho-\mu+\nu}{2}\right) \Gamma\left(\frac{\rho-\mu-\nu}{2}\right)$ $\times {}_4F_3\left(\frac{\rho+\mu+\nu}{2}, \frac{\rho+\mu-\nu}{2}, \frac{\rho-\mu+\nu}{2}, \frac{\rho-\mu-\nu}{2}; \frac{1}{2}, \frac{\rho}{2}, \frac{\rho+1}{2}; -a^2\right)$ <p style="text-align: right;"><math> \operatorname{Re} a  &lt; 1, \quad \operatorname{Re} \rho &gt;  \operatorname{Re} \mu  +  \operatorname{Re} \nu </math></p>                                         |
| (52) | $\int_0^{\infty} x^3 \cos\left(\frac{x^2}{2a}\right) Y_1(x) K_1(x) dx = -a^3 K_0(a)$ <p style="text-align: right;"><math>a &gt; 0</math></p>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 |

## 19.6. Modified Bessel functions of other arguments

|     |                                                                                                                                                                                                                                                                                                                                                                                                                                                  |
|-----|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $\int_0^\infty \cos\left(\frac{x^2}{2a}\right) K_{2\nu}(xe^{i\pi/4}) K_{2\nu}(xe^{-i\pi/4}) dx$ $= \frac{\Gamma(\frac{1}{4} + \nu) \Gamma(\frac{1}{4} - \nu)}{8a^{\frac{1}{2}} \pi^{-\frac{1}{2}}} W_{\frac{1}{4}, \nu}(ae^{i\pi/2}) W_{\frac{1}{4}, \nu}(ae^{-i\pi/2})$ $a > 0, \quad -\frac{1}{4} < \operatorname{Re} \nu < \frac{1}{4}$                                                                                                       |
| (2) | $\int_0^\infty x^{-\frac{1}{2}} I_\nu(x) K_\nu(x) K_\mu(2x) dx$ $= \frac{\Gamma(\frac{1}{4} + \frac{1}{2}\mu) \Gamma(\frac{1}{4} - \frac{1}{2}\mu) \Gamma(\frac{1}{4} + \nu + \frac{1}{2}\mu) \Gamma(\frac{1}{4} + \nu - \frac{1}{2}\mu)}{4\Gamma(\frac{3}{4} + \nu + \frac{1}{2}\mu) \Gamma(\frac{3}{4} + \nu - \frac{1}{2}\mu)}$ $ \operatorname{Re} \mu  < \frac{1}{2}, \quad 2\operatorname{Re} \nu >  \operatorname{Re} \mu  - \frac{1}{2}$ |
| (3) | $\int_0^\infty [J_0(ax) Y_1(bx) + 2\pi^{-1} I_0(ax) K_1(bx)] dx = 0 \quad 0 < a < b$                                                                                                                                                                                                                                                                                                                                                             |
| (4) | $\int_0^\infty x^\rho [Y_\mu(ax) \pm 2\pi^{-1} K_\mu(ax)] [Y_\nu(bx) \pm 2\pi^{-1} K_\nu(bx)] dx$ <p>See Dixon, A.L. and W.L. Ferrar, 1930: <i>Quart. J. Math.</i> Oxford Ser. 1, 122-145.</p>                                                                                                                                                                                                                                                   |
| (5) | $\int_0^\infty \sinh(cx) K_1(ax) J_0(bx) dx, \quad \int_0^\infty \cosh(cx) K_0(ax) J_0(bx) dx$ <p>See Watson, G.N., 1928: <i>J. London Math. Soc.</i> 3, 22-27.</p>                                                                                                                                                                                                                                                                              |
| (6) | $\int_0^\infty x J_0(ax) I_0(\beta x) K_0(\gamma x) dx = [(a^2 + \beta^2 + \gamma^2)^2 - 4\beta^2 \gamma^2]^{-\frac{1}{2}}$ $\operatorname{Re} \gamma >  \operatorname{Im} a  +  \operatorname{Re} \beta $                                                                                                                                                                                                                                       |
| (7) | $\int_0^\infty x J_0(ax) I_1(\beta x) K_1(\gamma x) dx = \frac{1}{2\beta\gamma} \{(a^2 + \beta^2 + \gamma^2)$ $\times [(a^2 + \beta^2 + \gamma^2)^2 - 4\beta^2 \gamma^2]^{\frac{1}{2}} - 1\}$ $\operatorname{Re} \gamma >  \operatorname{Im} a  +  \operatorname{Re} \beta $                                                                                                                                                                     |

## Modified functions of other arguments (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (8)  | $\int_0^\infty x^{\lambda-1} J_\mu(ax) J_\nu(\beta x) K_\rho(\gamma x) dx = \frac{2^{\lambda-2} a^\mu \beta^\nu \gamma^{-\lambda-\mu-\nu}}{\Gamma(\mu+1) \Gamma(\nu+1)}$ $\times \Gamma\left(\frac{\lambda+\mu+\nu-\rho}{2}\right) \Gamma\left(\frac{\lambda+\mu+\nu+\rho}{2}\right)$ $\times F_4\left(\frac{\lambda+\mu+\nu-\rho}{2}, \frac{\lambda+\mu+\nu+\rho}{2}; \mu+1, \nu+1; -\frac{a^2}{\gamma^2}, -\frac{\beta^2}{\gamma^2}\right)$ $\operatorname{Re}(\lambda+\mu+\nu) >  \operatorname{Re} \rho , \quad \operatorname{Re} \gamma >  \operatorname{Im} a  +  \operatorname{Im} \beta $ |
| (9)  | $\int_0^\infty x [J_0(ax) K_0(bx)]^2 dx = \frac{\pi}{8ab} - \frac{1}{4ab} \sin^{-1}\left(\frac{b^2 - a^2}{b^2 + a^2}\right)$ $a, b > 0$                                                                                                                                                                                                                                                                                                                                                                                                                                                           |
| (10) | $\int_0^\infty J_\nu(ax) J_\nu(bx) K_\nu(ax) K_\nu(bx) x^{2\nu+1} dx$ $= \frac{2^{\nu-3} a^{2\nu} \Gamma(\frac{\nu+1}{2}) \Gamma(\nu + \frac{1}{2}) \Gamma(\frac{3\nu+1}{2})}{b^{4\nu+2} \pi^{1/2} \Gamma(\nu+1)}$ $\times {}_2F_1\left(\nu + \frac{1}{2}, \frac{3\nu+1}{2}; 2\nu+1; 1 - \frac{a^4}{b^4}\right)$ $0 < a < b, \quad \operatorname{Re} \nu > -1/3$                                                                                                                                                                                                                                  |
|      | For other similar integrals see sec. 6.8.                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                         |
| (11) | $\int_0^\infty x^{-\mu} e^x P_\nu^{2\mu}(1+2x/a) K_\mu(x+a) dx$ $= \pi^{-1/2} 2^{\mu-1} \cos(\mu\pi) \Gamma(\mu+\nu+\frac{1}{2}) \Gamma(\mu-\nu+\frac{1}{2}) W_{\frac{1}{2}-\mu, \frac{1}{2}+\nu}(2a)$ $ \arg a  < \pi, \quad \operatorname{Re} \mu >  \operatorname{Re} \nu + \frac{1}{2} $                                                                                                                                                                                                                                                                                                      |

## Modified functions of other arguments (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                 |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (12) | $\int_0^\infty x^{-\frac{1}{2}\mu} (x+a)^{-\frac{1}{2}} e^{-x} P_{\nu-\frac{1}{2}}^\mu \left( \frac{a-x}{a+x} \right) K_\nu(a+x) dx$ $= (\frac{1}{2}\pi)^{\frac{1}{2}} a^{-\frac{1}{2}\mu} \Gamma(\mu, 2a)$ $a > 0, \quad \operatorname{Re} \mu < 1$                                                                            |
| (13) | $\int_0^\infty x^{\mu-1} (x+\beta)^{-\mu} I_\mu(x+\beta) K_\nu(x) dx$ <p>See MacRobert, T.M., 1950: <i>Functions of a complex variable</i>. Macmillan, p. 379.</p>                                                                                                                                                              |
| (14) | $\int_0^\infty x^{\mu-1}  x-b ^{-\mu} K_\mu( x-b ) K_\nu(x) dx$ $= \pi^{-\frac{1}{2}} (2b)^{-\mu} \Gamma(\frac{1}{2}-\mu) \Gamma(\mu+\nu) \Gamma(\mu-\nu) K_\nu(b)$ $b > 0, \quad \operatorname{Re} \mu < \frac{1}{2}, \quad \operatorname{Re} \mu >  \operatorname{Re} \nu $                                                   |
| (15) | $\int_0^\infty x^{\mu-1} (x+\beta)^{-\mu} K_\mu(x+\beta) K_\nu(x) dx = \frac{\pi^{\frac{1}{2}} \Gamma(\mu+\nu) \Gamma(\mu-\nu)}{2^\mu \beta^\mu \Gamma(\mu+\frac{1}{2})} K_\nu(\beta)$ $ \arg \beta  < \pi, \quad \operatorname{Re} \mu >  \operatorname{Re} \nu $                                                              |
| (16) | $\int_0^\infty x^{1+2\nu} J_{2\nu-1}(2ax) K_{2\nu-1}(2ax) J_\nu(x^2) dx$ $= \pi^{-1/2} 2^{\nu-2} a^{2\nu-1} K_{\nu-\frac{1}{2}}(2a^2)$ $ \arg a  < \frac{1}{4}\pi, \quad \operatorname{Re} \nu > 0$                                                                                                                             |
| (17) | $\int_0^\infty x^{1-2\nu} J_{2\nu+1}(2ax) K_{2\nu+1}(2ax) J_\nu(x^2) dx$ $= \pi^{\frac{1}{2}} 2^{-\nu-3} a^{-2\nu-1} \csc(\nu\pi) [I_{\nu+\frac{1}{2}}(2a^2) - \mathbf{L}_{\nu+\frac{1}{2}}(2a^2)]$ $ \arg a  < \frac{1}{4}\pi, \quad \operatorname{Re} \nu > -1$                                                               |
| (18) | $\int_0^\infty x^{1-2\nu} Y_{2\nu+1}(2ax) K_{2\nu+1}(2ax) J_\nu(x^2) dx = \pi^{\frac{1}{2}} 2^{-\nu-3} a^{-2\nu-1} \cot(\nu\pi)$ $\times [I_{\nu+\frac{1}{2}}(2a^2) - \mathbf{L}_{\nu+\frac{1}{2}}(2a^2) + 2\pi^{-1} \sec(\nu\pi) K_{\nu+\frac{1}{2}}(2a^2)]$ $ \arg a  < \frac{1}{4}\pi, \quad -1 < \operatorname{Re} \nu < 0$ |

## Modified functions of other arguments (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                                                                           |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (19) | $\int_0^\infty x^{3-2\nu} K_{2\nu}(2ax) [\cos(\nu\pi) J_{2\nu}(2ax) - \sin(\nu\pi) Y_{2\nu}(2ax)] J_\nu(x^2) dx$ $= \pi^{-\frac{1}{2}} 2^{-\nu-1} a^{1-2\nu} K_{\nu+\frac{1}{2}}(2a^2) \quad  \arg a  < \frac{1}{4}\pi, \quad -1 < \operatorname{Re} \nu < 1$                                                                                                                                                             |
| (20) | $\int_0^\infty x J_\nu(x^2) [\sin(\nu\pi) J_\nu(x^2) - \cos(\nu\pi) Y_\nu(x^2)] J_{4\nu}(4ax) dx$ $= \frac{1}{4} J_\nu(a^2) J_{-\nu}(a^2) \quad a > 0, \quad \operatorname{Re} \nu > -1$                                                                                                                                                                                                                                  |
| (21) | $\int_0^\infty x^{\rho-1} K_\mu(ax) J_\nu(bx^{-1}) dx = 2^{\rho-2} a^{-\rho} G_{04}^{30} \left( \frac{a^2 b^2}{16} \left  \begin{matrix} \nu, \frac{\rho+\mu}{2}, \frac{\rho-\mu}{2}, -\frac{\nu}{2} \end{matrix} \right. \right)$ $b > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \rho >  \operatorname{Re} \mu  - 3/2$                                                                                   |
| (22) | $\int_0^\infty x^{\rho-1} K_\mu(ax) Y_\nu(bx^{-1}) dx$ $= (-1)^{m+1} 2^{\rho-2} a^{-\rho} G_{15}^{40} \left( \frac{a^2 b^2}{16} \left  \begin{matrix} \frac{1-\nu}{2} - m \\ \nu, -\frac{\nu}{2}, \frac{\rho+\nu}{2}, \frac{\rho-\mu}{2}, \frac{1-\nu}{2} - m \end{matrix} \right. \right)$ $m \text{ integer}, \quad b > 0, \quad \operatorname{Re} a > 0, \quad \operatorname{Re} \rho >  \operatorname{Re} \mu  - 3/2$ |
| (23) | $\int_0^\infty K_\nu(ax) K_\nu(\beta x^{-1}) dx = \pi a^{-1} K_{2\nu}(2a^{\frac{1}{2}} \beta^{\frac{1}{2}})$ $\operatorname{Re} a > 0, \quad \operatorname{Re} \beta > 0$                                                                                                                                                                                                                                                 |
| (24) | $\int_0^\infty x^{2\nu-\frac{1}{2}} K_{\frac{1}{2}-\nu}(ax) K_\nu(\beta x^{-1}) dx$ $= (2\pi)^{\frac{1}{2}} a^{-\nu-\frac{1}{2}} \beta^\nu K_{2\nu}[(2a\beta)^{\frac{1}{2}} e^{\frac{1}{4}\pi i}] K_{2\nu}[(2a\beta)^{\frac{1}{2}} e^{-\frac{1}{4}\pi i}]$ $\operatorname{Re} a > 0, \quad \operatorname{Re} \beta > 0$                                                                                                   |
| (25) | $\int_0^\infty x^{\rho-1} K_\mu(ax) K_\nu(\beta x^{-1}) dx$ $= 2^{\rho-3} a^{-\rho} G_{04}^{40} \left( \frac{a^2 \beta^2}{16} \left  \begin{matrix} \nu, -\frac{\nu}{2}, \frac{\rho+\mu}{2}, \frac{\rho-\mu}{2} \end{matrix} \right. \right)$ $\operatorname{Re} a > 0, \quad \operatorname{Re} \beta > 0$                                                                                                                |

## Modified functions of other arguments (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                              |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (26) | $\int_0^\infty x^{-2} [K_\nu(ax)]^2 J_0(bx^{-1}) dx = -2\pi b^{-1} K_{2\nu}(2a^{1/2} b^{1/2})$ $\times [\sin(\nu\pi) J_{2\nu}(2a^{1/2} b^{1/2}) + \cos(\nu\pi) Y_{2\nu}(2a^{1/2} b^{1/2})]$ $b > 0, \quad \operatorname{Re} a > 0, \quad -\frac{1}{4} < \operatorname{Re} \nu < \frac{1}{4}$                                                                 |
| (27) | $\int_0^a x^{\mu+1} y^{-\mu-2} J_\mu(x) I_\nu(y) dx = \frac{\Gamma(\frac{1}{2}\nu - \frac{1}{2}\mu) (\frac{1}{2}a)^\mu}{\Gamma(\frac{1}{2}\nu + \frac{1}{2}\mu + 1)} J_\nu(a)$ $y = (a^2 - x^2)^{1/2}, \quad \operatorname{Re} \nu > \operatorname{Re} \mu > -1$                                                                                             |
| (28) | $\int_0^a x^{2\nu} y^{-2\nu-1} J_{2\nu-1}(2x) K_{2\mu}(2y) dx = -\frac{1}{2} \Gamma(\frac{1}{2} + \mu - \nu) \Gamma(\frac{1}{2} - \mu - \nu)$ $\times a^{2\nu-1} \{ \sin[(\mu - \nu)\pi] J_{2\mu}(2a) + \cos[(\mu - \nu)\pi] Y_{2\mu}(2a) \}$ $y = (a^2 - x^2)^{1/2}, \quad 0 < \operatorname{Re} \nu < \frac{1}{2} -  \operatorname{Re} \mu $               |
| (29) | $\int_0^a x^{1-3\nu} y^{2\nu-1} J_{-3\nu}(2x) I_\nu(y) I_{\nu-1}(y) dx = \frac{\Gamma(\nu + \frac{1}{2})}{2\pi^{1/2} a^{2\nu}} J_\nu(a) J_{-\nu}(a)$ $y = (a^2 - x^2)^{1/2}, \quad 0 < \operatorname{Re} \nu < 1/3$                                                                                                                                          |
| (30) | $\int_0^a x^{1-2\nu} y^{2\nu-3/2} I_{-\nu}(x) K_\nu(x) J_{2\nu-3/2}(2y) dx$ $= -\frac{1}{4} \Gamma(\frac{1}{2} - \nu) a^{\nu-1} Y_\nu(2a) \quad y = (a^2 - x^2)^{1/2}, \quad \frac{1}{4} < \operatorname{Re} \nu < \frac{1}{2}$                                                                                                                              |
| (31) | $\int_0^a x J_\lambda(2x) I_\lambda(2x) J_\mu(2y) I_\mu(2y) dx = \frac{a^{2\lambda+2\mu+2}}{2\Gamma(\lambda+1)\Gamma(\mu+1)\Gamma(\lambda+\mu+2)}$ $\times {}_1F_4\left(\frac{\lambda+\mu+1}{2}; \lambda+1, \mu+1, \lambda+\mu+1, \frac{\lambda+\mu+3}{2}; -a^4\right)$ $y = (a^2 - x^2)^{1/2}, \quad \operatorname{Re} \lambda, \operatorname{Re} \mu > -1$ |

## Modified functions of other arguments (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                           |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (32) | $\int_0^a x^{\kappa-\frac{1}{2}} (a-x)^{-\kappa-\frac{1}{2}} e^{-x \sinh t} I_{2\mu} [x^{\frac{1}{2}} (a-x)^{\frac{1}{2}}] dx$ $= \frac{2\Gamma(\frac{1}{2} + \kappa + \mu) \Gamma(\frac{1}{2} - \kappa + \mu)}{a [\Gamma(2\mu + 1)]^2} M_{\kappa, \mu}(\frac{1}{2} a e^t) M_{-\kappa, \mu}(\frac{1}{2} a e^{-t})$ $\text{Re } \mu >  \text{Re } \kappa  - \frac{1}{2}$                                                                                   |
| (33) | $\int_0^\infty x^{\nu-1} (a^2 + x^2)^{-\frac{1}{2}\nu} K_\nu [(a^2 + x^2)^{\frac{1}{2}}] K_\mu(x) dx$ $= 2^{\nu-2} a^{-\nu} \Gamma\left(\frac{\nu-\mu}{2}\right) \Gamma\left(\frac{\nu+\mu}{2}\right) K_\mu(a)$ $\text{Re } \nu >  \text{Re } \mu , \quad \text{Re } a > 0$                                                                                                                                                                               |
| (34) | $\int_0^\infty x^{\nu-1} y^{-\nu} K_\nu(y) K_\mu(x) dx = 2^{-\frac{1}{2}} \pi^{\frac{1}{2}} \beta^{\frac{1}{2}-\nu} c^{-\frac{1}{2}} \Gamma(\nu + \mu) \Gamma(\nu - \mu)$ $\times P_{\mu-\frac{1}{2}}^{-\nu}(a/c) K_\mu(c) \quad y = [(x+a)^2 + \beta^2]^{\frac{1}{2}}, \quad c = (a^2 + \beta^2)^{\frac{1}{2}}$ $\text{Re } \beta >  \text{Im } a , \quad \text{Re } \nu >  \text{Re } \mu $                                                             |
| (35) | $\int_0^\infty x^{-\kappa-\frac{1}{2}} (a+x)^{\kappa-\frac{1}{2}} K_{2\mu} [x^{\frac{1}{2}} (a+x)^{\frac{1}{2}}] dx$ $= a^{-1} \Gamma(\frac{1}{2}-\kappa+\mu) \Gamma(\frac{1}{2}-\kappa-\mu) W_{\kappa, \mu}(\frac{1}{2} a e^{i\pi/2}) W_{\kappa, \mu}(\frac{1}{2} a e^{-i\pi/2})$ $ \arg a  < \pi, \quad \text{Re } \kappa +  \text{Re } \mu  < \frac{1}{2}$                                                                                             |
| (36) | $\int_0^\infty x^{-\frac{1}{2}} (a+x)^{-\frac{1}{2}} e^{-x \cosh t} K_\nu [x^{\frac{1}{2}} (a+x)^{\frac{1}{2}}] dx$ $= \frac{1}{2} \sec(\frac{1}{2} \nu \pi) e^{\frac{1}{2} a \cosh t} K_{\frac{1}{2}\nu}(\frac{1}{4} a e^t) K_{\frac{1}{2}\nu}(\frac{1}{4} a e^{-t}) \quad -1 < \text{Re } \nu < 1$                                                                                                                                                      |
| (37) | $\int_0^\infty x^{-\kappa-\frac{1}{2}} (a+x)^{\kappa-\frac{1}{2}} \exp(-\beta x) K_{2\mu} [x^{\frac{1}{2}} (a+x)^{\frac{1}{2}}] dx$ $= a^{-1} e^{\frac{1}{2} a \beta} \Gamma(\frac{1}{2}-\kappa+\mu) \Gamma(\frac{1}{2}-\kappa-\mu) W_{\kappa, \mu}(z_1) W_{\kappa, \mu}(z_2)$ $ \arg a  < \pi, \quad \text{Re } \beta > -1, \quad \text{Re } \kappa +  \text{Re } \mu  < \frac{1}{2}$ $z_1, z_2 = \frac{1}{2} a [\beta \pm (\beta^2 - 1)^{\frac{1}{2}}]$ |

## Modified functions of other arguments (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                         |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (38) | $\int_0^\infty x^{\rho-1} \left( \frac{a+\beta x}{ax+\beta} \right)^\nu K_{2\nu} [x^{-1/2}(a+\beta x)^{1/2} (ax+\beta)^{1/2}] dx$ $= 2K_{\nu+\rho}(a) K_{\nu-\rho}(\beta) \quad \text{Re } a > 0, \quad \text{Re } \beta > 0$                                                                                                                           |
| (39) | $\int_0^{\frac{1}{2}\pi} \cos[(\mu-\nu)x] I_{\mu+\nu}(2a \cos x) dx = \frac{1}{2}\pi I_\mu(a) I_\nu(a)$ $\text{Re}(\mu+\nu) > -1$                                                                                                                                                                                                                       |
| (40) | $\int_0^{\frac{1}{2}\pi} \cos[(\mu-\nu)x] K_{\mu+\nu}(2a \cos x) dx = \frac{1}{2}\pi \csc[(\mu+\nu)\pi]$ $\times [I_{-\mu}(a) I_{-\nu}(a) - I_\mu(a) I_\nu(a)] \quad -1 < \text{Re}(\mu+\nu) < 1$                                                                                                                                                       |
| (41) | $\int_0^{\frac{1}{2}\pi} \sec x \cos(2\kappa x) K_{2\mu}(a \sec x) dx = \frac{\pi}{2a} W_{\kappa,\mu}(a) W_{-\kappa,\mu}(a)$ $\text{Re } a > 0$                                                                                                                                                                                                         |
| (42) | $\int_0^\infty \cosh(2\mu x) K_{2\nu}(2a \cosh x) dx = \frac{1}{2} K_{\mu+\nu}(a) K_{\mu-\nu}(a) \quad \text{Re } a > 0$                                                                                                                                                                                                                                |
| (43) | $\int_0^\infty \text{sech } x \cosh(2\kappa x) I_{2\mu}(a \text{sech } x)$ $= \frac{\Gamma(\frac{1}{2} + \kappa + \mu) \Gamma(\frac{1}{2} - \kappa + \mu)}{2a [\Gamma(2\mu + 1)]^2} M_{\kappa,\mu}(a) M_{-\kappa,\mu}(a) \quad  \text{Re } \kappa  - \text{Re } \mu < \frac{1}{2}$                                                                      |
| (44) | $\int_0^\infty (\sinh x)^{\mu+1} (\cosh x)^{-2\mu-3/2} P_\nu^{-\mu}[\cosh(2x)] I_{\mu-\frac{1}{2}}(a \text{sech } x) dx$ $= \frac{2^{\mu-\frac{1}{2}} \Gamma(\mu-\nu) \Gamma(\mu+\nu+1)}{\pi^{1/2} a^{\mu+3/2} [\Gamma(\mu+1)]^2} M_{\nu+1/2,\mu}(a) M_{-\nu-1/2,\mu}(a)$ $\text{Re } \mu > \text{Re } \nu, \quad \text{Re } \mu > -\text{Re } \nu - 1$ |



## Modified functions of other arguments (cont'd)

|      |                                                                                                                                                                                                     |                                                          |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------|
| (45) | $\int_{-\infty}^{\infty} e^{\rho x} \left( \frac{a + \beta e^x}{a e^x + \beta} \right)^{\nu} K_{2\nu}[(a^2 + \beta^2 + 2a\beta \cosh x)^{\frac{1}{2}}] dx$ $= 2K_{\nu+\rho}(a) K_{\nu-\rho}(\beta)$ | $\operatorname{Re} a, \quad \operatorname{Re} \beta > 0$ |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------|

## 19.7. Bessel functions and modified Bessel functions of variable order

|     |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |                                                                                                          |
|-----|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------|
| (1) | $\int_{-\infty}^{\infty} J_{\nu-x}(a) J_{\mu+x}(a) dx = J_{\mu+\nu}(2a)$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                              | $\operatorname{Re}(\mu + \nu) > 1$                                                                       |
| (2) | $\int_{-\infty}^{\infty} a^{-\mu-x} b^{-\nu+x} e^{cx} J_{\mu+x}(a) J_{\nu-x}(b) dx$ $= \left[ \frac{2 \cos(\frac{1}{2}c)}{a^2 e^{-\frac{1}{2}ci} + b^2 e^{\frac{1}{2}ci}} \right]^{\frac{1}{2}\mu + \frac{1}{2}\nu} \exp[\frac{1}{2}c(\nu - \mu)i]$ $\times J_{\mu+\nu}\{[2 \cos(\frac{1}{2}c)(a^2 e^{-\frac{1}{2}ci} + b^2 e^{\frac{1}{2}ci})]^{\frac{1}{2}}\}$ $= 0$                                                                                                                                                                                                                                | $-\pi < c < \pi$<br>$c \geq \pi \quad \text{or} \quad c \leq -\pi$<br>$\operatorname{Re}(\mu + \nu) > 1$ |
| (3) | $\int_{-\infty}^{\infty} J_{\kappa+x}(a) J_{\lambda-x}(a) J_{\mu+x}(a) J_{\nu-x}(a) dx$ $= \frac{\Gamma(\kappa + \lambda + \mu + \nu + 1)}{\Gamma'(\kappa + \lambda + 1) \Gamma'(\lambda + \mu + 1) \Gamma'(\mu + \nu + 1) \Gamma'(\nu + \kappa + 1)}$ $\times {}_4F_5\left(\frac{\kappa + \lambda + \mu + \nu + 1}{2}, \frac{\kappa + \lambda + \mu + \nu + 1}{2}, \frac{\kappa + \lambda + \mu + \nu}{2} + 1, \frac{\kappa + \lambda + \mu + \nu}{2} + 1;\right.$ $\left. \kappa + \lambda + \mu + \nu + 1, \kappa + \lambda + 1, \lambda + \mu + 1, \mu + \nu + 1, \nu + \kappa + 1; -4a^2\right)$ | $\operatorname{Re}(\kappa + \lambda + \mu + \nu) > -1$                                                   |

For similar integrals see Vol. I, p. 59 ff. and p. 123 ff.

## Variable order (cont'd)

|      |                                                                                                                                                                                                                                                                                                       |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (4)  | $\int_0^\infty J_x(xz) J_{-x}(xz) \cos(\pi x) dx = \frac{1}{4}(1-z^2)^{-\frac{1}{2}} \quad  z  < 1$                                                                                                                                                                                                   |
| (5)  | $\int_0^\infty [J_x(xz) J_{-x}(xz) \cos(\pi x) - 1] x^{-2} dx = -\frac{1}{2}\pi^2$                                                                                                                                                                                                                    |
| (6)  | $\int_{-\infty}^\infty \operatorname{sech}(\frac{1}{2}\pi x) J_{ix}(a) dx = 2 \sin a \quad a > 0$                                                                                                                                                                                                     |
| (7)  | $\int_{-\infty}^\infty \operatorname{csch}(\frac{1}{2}\pi x) J_{ix}(a) dx = -2i \cos a \quad a > 0$                                                                                                                                                                                                   |
| (8)  | $\int_{-\infty}^\infty \frac{e^{\frac{1}{2}\pi x} \cos(bx)}{\sinh(\pi x)} J_{ix}(a) dx = -i \exp(ia \cosh b) \quad a, b > 0$                                                                                                                                                                          |
| (9)  | $\int_{-\infty}^\infty e^{-cx i} [J_{\nu-ix}(a) Y_{\nu+ix}(b) + Y_{\nu-ix}(a) J_{\nu+ix}(b)] dx$ $= -2(h/k)^{2\nu} J_{2\nu}(hk)$ $h = (ae^{\frac{1}{2}c} + be^{-\frac{1}{2}c})^{\frac{1}{2}}, \quad k = (ae^{-\frac{1}{2}c} + be^{\frac{1}{2}c})^{\frac{1}{2}}, \quad a, b > 0, \quad c \text{ real}$ |
| (10) | $\int_{-\infty}^\infty e^{-cx i} [J_{\nu-ix}(a) J_{\nu+ix}(b) - Y_{\nu-ix}(a) Y_{\nu+ix}(b)] dx$ $= 2(h/k)^{2\nu} Y_{2\nu}(hk)$ $h = (ae^{\frac{1}{2}c} + be^{-\frac{1}{2}c}), \quad k = (ae^{-\frac{1}{2}c} + be^{\frac{1}{2}c})^{\frac{1}{2}}, \quad a, b > 0, \quad c \text{ real}$                |
| (11) | $\int_{-\infty}^\infty e^{-cx i} H_{\nu-ix}^{(2)}(a) H_{\nu+ix}^{(2)}(b) dx = 2i(h/k)^{2\nu} H_{2\nu}^{(2)}(hk)$ $a, b > 0, \quad c \text{ real}, \quad h = (ae^{\frac{1}{2}c} + be^{-\frac{1}{2}c})^{\frac{1}{2}}, \quad k = (ae^{-\frac{1}{2}c} + be^{\frac{1}{2}c})^{\frac{1}{2}}$                 |
| (12) | $\int_0^\infty \operatorname{sech}(\pi x) \{[J_{ix}(a)]^2 + [Y_{ix}(a)]^2\} dx = -Y_0(2a) - \mathbf{E}_0(2a) \quad a > 0$                                                                                                                                                                             |

## Variable order (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                                  |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (13) | $\int_0^\infty x e^{\pi x} \tanh(\pi x) H_{ix}^{(2)}(a) H_{ix}^{(2)}(b) dx = -\frac{2(ab)^{\frac{1}{2}}}{\pi(a+b)} \exp[-ik(a+b)]$ $a, b > 0$                                                                                                                                                                                                                                    |
| (14) | $\int_0^\infty x e^{\pi x} \sinh(\pi x) \Gamma(\nu + ix) \Gamma(\nu - ix) H_{ix}^{(2)}(a) H_{ix}^{(2)}(b) dx$ $= i 2^\nu \pi^{\frac{1}{2}} \Gamma(\frac{1}{2} + \nu) (ab)^\nu (a+b)^{-\nu} K_\nu(a+b)$ $a, b > 0, \quad \operatorname{Re} \nu > 0$                                                                                                                               |
| (15) | $\int_0^\infty x e^{\pi x} \sinh(\pi x) \cosh(\pi x) \Gamma(\nu + ix) \Gamma(\nu - ix) H_{ix}^{(2)}(a) H_{ix}^{(2)}(b) dx$ $= \frac{i \pi^{3/2} 2^\nu}{\Gamma(\frac{1}{2} - \nu)} (b-a)^{-\nu} H_\nu^{(2)}(b-a) \quad 0 < a < b, \quad 0 < \operatorname{Re} \nu < \frac{1}{2}$                                                                                                  |
| (16) | $\int_0^\infty x e^{\pi x} \sinh(\pi x) \Gamma\left(\frac{\nu + ix}{2}\right) \Gamma\left(\frac{\nu - ix}{2}\right) H_{ix}^{(2)}(a) H_{ix}^{(2)}(b) dx$ $= i \pi 2^{2-\nu} (ab)^\nu (a^2 + b^2)^{-\frac{1}{2}\nu} H_\nu^{(2)}[(a^2 + b^2)^{\frac{1}{2}}]$ $a, b > 0, \quad \operatorname{Re} \nu > 0$                                                                            |
| (17) | $\int_0^\infty x e^{\pi x} \tanh(\pi x) P_{-\frac{1}{2} + ix}(-\cos \phi) H_{ix}^{(2)}(a) H_{ix}^{(2)}(b) dx$ $= -\frac{2(ab)^{\frac{1}{2}}}{\pi R} e^{-iR}$ $a, b > 0, \quad 0 < \phi < \pi, \quad R = (a^2 + b^2 - 2ab \cos \phi)^{\frac{1}{2}}$                                                                                                                               |
| (18) | $\int_0^\infty x e^{\pi x} \sinh(\pi x) \Gamma(\nu + ix) \Gamma(\nu - ix) P_{-\frac{1}{2} + ix}^{-\nu}(-\cos \phi)$ $\times H_{ix}^{(2)}(a) H_{ix}^{(2)}(b) dx = i (2\pi)^{\frac{1}{2}} (\sin \phi)^{\nu - \frac{1}{2}} (ab)^\nu R^{-\nu} H_\nu^{(2)}(R)$ $a, b > 0, \quad 0 < \phi < \pi, \quad R = (a^2 + b^2 - 2ab \cos \phi)^{\frac{1}{2}}, \quad \operatorname{Re} \nu > 0$ |

## Variable order (cont'd)

|      |                                                                                                                                                                                                                                                                                   |                                                    |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------|
| (19) | $\int_0^\infty \cosh(\frac{1}{2}\pi x) K_{ix}(a) dx = \frac{1}{2}\pi$                                                                                                                                                                                                             | $a > 0$                                            |
| (20) | $\int_0^\infty x \sinh(\frac{1}{2}\pi x) K_{ix}(a) dx = \frac{1}{2}\pi a$                                                                                                                                                                                                         | $a > 0$                                            |
| (21) | $\int_{-\infty}^\infty K_{ix+iy}(a) K_{ix+iz}(\beta) dx = \pi K_{iy-iz}(a+\beta)$                                                                                                                                                                                                 | $ \arg a  +  \arg \beta  < \pi$                    |
| (22) | $\int_{-\infty}^\infty e^{\pi x} K_{ix+iy}(a) K_{ix+iz}(b) dx = \pi e^{-\pi z} K_{iy-iz}(a-b)$                                                                                                                                                                                    | $a > b > 0$                                        |
| (23) | $\int_{-\infty}^\infty e^{i\rho x} K_{\nu+ix}(a) K_{\nu-ix}(\beta) dx = \pi \left( \frac{a + \beta e^\rho}{a e^\rho + \beta} \right)^\nu K_{2\nu}(w)$<br>$ \arg a  +  \arg \beta  +  \operatorname{Im} \rho  < \pi, \quad w = (a^2 + \beta^2 + 2a\beta \cosh \rho)^{\frac{1}{2}}$ |                                                    |
| (24) | $\int_{-\infty}^\infty \exp[(\pi - \gamma)x] K_{ix+iy}(a) K_{ix+iz}(b) dx = \pi e^{-\beta y - \alpha z} K_{iy-iz}(c)$<br>where $0 < \gamma < \pi$ , $a, b, c > 0$ , and $a, \beta, \gamma$ are angles of the triangle with the sides $a, b, c$ .                                  |                                                    |
| (25) | $\int_{-\infty}^\infty (n + \nu + ix)^{-1} \sin[(\nu + ix)\pi] K_{\nu+ix}(a) K_{\nu-ix}(b) dx$<br>$= \pi^2 I_n(a) K_{n+2\nu}(b)$<br>$= \pi^2 K_{n+2\nu}(a) I_n(b)$                                                                                                                | $0 < a < b$<br>$0 < b < a$<br>$n = 0, 1, 2, \dots$ |
| (26) | $\int_0^\infty \sin(bx) \sinh(\pi x) [K_{ix}(a)]^2 dx = \frac{1}{4}\pi^2 J_0[2a \sinh(\frac{1}{2}b)]$                                                                                                                                                                             | $a, b > 0$                                         |

## Variable order (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                             |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (27) | $\int_0^\infty \cos(bx) \cosh(\pi x) [K_{ix}(a)]^2 dx = -\frac{1}{4} \pi^2 Y_0[2a \sinh(\frac{1}{2}b)]$ $a, b > 0$                                                                                                                                                                                                                                          |
| (28) | $\int_0^\infty \cosh(\rho x) K_{\nu+ix}(a) K_{\nu-ix}(a) dx = \frac{1}{2} \pi K_{2\nu}[2a \cos(\frac{1}{2}\rho)]$ $2 \arg a  +  \operatorname{Re} \rho  < \pi$                                                                                                                                                                                              |
| (29) | $\int_{-\infty}^\infty (\nu - \frac{1}{2} + ix) \Gamma(\frac{1}{2} - ix) \Gamma(2\nu - \frac{1}{2} + ix) P_{\nu+ix-1}^{\frac{1}{2}-\nu}(\cos \phi)$ $\times I_{\nu-\frac{1}{2}+ix}(a) K_{\nu-\frac{1}{2}+ix}(b) dx = (2\pi)^{\frac{1}{2}} (\sin \phi)^{\nu-\frac{1}{2}} (ab/\omega)^\nu K_\nu(\omega)$ $\omega = (a^2 + b^2 + 2ab \cos \phi)^{\frac{1}{2}}$ |
|      | See also Chapter XII for similar integrals.                                                                                                                                                                                                                                                                                                                 |

## 19.8. Functions related to Bessel functions

|     |                                                                                                                                                                                                                                     |
|-----|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $\int_0^\infty x J_\nu(ax) [\mathbf{J}_\nu(x) - J_\nu(x)] dx = \frac{\sin(\nu\pi)}{\pi a(a+1)} \quad a > 0, \quad \operatorname{Re} \nu > -1$                                                                                       |
| (2) | $\int_0^\infty x^{-\nu-1} \mathbf{H}_\nu(x) dx = \frac{2^{-\nu-1} \pi}{\Gamma(\nu+1)} \quad \operatorname{Re} \nu > -3/2$                                                                                                           |
| (3) | $\int_{-\infty}^\infty x^{-\nu-1} \frac{\sin[a(x+\lambda)]}{x+\lambda} \mathbf{H}_\nu(x) dx = \pi \lambda^{-\nu-1} \mathbf{H}_\nu(\lambda)$ $a \geq 1, \quad \operatorname{Re} \nu > 5/2$                                           |
| (4) | $\int_0^\infty x^{1-\mu-\nu} J_\nu(x) \mathbf{H}_\mu(x) dx = \frac{(2\nu-1) 2^{-\mu-\nu}}{(\mu+\nu-1) \Gamma(\mu+\frac{1}{2}) \Gamma(\nu+\frac{1}{2})}$ $\operatorname{Re} \nu > \frac{1}{2}, \quad \operatorname{Re}(\mu+\nu) > 1$ |

## Related functions (cont'd)

|                                                                                                                                                        |                                                                                                                                                                                                                                                                                                                                                 |
|--------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (5)                                                                                                                                                    | $\int_0^\infty [\cos(\tfrac{1}{2}\nu\pi) J_\nu(x) + \sin(\tfrac{1}{2}\nu\pi) \mathbf{H}_\nu(x)] \frac{dx}{x^2 + a^2}$ $= \frac{\pi}{2a} [I_\nu(a) - \mathbf{L}_\nu(a)] \quad \text{Re } a > 0, \quad -\tfrac{1}{2} < \text{Re } \nu < 2$                                                                                                        |
| (6)                                                                                                                                                    | $\int_a^\infty x^{\frac{1}{2}} (x^2 - a^2)^{-\frac{1}{4} - \frac{1}{2}\nu} p_\mu^{\nu + \frac{1}{2}} (2x^2 a^{-2} - 1) [\mathbf{H}_\nu(x) - Y_\nu(x)] dx$ $= 2^{-\nu-2} \pi^{\frac{1}{2}} a \csc(\mu\pi) \cos(\nu\pi) \{ [Y_\nu(\tfrac{1}{2}a)]^2 - [J_\nu(\tfrac{1}{2}a)]^2 \}$ $-1 < \text{Re } \mu < 0, \quad \text{Re } \nu < \tfrac{1}{2}$ |
| (7)                                                                                                                                                    | $\int_0^\infty x^{\rho-1} K_\mu(ax) K_\nu(ax) \mathbf{H}_\lambda(x) dx$ <p>See Mohan, B., 1942: <i>Bull. Calcutta Math. Soc.</i> 34, 55-59.</p>                                                                                                                                                                                                 |
| <p>For other integrals involving Bessel functions and Struve functions see McLachlan, N.W. and A.L. Meyers, 1936: <i>Philos. Mag.</i> 21, 425-448.</p> |                                                                                                                                                                                                                                                                                                                                                 |
| (8)                                                                                                                                                    | $\int_0^\infty x^{-\mu-\nu} \mathbf{H}_\mu(x) \mathbf{H}_\nu(x) dx = \frac{2^{-\mu-\nu} \pi^{\frac{1}{2}} \Gamma(\mu + \nu)}{\Gamma(\mu + \tfrac{1}{2}) \Gamma(\nu + \tfrac{1}{2}) \Gamma(\mu + \nu + \tfrac{1}{2})}$ $\text{Re}(\mu + \nu) > 0$                                                                                                |
| (9)                                                                                                                                                    | $\int_0^{\frac{1}{2}\pi} \cos[(\nu + 1)x] \mathbf{H}_\nu(a \cos x) dx = \pi^{\frac{1}{2}} a^{-\frac{1}{2}} \sin(\tfrac{1}{2}a) J_{\nu+\frac{1}{2}}(\tfrac{1}{2}a)$ $\text{Re } \nu > -2$                                                                                                                                                        |
| (10)                                                                                                                                                   | $\int_0^{\frac{1}{2}\pi} \frac{\cos(2\nu x)}{\cos x} [\mathbf{H}_{2\nu}(a \sec x) - Y_{2\nu}(a \sec x)] dx$ $= \frac{\pi^{\frac{1}{2}} a^{-1}}{\Gamma(2\nu + \tfrac{1}{2})} \mathbb{W}_{\nu,\nu}(ae^{i\pi/2}) \mathbb{W}_{\nu,\nu}(ae^{-i\pi/2}) \quad \text{Re } \nu < \tfrac{1}{2}$                                                           |

## Related functions (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                         |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (11) | $\int_0^\infty \exp[(\nu+1)x] \mathbf{H}_\nu(a \sinh x) dx = \pi^{\frac{1}{2}} a^{-\frac{1}{2}} \csc(\nu\pi)$ $\times [\sinh(\frac{1}{2}a) I_{\nu+\frac{1}{2}}(\frac{1}{2}a) - \cosh(\frac{1}{2}a) I_{-\nu-\frac{1}{2}}(\frac{1}{2}a)]$ $\text{Re } a > 0, \quad -2 < \text{Re } \nu < 0$                                                               |
| (12) | $\int_0^\infty x^{-1} \cos(2ax^{-1}) [I_0(x) - \mathbf{L}_0(x)] dx = 2 J_0(2a^{\frac{1}{2}}) K_0(2a^{\frac{1}{2}}) \quad a > 0$                                                                                                                                                                                                                         |
| (13) | $\int_0^\infty x^{\nu-\frac{1}{2}} \exp[-(1+a)x] K_0(ax) \mathbf{L}_\nu(x) dx$ $= \frac{\pi^{\frac{1}{2}} [\Gamma(\nu+\frac{1}{2})]^2}{(2a)^{\nu+\frac{1}{2}} \Gamma(\nu+1)} P_{\nu-\frac{1}{2}}(1+a^{-1}) \quad \text{Re } a > 0, \quad \text{Re } \nu > -\frac{1}{2}$                                                                                 |
| (14) | $\int_0^a x P_n(1-2x^2 a^{-2}) [I_0(x) - \mathbf{L}_0(x)] dx$ $= (-1)^n a [I_{2n+1}(a) - \mathbf{L}_{2n+1}(a)] \quad n = 0, 1, 2, \dots$                                                                                                                                                                                                                |
| (15) | $\int_a^\infty x^{\frac{1}{2}} (x^2 - a^2)^{-\frac{1}{4}-\frac{1}{2}\nu} P_\mu^{\nu+\frac{1}{2}}(2x^2 a^{-2} - 1) [I_{-\nu}(x) - \mathbf{L}_\nu(x)] dx$ $= 2^{-\nu-1} \pi^{\frac{1}{2}} a \csc(2\mu\pi) \cos(\nu\pi) \{ [I_\nu(\frac{1}{2}a)]^2 - [\mathbf{I}_{-\nu}(\frac{1}{2}a)]^2 \}$ $-1 < \text{Re } \mu < 0, \quad \text{Re } \nu < \frac{1}{2}$ |
| (16) | $\int_0^{\frac{1}{2}\pi} \frac{\cos(2\nu x)}{\cos x} [I_{2\nu}(a \sec x) - \mathbf{L}_{2\nu}(a \sec x)] dx$ $= \frac{\pi^{\frac{1}{2}} a^{-1}}{\Gamma(2\nu+1)} W_{\nu,\nu}(a) M_{-\nu,\nu}(a) \quad \text{Re } \nu < \frac{1}{2}$                                                                                                                       |
| (17) | $\int_0^\infty x^{\lambda-1} s_{\mu,\nu}(x) dx = \frac{\Gamma(\frac{1+\lambda+\mu}{2}) \Gamma(\frac{1-\lambda-\mu}{2}) \Gamma(\frac{1+\mu+\nu}{2}) \Gamma(\frac{1+\mu-\nu}{2})}{2^{2-\lambda-\mu} \Gamma(1-\frac{\lambda+\nu}{2}) \Gamma(1-\frac{\lambda-\nu}{2})}$ $-\text{Re } \mu < \text{Re } \lambda + 1 < 5/2$                                    |

## Related functions (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (18) | $\int_0^\infty x^{-\mu-1} \cos(ax) s_{\mu,\nu}(x) dx$ $= 0 \quad a > 1$ $= 2^{\mu-\frac{1}{2}} \pi^{\frac{1}{2}} \Gamma\left(\frac{\mu+\nu+1}{2}\right) \Gamma\left(\frac{\mu-\nu+1}{2}\right) (1-a^2)^{\frac{1}{2}\mu+\frac{1}{4}} P_{\nu-\frac{1}{2}}^{-\mu-\frac{1}{2}}(a)$ $0 < a < 1$                                                                                                                                                                                                    |
| (19) | $\int_0^\infty x^{-\mu} J_\nu(ax) s_{\mu+\nu,\mu-\nu+1}(x) dx$ $= 0 \quad a > 1$ $= 2^{\nu-1} \Gamma(\nu) a^{-\nu} (1-a^2)^\mu$ $0 < a < 1$ $\operatorname{Re} \nu > -1$                                                                                                                                                                                                                                                                                                                      |
| (20) | $\int_0^\infty x^\nu K_{1-2\nu}(ax^{\frac{1}{2}} e^{i\pi/4}) K_{1-2\nu}(ax^{\frac{1}{2}} e^{-i\pi/4}) s_{\mu,\nu}(x) dx$ $= 2^{2\mu+2\nu-2} \pi^{-\frac{1}{2}} \Gamma\left(\frac{1+\mu+\nu}{2}\right) \Gamma\left(\frac{1+\mu-\nu}{2}\right)$ $\times \Gamma\left(\frac{\mu+3\nu+1}{2}\right) \Gamma\left(\frac{\mu+\nu}{2}+1\right) S_{-\mu-2\nu-\frac{1}{2},\frac{1}{2}-\nu}(a)$ $ \arg a  < \frac{1}{4}\pi, \quad \operatorname{Re}(\mu-\nu) > -3, \quad \operatorname{Re}(\mu+3\nu) > -1$ |
| (21) | $\int_0^{\frac{1}{2}\pi} \cos[(\mu+1)x] s_{\mu,\nu}(a \cos x) dx = 2^{\mu-2} \pi \Gamma\left(\frac{\mu+\nu+1}{2}\right) \Gamma\left(\frac{\mu-\nu+1}{2}\right)$ $\times J_{\frac{1}{2}(\mu+\nu+1)}(\frac{1}{2}a) J_{\frac{1}{2}(\mu-\nu+1)}(\frac{1}{2}a)$ $\operatorname{Re} \mu > -2$                                                                                                                                                                                                       |
| (22) | $\int_0^\infty \exp[(\mu+1)x] s_{\mu,\nu}(a \sinh x) dx = 2^{\mu-2} \pi \csc(\mu\pi) \Gamma(\rho) \Gamma(\sigma)$ $\times [I_\rho(\frac{1}{2}a) I_\sigma(\frac{1}{2}a) - I_{-\rho}(\frac{1}{2}a) I_{-\sigma}(\frac{1}{2}a)]$ $2\rho = \mu + \nu + 1, \quad 2\sigma = \mu - \nu + 1, \quad a > 0, \quad -2 < \operatorname{Re} \mu < 0$                                                                                                                                                        |



## Related functions (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (23) | $\int_0^\infty x^{-\mu} \sin(ax) S_{\mu,\nu}(x) dx = 2^{-\mu-\frac{1}{2}} \pi^{\frac{1}{2}} \Gamma\left(1 - \frac{\mu+\nu}{2}\right) \Gamma\left(1 - \frac{\mu-\nu}{2}\right) \\ \times (a^2 - 1)^{\frac{1}{2}\mu-\frac{1}{4}} P_{\nu-\frac{1}{2}}^{\mu-\frac{1}{2}}(a) \quad a > 1, \quad \operatorname{Re} \mu < 1 -  \operatorname{Re} \nu $                                                                                                                                                                                                                                |
| (24) | $\int_0^a x^{\frac{1}{2}(\nu-\mu-1)} (a^2 - x^2)^{\frac{1}{4}(\nu-\mu-2)} P_{\nu-\frac{1}{2}}^{\frac{1}{2}(\mu-\nu+2)}(x/a) S_{\mu,\nu}(x) dx \\ = 2^{\mu-3/2} \pi^{1/2} a^{(\nu-\mu)/2} \Gamma\left(\frac{\mu+\nu+3}{4}\right) \Gamma\left(\frac{\mu-3\nu+3}{4}\right) \cos[\frac{1}{2}(\mu-\nu)\pi] \\ \times [J_{\nu}(\frac{1}{2}a) Y_{-\frac{1}{2}(\mu-\nu+1)}(\frac{1}{2}a) - Y_{\nu}(\frac{1}{2}a) J_{-\frac{1}{2}(\mu-\nu+1)}(\frac{1}{2}a)] \\ \operatorname{Re}(\mu - \nu) < 0, \quad -1 < \operatorname{Re}(\mu + \nu) < 1, \quad \operatorname{Re}(\mu - 3\nu) < 1$ |
| (25) | $\int_a^\infty x^{\frac{1}{2}} (x^2 - a^2)^{-\frac{1}{2}\beta} P_{\nu}^{\beta}(x/a) S_{\mu,\frac{1}{2}}(x) dx \\ = \frac{a^{\frac{1}{2}} \Gamma(\frac{\beta-\mu+\nu}{2} + \frac{1}{4}) \Gamma(\frac{\beta-\mu-\nu}{2} - \frac{1}{4})}{\pi^{1/2} 2^{3/2-\beta+\mu} \Gamma(\frac{1}{2}-\mu)} S_{\mu-\beta+1, \nu+\frac{1}{2}}(a) \\ \operatorname{Re} \beta < 1, \quad \operatorname{Re}(\mu + \nu - \beta) < -\frac{1}{2}, \quad \operatorname{Re}(\mu - \nu - \beta) < \frac{1}{2}$                                                                                            |
| (26) | $\int_a^\infty x(x^2 - a^2)^{-\frac{1}{2}\nu} P_{\lambda}^{\nu}(2x^2 a^{-2} - 1) S_{\mu,\nu}(x) dx \\ = \frac{a \Gamma(\frac{\nu-\mu-1}{2} - \lambda) \Gamma(\frac{\nu-\mu+1}{2} + \lambda)}{2 \Gamma(\frac{1-\mu-\nu}{2}) \Gamma(\frac{1-\mu+\nu}{2})} S_{\mu-\nu+1, 2\lambda+1}(a) \\ \operatorname{Re} \nu < 1, \quad \operatorname{Re}(\mu - \nu + \lambda) < -1, \quad \operatorname{Re}(\mu - \nu - \lambda) < 0$                                                                                                                                                        |
| (27) | $\int_a^\infty x^{-\nu} (x^2 - a^2)^{\frac{1}{4}-\frac{1}{2}\nu} P_{\frac{1}{2}\mu-\frac{1}{2}\nu}^{\nu-\frac{1}{2}}(2a^2 x^{-2} - 1) S_{\mu,\nu}(x) dx \\ = \frac{\pi^{\frac{1}{2}} 2^{\mu-\nu} \Gamma(\frac{3\nu-\mu-1}{2})}{a^{\nu+\frac{1}{2}} \Gamma(\frac{1+\nu-\mu}{2})} \mathfrak{H}_{\rho,\sigma}^{\nu}(ae^{i\pi/2}) \mathfrak{H}_{\rho,\sigma}^{\nu}(ae^{-i\pi/2}) \\ \rho = \frac{1}{2}(\mu + 1 - \nu), \quad \sigma = \nu - \frac{1}{2} \\ \operatorname{Re}(\mu - \nu) < 0, \quad \operatorname{Re} \nu < 3/2, \quad \operatorname{Re}(3\nu - \mu) > 1$           |

## Related functions (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                     |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (28) | $\int_0^\infty x^{1-\mu-\nu} J_\nu(ax) S_{\mu, -\mu-2\nu}(x) dx = \frac{\pi^{\frac{1}{2}} a^{\nu-1} \Gamma(1-\mu-\nu)}{2^{\mu+2\nu} \Gamma(\nu+\frac{1}{2})}$ $\times (a^2-1)^{\frac{1}{2}(\mu+\nu-1)} P_{\mu+\nu}^{\mu+\nu-1}(a)$ $a > 1, \quad \operatorname{Re} \nu > -\frac{1}{2}, \quad \operatorname{Re}(\mu+\nu) < 1$                        |
| (29) | $\int_0^{\frac{1}{2}\pi} \cos(2\mu x) S_{2\mu-1, 2\nu}(a \cos x) dx = \frac{\pi 2^{2\mu-3} a^{2\mu} \csc(2\nu\pi)}{\Gamma(1-\mu-\nu) \Gamma(1-\mu+\nu)}$ $\times [J_{\mu+\nu}(\frac{1}{2}a) Y_{\mu-\nu}(\frac{1}{2}a) - J_{\mu-\nu}(\frac{1}{2}a) Y_{\mu+\nu}(\frac{1}{2}a)]$ $\operatorname{Re} \mu > -2, \quad -1 < \operatorname{Re} \nu < 1$    |
| (30) | $\int_0^{\frac{1}{2}\pi} \frac{\cos(2\mu x)}{\cos x} S_{2\mu, 2\nu}(a \sec x) dx = \frac{\pi 2^{2\mu-1}}{a} W_{\mu, \nu}(ae^{i\pi/2})$ $\times W_{\mu, \nu}(ae^{-i\pi/2}) \quad  \arg a  < \pi, \quad \operatorname{Re} \mu < 1$                                                                                                                    |
| (31) | $\int_0^\infty (\sinh x)^{\frac{1}{2}} \cosh(\nu x) S_{\mu, \frac{1}{2}}(a \cosh x) dx$ $= \frac{\Gamma(\frac{1}{4} - \frac{\mu+\nu}{2}) \Gamma(\frac{1}{4} - \frac{\mu-\nu}{2})}{2^{\mu+3/2} a^{1/2} \Gamma(\frac{1}{2} - \mu)} S_{\mu+\frac{1}{2}, \nu}(a)$ $ \arg a  < \pi, \quad \operatorname{Re} \mu +  \operatorname{Re} \nu  < \frac{1}{2}$ |
| (32) | $\int_0^\infty x^{2\nu-1} U_\nu(w, x) dx = 2^{\nu-1} \Gamma(\nu) w^\nu \cos(\frac{1}{2}w) \quad \operatorname{Re} \nu > 0$                                                                                                                                                                                                                          |
| (33) | $\int_0^\infty x^{2\nu-3} U_\nu(w, x) dx = 2^{\nu-2} \Gamma(\nu-1) w^{\nu-1} \sin(\frac{1}{2}w) \quad \operatorname{Re} \nu > 1$                                                                                                                                                                                                                    |
| (34) | $\int_0^\infty x^{1-\nu} \sin(\frac{1}{2}ax) U_\nu(x, z) dx$ $= 0 \quad a > 1$ $= \frac{1}{2}\pi(1-a)^{\frac{1}{2}\nu-1} z^{-2\nu} J_{\nu-2}[z(1-a)^{\frac{1}{2}}] \quad 0 < a < 1$                                                                                                                                                                 |

**Related functions (cont'd)**

|      |                                                                                                                                                                                                  |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (35) | $\int_0^\infty x^{-\nu} \cos(\tfrac{1}{2}ax) U_\nu(x, z) dx$ $= 0 \qquad a > 1$ $= \tfrac{1}{2}\pi(1-a)^{\frac{1}{2}\nu-\frac{1}{2}} z^{1-\nu} J_{\nu-1}[z(1-a)^{\frac{1}{2}}] \qquad 0 < a < 1$ |
| (36) | $\int_0^\infty x^{\nu-1} J_{\frac{1}{2}\nu-1}\left(\frac{x^2}{2w}\right) U_\nu(w, x) dx = \frac{w^\nu}{2(\nu-1)} J_{\frac{1}{2}\nu}\left(\frac{w}{2}\right)$ $\text{Re } \nu > 1$                |
| (37) | $\int_{-\infty}^\infty \frac{\sin[a(x+z)]}{x+z} U_\nu(w, x) dx = \pi U_\nu(w, z) \qquad a > 1$                                                                                                   |



## CHAPTER XX

### HYPERGEOMETRIC FUNCTIONS

Most of the higher transcendental functions of Chapters XVI to XIX are special hypergeometric functions. In the present chapter we list those special hypergeometric functions not included in the former chapters, and also generalized hypergeometric functions. The integrals listed in sections 20.4 and 20.5 are key formulas from which an enormous number of integrals involving special hypergeometric functions may be derived. For particular cases of the  $E$ -function and of the  $G$ -function see the Appendix. We do not list in this chapter integrals involving the generalized hypergeometric series  ${}_pF_q$ ; since

$$\begin{aligned} & {}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x) \\ &= \frac{\Gamma(b_1) \dots \Gamma(b_q)}{\Gamma(a_1) \dots \Gamma(a_p)} E(a_1, \dots, a_p; b_1, \dots, b_q; -x^{-1}) \\ &= \frac{\Gamma(b_1) \dots \Gamma(b_q)}{\Gamma(a_1) \dots \Gamma(a_p)} G_{q+1, p}^{p, 1} \left( -\frac{1}{x} \left| \begin{matrix} 1, b_1, \dots, b_q \\ a_1, \dots, a_p \end{matrix} \right. \right) \end{aligned}$$

such integrals may be derived from those given in sections 20.4 and 20.5. Because of the great importance of integrals involving  $E$ -functions and  $G$ -functions, we have repeated integrals given in the earlier chapters, and in some cases have given more elaborate conditions of validity.

#### Parabolic cylinder functions

For the theory of these functions see H.T.F. vol. II, Chapter VIII and the literature quoted there, also Buchholz, Herbert, 1953: *Die konfluente hypergeometrische Funktion*. Springer Verlag.

$$\begin{aligned}
 D_\nu(z) &= 2^{\frac{1}{2}\nu + \frac{1}{4}} z^{-\frac{1}{2}} W_{\frac{1}{2}\nu + \frac{1}{4}, \pm \frac{1}{4}} \left( \frac{z^2}{2} \right) \\
 &= \frac{2^{-\nu-1} z^\nu}{\pi^{\frac{1}{2}} \Gamma(-\nu)} \exp \left( -\frac{z^2}{4} \right) E \left( -\frac{\nu}{2}, \frac{1-\nu}{2} :: \frac{z^2}{2} \right) \\
 &= 2^{\frac{1}{2}\nu} \exp \left( \frac{z^2}{4} \right) G_{12}^{20} \left( \frac{z^2}{2} \middle| \frac{1}{2} - \frac{1}{2}\nu \right) \\
 &= \frac{2^{-\frac{1}{2}\nu-1}}{\pi^{\frac{1}{2}} \Gamma(-\nu)} \exp \left( -\frac{z^2}{4} \right) G_{12}^{21} \left( \frac{z^2}{2} \middle| 1 + \frac{1}{2}\nu \right)
 \end{aligned}$$

Other expressions in terms of the  $G$ -function, and expressions for products of parabolic cylinder functions may be derived by means of the formulas given in the Appendix.

### Gauss' hypergeometric series

For the theory of these series see H.T.F. vol. I, Chapter II and the literature quoted there, especially the monographs by Goursat, Kampé de Fériet, Klein, Snow (now available in a second edition), and Chapter XIV of Whittaker and Watson.

$$\begin{aligned}
 {}_2F_1(a, b; c; x) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} E(a, b; c; -x^{-1}) \\
 &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} G_{22}^{21} \left( -\frac{1}{x} \middle| 1, c \right)_{a, b}
 \end{aligned}$$

The evaluation of integrals involving Gauss' series is often facilitated by the use of the transformation formulas: for these see H.T.F. vol. I, sections 2.9 and 2.11.

### Confluent hypergeometric functions

For the theory of these functions see H.T.F. vol. I, Chapter VI and the literature quoted there, especially Chapter XVI of Whittaker and Watson, and also Tricomi, F.G., 1952: *Lezioni sulle funzioni ipergeometriche confluenti*. Torino, Geroni and Buchholz, Herbert, 1953: *Die konfluente hypergeometrische Funktion*. Springer Verlag.

$$\begin{aligned}
M_{\kappa, \mu}(z) &= z^{\mu+\frac{1}{2}} e^{-\frac{1}{2}z} {}_1F_1\left(\frac{1}{2}-\kappa+\mu; 2\mu+1; z\right) \\
&= z^{\mu+\frac{1}{2}} e^{\frac{1}{2}z} {}_1F_1\left(\frac{1}{2}+\kappa+\mu; 2\mu+1; -z\right) \\
&= \frac{\Gamma(2\mu+1)}{\Gamma(\frac{1}{2}-\kappa+\mu)} z^{\mu+\frac{1}{2}} e^{-\frac{1}{2}z} E\left(\frac{1}{2}-\kappa+\mu; 2\mu+1; -z^{-1}\right) \\
&= \frac{\Gamma(2\mu+1)}{\Gamma(\frac{1}{2}+\kappa+\mu)} z^{\mu+\frac{1}{2}} e^{\frac{1}{2}z} E\left(\frac{1}{2}+\kappa+\mu; 2\mu+1; z^{-1}\right) \\
&= \frac{\Gamma(2\mu+1)}{\Gamma(\frac{1}{2}+\kappa+\mu)} e^{\frac{1}{2}z} G_{12}^{11}\left(z \left| \begin{matrix} 1-\kappa \\ \frac{1}{2}+\mu, \frac{1}{2}-\mu \end{matrix} \right. \right) \\
W_{\kappa, \mu}(z) &= \frac{z^{\kappa} e^{-\frac{1}{2}z}}{\Gamma(\frac{1}{2}-\kappa+\mu) \Gamma(\frac{1}{2}-\kappa-\mu)} E\left(\frac{1}{2}-\kappa+\mu, \frac{1}{2}-\kappa-\mu; z\right) \\
&= \frac{e^{-\frac{1}{2}z}}{\Gamma(\frac{1}{2}-\kappa+\mu) \Gamma(\frac{1}{2}-\kappa-\mu)} G_{12}^{21}\left(z \left| \begin{matrix} 1+\kappa \\ \frac{1}{2}+\mu, \frac{1}{2}-\mu \end{matrix} \right. \right) \\
&= e^{\frac{1}{2}z} G_{12}^{20}\left(z \left| \begin{matrix} 1-\kappa \\ \frac{1}{2}+\mu, \frac{1}{2}-\mu \end{matrix} \right. \right)
\end{aligned}$$

For other expressions in terms of the  $G$ -function, and for expressions of products of confluent hypergeometric functions see the Appendix.

### MacRobert's $E$ -function

A brief introduction to this function is given in H.T.F. vol. I, sections 5.2-5.2.2, and a more detailed presentation of its theory may be found in MacRobert, T.M., 1950: *Functions of a complex variable*, Macmillan, Appendix V and Miscellaneous Examples III. See also the papers by MacRobert listed on p. 246 ff. of H.T.F. vol. I, and further papers by Professor MacRobert and his pupils in *Proc. Glasgow Math. Ass.* vol. I, 1953.

$$E(a_1, \dots, a_p; b_1, \dots, b_q; x) = G_{q+1, p}^{p, 1}\left(z \left| \begin{matrix} 1, b_1, \dots, b_q \\ a_1, \dots, a_p \end{matrix} \right. \right)$$

Numerous higher transcendental functions and some of their combinations are special instances of the  $E$ -function: a selection of these is given in the Appendix.

**Meijer's  $G$ -function**

For the theory of this function see H.T.F. vol. I, sections 5.3-5.6 and the papers by Meijer listed on p.247 of H.T.F. vol. I, and also further papers by Professor Meijer in recent volumes of *Proc. Nederl. Akad. Wetensch.* It has already been mentioned that a very large number of integrals involving special functions may be reduced to integrals involving the  $G$ -function. Examples of this process, and the necessary formulas, are given in Professor Meijer's papers: a selection of reduction formulas is also given in the Appendix.



20.1. Parabolic cylinder functions

See also under confluent hypergeometric functions, *E*-function, *G*-function.

|     |                                                                                                                                                                                                                                                      |                                                                  |
|-----|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------|
| (1) | $\int_0^\infty x^{\nu-1} \exp\left(-\frac{3x^2}{4}\right) D_\nu(x) dx$ $= 2^{-\frac{1}{2}\nu} \Gamma(\nu) \cos(\frac{1}{4}\nu\pi)$                                                                                                                   | $\operatorname{Re} \nu > 0$                                      |
| (2) | $\int_0^\infty x^\nu \exp\left(-\frac{3x^2}{4}\right) D_{\nu-1}(x) dx$ $= 2^{-\frac{1}{2}\nu-1} \Gamma(\nu) \sin(\frac{1}{4}\nu\pi)$                                                                                                                 | $\operatorname{Re} \nu > -1$                                     |
| (3) | $\int_0^a x^{2\nu-1} (a^2 - x^2)^{\lambda-1} \exp\left(\frac{x^2}{4}\right) D_{-2\lambda-2\nu}(x) dx$ $= \frac{\Gamma(\lambda) \Gamma(2\nu)}{\Gamma(2\lambda + 2\nu)} 2^{\lambda-1} a^{2\lambda+2\nu-2} \exp\left(\frac{a^2}{4}\right) D_{-2\nu}(a)$ | $\operatorname{Re} \lambda > 0, \quad \operatorname{Re} \nu > 0$ |
| (4) | $\int_a^\infty x^\nu (x-a)^{\frac{1}{2}\mu-\frac{1}{2}\nu-1} \exp[-\frac{1}{4}(x-a)^2] D_\mu(x) dx$ $= 2^{\mu-\nu-2} a^{\mu-1} \Gamma\left(\frac{\mu-\nu}{2}\right) D_\nu(a)$                                                                        | $\operatorname{Re}(\mu - \nu) > 0$                               |
| (5) | $\int_{-\infty}^\infty (x - ia)^{-1} \exp\left(-\frac{x^2}{4}\right) D_n(x) dx = -(2\pi)^{\frac{1}{2}} (-i)^n n!$ $\times \exp\left(-\frac{a^2}{4}\right) D_{-n-1}(a)$                                                                               | $n = 0, 1, 2, \dots, \quad \operatorname{Re} a > 0$              |

## Parabolic cylinder functions (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                            |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (6)  | $\int_0^\infty x^\nu (x^2 + a^2)^{-1} \exp\left(-\frac{x^2}{4}\right) D_\nu(x) dx$ $= (\tfrac{1}{2}\pi)^{\frac{1}{2}} a^{\nu-1} \Gamma(\nu+1) \exp\left(\frac{a^2}{4}\right) D_{-\nu-1}(a)$ $\text{Re } a > 0, \quad 0 < \text{Re } \nu < 1$                                                                                                               |
| (7)  | $\int_0^\infty x^{\nu-1} (x^2 + a^2)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{4}\right) D_\nu(x) dx$ $= a^{\nu-1} \Gamma(\nu) \exp\left(\frac{a^2}{4}\right) D_{-\nu}(a)$ $\text{Re } a > 0, \quad \text{Re } \nu > 0$                                                                                                                                         |
| (8)  | $\int_0^\infty x^{2\rho-1} \sin(ax) \exp\left(-\frac{x^2}{4}\right) D_{2\nu}(x) dx$ $= 2^{\nu-\rho-\frac{1}{2}} \pi^{\frac{1}{2}} a \frac{\Gamma(2\rho+1)}{\Gamma(\rho-\nu+1)} {}_2F_2\left(\rho+\frac{1}{2}, \rho+1; \frac{3}{2}, \rho-\nu+1; -\frac{a^2}{2}\right)$ $\text{Re } \rho > -\tfrac{1}{2}$                                                    |
| (9)  | $\int_0^\infty x^{2\rho-1} \sin(ax) \exp\left(\frac{x^2}{4}\right) D_{2\nu}(x) dx$ $= \frac{2^{\rho-\nu-2}}{\Gamma(-2\nu)} G_{23}^{22}\left(\frac{a^2}{2} \middle  \begin{matrix} \frac{1}{2}-\rho, 1-\rho \\ -\rho-\nu, \frac{1}{2}, 0 \end{matrix} \right)$ $a > 0, \quad \text{Re } \rho > -\tfrac{1}{2}, \quad \text{Re } (\rho + \nu) < \tfrac{1}{2}$ |
| (10) | $\int_0^\infty x^{2\rho-1} \cos(ax) \exp\left(-\frac{x^2}{4}\right) D_{2\nu}(x) dx$ $= \frac{2^{\nu-\rho} \Gamma(2\rho) \pi^{\frac{1}{2}}}{\Gamma(\rho-\nu+\frac{1}{2})} {}_2F_2\left(\rho, \rho+\tfrac{1}{2}; \tfrac{1}{2}, \rho-\nu+\tfrac{1}{2}; -\frac{a^2}{2}\right)$ $\text{Re } \nu > 0$                                                            |
| (11) | $\int_0^\infty x^{2\rho-1} \cos(ax) \exp\left(\frac{x^2}{4}\right) D_{2\nu}(x) dx$ $= \frac{2^{\rho-\nu-2}}{\Gamma(-2\nu)} G_{23}^{22}\left(\frac{a^2}{2} \middle  \begin{matrix} \frac{1}{2}-\rho, 1-\rho \\ -\rho-\nu, 0, \frac{1}{2} \end{matrix} \right)$ $a > 0, \quad \text{Re } \rho > 0, \quad \text{Re } (\rho + \nu) < \tfrac{1}{2}$             |

## Parabolic cylinder functions (cont'd)

|      |                                                                                                                                                                                                                  |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (12) | $\int_0^\infty x^{\nu-1/2} \exp[-(x+a)^2] I_{\nu-1/2}(2ax) D_\nu(2x) dx$ $= \frac{1}{2} \pi^{-1/2} \Gamma(\nu) a^{\nu-1/2} D_{-\nu}(2a) \quad \text{Re } a > 0, \quad \text{Re } \nu > 0$                        |
| (13) | $\int_0^\infty x^{\nu-3/2} \exp[-(x+a)^2] I_{\nu-3/2}(2ax) D_\nu(2x) dx$ $= \frac{1}{2} \pi^{-1/2} \Gamma(\nu) a^{\nu-3/2} D_{-\nu}(2a) \quad \text{Re } a > 0, \quad \text{Re } \nu > 1$                        |
| (14) | $\int_0^\infty [D_\nu(x)]^2 dx = (\frac{1}{2}\pi)^{1/2} \Gamma(\nu+1) + \frac{\pi^{1/2}}{2^{3/2} \Gamma(-\nu)}$ $\times \left[ \psi\left(\frac{\nu+1}{2}\right) - \psi\left(\frac{\nu}{2}+1\right) \right]$      |
| (15) | $\int_0^\infty D_\nu(x) D_\mu(x) dx = \frac{\pi 2^{1/2}(\mu+\nu+1)}{\mu-\nu} \left[ \frac{1}{\Gamma(-\frac{\nu}{2}) \Gamma(\frac{1-\mu}{2})} - \frac{1}{\Gamma(-\frac{\mu}{2}) \Gamma(\frac{1-\nu}{2})} \right]$ |
| (16) | $\int_0^\infty J_0(xy) D_\nu(x) D_{\nu-1}(-x) dx = y^{-1} [D_\nu(y) D_{\nu-1}(y)$ $+ \frac{1}{2} D_\nu(y) D_{\nu-1}(-y) + \frac{1}{2} D_\nu(-y) D_{\nu-1}(y)] \quad y > 0$                                       |
| (17) | $\int_0^\infty J_0(xy) D_\nu(x) D_{\nu-1}(x) dx$ $= \frac{1}{2y} [D_\nu(y) D_{\nu-1}(-y) - D_\nu(-y) D_{\nu-1}(y)]$                                                                                              |
| (18) | $\int_0^\infty J_0(xy) D_\nu(-x) D_{\nu-1}(x) dx = y^{-1} [\frac{1}{2} D_\nu(y) D_{\nu-1}(-y)$ $+ \frac{1}{2} D_\nu(-y) D_{\nu-1}(y) - D_\nu(y) D_{\nu-1}(y)]$                                                   |
| (19) | $\int_0^{\frac{1}{2}\pi} (\sin x)^{-\nu} (\cos x)^{-\mu-2} D_\nu(a \sin x) D_\mu(a \cos x) dx$ $= -(\frac{1}{2}\pi)^{1/2} (\mu+1)^{-1} D_{\mu+\nu+1}(a) \quad \text{Re } \nu < 1, \quad \text{Re } \mu < -1$     |

**Parabolic cylinder functions (cont'd)**

|      |                                                                                                                                                                                                                                                                                   |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (20) | $\int_0^\infty \cosh(2\mu x) \exp[-(a \sinh x)^2] D_{2\kappa}(2a \cosh x) dx$ $= 2^{\kappa-3/2} \pi^{1/2} a^{-1} W_{\kappa, \mu}(2a^2)$ $\text{Re } a^2 > 0$                                                                                                                      |
| (21) | $\int_0^\infty \cosh(2\mu x) \exp[(a \sinh x)^2] D_{2\kappa}(2a \cosh x) dx$ $= \frac{\Gamma(\mu - \kappa) \Gamma(-\mu - \kappa)}{2^{\kappa+5/2} a \Gamma(-2\kappa)} W_{\kappa+1/2, \mu}(2a^2)$ $ \arg a  < 3\pi/4, \quad \text{Re } \kappa +  \text{Re } \mu  < 0$               |
| (22) | $\int_0^\infty \cos(ax) D_{x-1/2}(\beta) D_{-x-1/2}(\beta) dx$ $= \frac{1}{2} \left( \frac{\pi}{\cos a} \right)^{1/2} \exp\left(-\frac{\beta^2}{2 \sec a}\right)$ $-\frac{1}{2}\pi < a < \frac{1}{2}\pi$ $= 0 \quad a < -\frac{1}{2}\pi \quad \text{or} \quad a > \frac{1}{2}\pi$ |

**20.2. Gauss' hypergeometric series**

See also under Legendre functions,  $E$ -function, and  $G$ -function.

|     |                                                                                                                                                                                                                                                                                                                                                                                          |
|-----|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $\int_0^1 x^{a-\gamma} (1-x)^{\gamma-\beta-1} {}_2F_1(a, \beta; \gamma; x) dx$ $= \frac{\Gamma(1 + \frac{1}{2}a) \Gamma(\gamma) \Gamma(a - \gamma + 1) \Gamma(\gamma - \frac{1}{2}a - \beta)}{\Gamma(1+a) \Gamma(1 + \frac{1}{2}a - \beta) \Gamma(\gamma - \frac{1}{2}a)}$ $\text{Re } a + 1 > \text{Re } \gamma > \text{Re } \beta, \quad \text{Re}(\gamma - \frac{1}{2}a - \beta) > 0$ |
| (2) | $\int_0^1 x^{\rho-1} (1-x)^{\beta-\gamma-n} {}_2F_1(-n, \beta; \gamma; x) dx$ $= \frac{\Gamma(\gamma) \Gamma(\rho) \Gamma(\beta - \gamma + 1) \Gamma(\gamma - \rho + n)}{\Gamma(\gamma + n) \Gamma(\gamma - \rho) \Gamma(\beta - \gamma + \rho + 1)}$ $n = 0, 1, 2, \dots, \quad \text{Re } \rho > 0, \quad \text{Re}(\beta - \gamma) > n - 1$                                           |

**Gauss' series (cont'd)**

|     |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |
|-----|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (3) | $\int_0^1 x^{\rho-1} (1-x)^{\beta-\rho-1} {}_2F_1(a, \beta; \gamma; x) dx$ $= \frac{\Gamma(\gamma) \Gamma(\rho) \Gamma(\beta-\rho) \Gamma(\gamma-a-\rho)}{\Gamma(\beta) \Gamma(\gamma-a) \Gamma(\gamma-\rho)}$ $\operatorname{Re} \rho > 0, \quad \operatorname{Re}(\beta-\rho) > 0, \quad \operatorname{Re}(\gamma-a-\rho) > 0$                                                                                                                                                  |
| (4) | $\int_0^1 x^{\gamma-1} (1-x)^{\rho-1} {}_2F_1(a, \beta; \gamma; x) dx$ $= \frac{\Gamma(\gamma) \Gamma(\rho) \Gamma(\gamma+\rho-a-\beta)}{\Gamma(\gamma+\rho-a) \Gamma(\gamma+\rho-\beta)}$ $\operatorname{Re} \gamma > 0, \quad \operatorname{Re} \rho > 0, \quad \operatorname{Re}(\gamma+\rho-a-\beta) > 0$                                                                                                                                                                     |
| (5) | $\int_0^1 x^{\rho-1} (1-x)^{\sigma-1} {}_2F_1(a, \beta; \gamma; x) dx$ $= \frac{\Gamma(\rho) \Gamma(\sigma)}{\Gamma(\rho+\sigma)} {}_3F_2(a, \beta, \rho; \gamma, \rho+\sigma; 1)$ $\operatorname{Re} \rho > 0, \quad \operatorname{Re} \sigma > 0, \quad \operatorname{Re}(\gamma+\sigma-a-\beta) > 0$                                                                                                                                                                           |
| (6) | $\int_0^1 x^{\gamma-1} (1-x)^{\rho-1} (1-zx)^{-\sigma} {}_2F_1(a, \beta; \gamma; x) dx$ $= \frac{\Gamma(\gamma) \Gamma(\rho) \Gamma(\gamma+\rho-a-\beta)}{\Gamma(\gamma+\rho-a) \Gamma(\gamma+\rho-\beta)} (1-z)^{\sigma}$ $\times {}_3F_2\left(\rho, a, \gamma+\rho-a-\beta; \gamma+\rho-a, \gamma+\rho-\beta; \frac{z}{z-1}\right)$ $\operatorname{Re} \gamma > 0, \quad \operatorname{Re} \rho > 0, \quad \operatorname{Re}(\gamma+\rho-a-\beta) > 0, \quad  \arg(1-z)  < \pi$ |
| (7) | $\int_0^1 x^{\rho-1} (1-x)^{\sigma-1} {}_2F_1(a, \beta; \gamma; xz) dx$ $= \frac{\Gamma(\rho) \Gamma(\sigma)}{\Gamma(\rho+\sigma)} {}_3F_2(a, \beta, \rho; \gamma, \sigma; z)$ $\operatorname{Re} \rho > 0, \quad \operatorname{Re} \sigma > 0, \quad  \arg(1-z)  < \pi$                                                                                                                                                                                                          |

## Gauss' series (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                  |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (8)  | $\int_0^1 x^{\gamma-1} (1-x)^{\rho-1} e^{-xz} {}_2F_1(a, \beta; \gamma; x) dx$ $= \frac{\Gamma(\gamma) \Gamma(\rho) \Gamma(\gamma + \rho - a - \beta)}{\Gamma(\gamma + \rho - a) \Gamma(\gamma + \rho - \beta)} e^{-z}$ $\times {}_2F_2(\rho, \gamma + \rho - a - \beta; \gamma + \rho - a, \gamma + \rho - \beta; z)$ $\text{Re } \gamma > 0, \quad \text{Re } \rho > 0, \quad \text{Re}(\gamma + \rho - a - \beta) > 0$                                        |
| (9)  | $\int_0^\infty x^{\gamma-1} (1+x)^{-\sigma} {}_2F_1(a, \beta; \gamma; -x) dx$ $= \frac{\Gamma(\gamma) \Gamma(a - \gamma + \sigma) \Gamma(\beta - \gamma + \sigma)}{\Gamma(\sigma) \Gamma(a + \beta - \gamma + \sigma)}$ $\text{Re } \gamma > 0, \quad \text{Re}(a - \gamma + \sigma) > 0, \quad \text{Re}(\beta - \gamma + \sigma) > 0$                                                                                                                          |
| (10) | $\int_0^\infty x^{\gamma-1} (x+z)^{-\sigma} {}_2F_1(a, \beta; \gamma; -x) dx$ $= \frac{\Gamma(a - \gamma + \sigma) \Gamma(\beta - \gamma + \sigma) \Gamma(\gamma)}{\Gamma(a + \beta - \gamma + \sigma) \Gamma(\sigma)}$ $\times {}_2F_1(a - \gamma + \sigma, \beta - \gamma + \sigma; a + \beta - \gamma + \sigma; 1 - z)$ $\text{Re } \gamma > 0, \quad \text{Re}(a - \gamma + \sigma) > 0, \quad \text{Re}(\beta - \gamma + \sigma) > 0, \quad  \arg z  < \pi$ |
| (11) | $\int_0^1 x^{\gamma-1} (1-x)^{\delta-\gamma-1} {}_2F_1(a, \beta; \gamma; xz) dx$ $\times {}_2F_1[\delta - a, \delta - \beta; \delta - \gamma; (1-x)\zeta] dx$ $= \frac{\Gamma(\gamma) \Gamma(\delta - \gamma)}{\Gamma(\delta)} (1 - \zeta)^{2a-\delta} {}_2F_1(a, \beta; \delta; z + \zeta - z\zeta)$ $0 < \text{Re } \gamma < \text{Re } \delta, \quad  \arg(1 - z)  < \pi, \quad  \arg(1 - \zeta)  < \pi$                                                      |
| (12) | $\int_0^1 x^{\gamma-1} (1-x)^{\epsilon-1} (1-xz)^{-\delta} {}_2F_1(a, \beta; \gamma; xz) dx$ $\times {}_2F_1\left[\delta, \beta - \gamma; \epsilon; \frac{(1-x)z}{1-xz}\right] dx = \frac{\Gamma(\gamma) \Gamma(\epsilon)}{\Gamma(\gamma + \epsilon)} {}_2F_1(a + \delta, \beta; \gamma + \epsilon; z)$ $\text{Re } \gamma > 0, \quad \text{Re } \epsilon > 0, \quad  \arg(z - 1)  < \pi$                                                                        |

**Gauss' series (cont'd)**

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (13) | $\int_0^\infty e^{-\lambda x} {}_2F_1(a, \beta; \frac{1}{2}; -x^2) dx = \lambda^{\alpha+\beta-1} S_{1-\alpha-\beta, \alpha-\beta}(\lambda)$ $\text{Re } \lambda > 0$                                                                                                                                                                                                                                                                                                                                                                        |
| (14) | $\int_0^\infty x e^{-\lambda x} {}_2F_1(a, \beta; 3/2; -x^2) dx = \lambda^{\alpha+\beta-2} S_{1-\alpha-\beta, \alpha-\beta}(\lambda)$ $\text{Re } \lambda > 0$                                                                                                                                                                                                                                                                                                                                                                              |
| (15) | $\int_0^\infty x^{\gamma-1} (x+y)^{-\alpha} (x+z)^{-\beta} e^{-x} {}_2F_1\left[a, \beta; \gamma; \frac{x(x+y+z)}{(x+y)(x+z)}\right] dx$ $= \Gamma(\gamma) (xy)^{-\frac{1}{2}-\mu} e^{\frac{1}{2}y+\frac{1}{2}z} W_{\kappa, \mu}(\gamma) W_{\lambda, \mu}(z)$ $2\kappa = 1 - \alpha + \beta - \gamma, \quad 2\lambda = 1 + \alpha - \beta - \gamma, \quad 2\mu = \alpha + \beta - \gamma$ $\text{Re } \gamma > 0, \quad  \arg \gamma  < \pi, \quad  \arg z  < \pi$                                                                           |
| (16) | $\int_0^\infty x^{\alpha+\beta-2\nu-1} (x+1)^{-\nu} e^{xz} K_\nu[(x+1)z] {}_2F_1(a, \beta; \alpha+\beta-2\nu; -x) dx$ $= \pi^{-\frac{1}{2}} \cos(\nu\pi) \Gamma(\frac{1}{2}-\alpha+\nu) \Gamma(\frac{1}{2}-\beta+\nu) \Gamma(\gamma)$ $\times (2z)^{-\frac{1}{2}-\frac{1}{2}\gamma} W_{\frac{1}{2}\gamma, \frac{1}{2}(\beta-\alpha)}(2z)$ $\text{Re}(\alpha + \beta - 2\nu) > 0, \quad \text{Re}(\frac{1}{2} - \alpha + \nu) > 0, \quad \text{Re}(\frac{1}{2} - \beta + \nu) > 0$ $ \arg z  < 3\pi/2, \quad \gamma = \alpha + \beta - 2\nu$ |

**20.3. Confluent hypergeometric functions**

See also under *E*-function, *G*-function.

For special confluent hypergeometric functions see also sections 16.5, 16.6, 17.3, 20.1, and Chapter XIX.

|     |                                                                                                                                                                                                                                             |
|-----|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $\int_0^a x^{\beta-1} (a-x)^{\gamma-1} {}_1F_1(a; \beta; x) dx = \frac{\Gamma(\beta) \Gamma(\gamma)}{\Gamma(\beta + \gamma)} a^{\beta+\gamma-1}$ $\times {}_1F_1(a; \beta + \gamma; a)$ $\text{Re } \beta > 0, \quad \text{Re } \gamma > 0$ |
|-----|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|

## Confluent hypergeometric functions (cont'd)

|     |                                                                                                                                                                                                                                                                                                                           |
|-----|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (2) | $\int_0^a x^{\beta-1} (a-x)^{\delta-1} {}_1F_1(a; \beta; x) {}_1F_1(\gamma; \delta; a-x) dx$ $= \frac{\Gamma(\beta) \Gamma(\delta)}{\Gamma(\beta + \delta)} a^{\beta+\delta-1} {}_1F_1(a + \gamma; \beta + \delta; a) \quad \text{Re } \beta > 0, \quad \text{Re } \delta > 0$                                            |
| (3) | $\int_0^1 x^{\beta-1} (1-x)^{\sigma-\beta-1} {}_1F_1(a; \beta; \lambda x) {}_1F_1[\sigma-a; \sigma-\beta; \mu(1-x)] dx$ $= \frac{\Gamma(\beta) \Gamma(\sigma-\beta)}{\Gamma(\sigma)} e^{\lambda} {}_1F_1(a; \sigma; \mu-\lambda) \quad 0 < \text{Re } \beta < \text{Re } \sigma$                                          |
| (4) | $\int_0^\infty \cos(ax) {}_1F_1(\nu+1; 1; ix) {}_1F_1(\nu+1; 1; -ix) dx$ $= -a^{-1} \sin(\nu\pi) P_\nu(2a^{-2} - 1) \quad 0 < a < 1$ $= 0 \quad 1 < a < \infty$ $-1 < \text{Re } \nu < 0$                                                                                                                                 |
| (5) | $\int_0^a x^{-1} (a-x)^{\kappa-1} e^{\frac{1}{2}(a-x)} M_{\kappa, \mu}(x) dx = \frac{\Gamma(\kappa) \Gamma(2\mu+1)}{\Gamma(\kappa + \mu + \frac{1}{2})} \pi^{\frac{1}{2}} a^{\kappa-\frac{1}{2}} I_\mu(\frac{1}{2}a)$ $\text{Re } \kappa > 0, \quad \text{Re } \mu > -\frac{1}{2}$                                        |
| (6) | $\int_0^a x^{\kappa-1} (a-x)^{\lambda-1} e^{\frac{1}{2}(a-x)} M_{\kappa+\lambda, \mu}(x) dx$ $= \frac{\Gamma(\lambda) \Gamma(\kappa + \mu + \frac{1}{2})}{\Gamma(\kappa + \lambda + \mu + \frac{1}{2})} a^{\kappa+\lambda-1} M_{\kappa, \mu}(a)$ $\text{Re } (\kappa + \mu) > -\frac{1}{2}, \quad \text{Re } \lambda > 0$ |
| (7) | $\int_0^a x^{\mu-\frac{1}{2}} (a-x)^{\nu-\frac{1}{2}} M_{\kappa, \mu}(x) M_{\lambda, \nu}(a-x) dx$ $= \frac{\Gamma(2\mu+1) \Gamma(2\nu+1)}{\Gamma(2\mu+2\nu+2)} a^{\mu+\nu} M_{\kappa+\lambda, \mu+\nu+\frac{1}{2}}(a)$ $\text{Re } \mu > -\frac{1}{2}, \quad \text{Re } \nu > -\frac{1}{2}$                              |



## Confluent hypergeometric functions (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                         |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (8)  | $\int_0^\infty x^{\rho-1} [x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}}]^{2\sigma} e^{-\frac{1}{2}x} M_{\kappa,\mu}(x) dx$ $= -\frac{\sigma \Gamma(2\mu+1) a^\sigma}{\pi^{\frac{1}{2}} \Gamma(\frac{1}{2} + \kappa + \mu)} G_{34}^{23} \left( a \left  \begin{matrix} \frac{1}{2}, 1, 1 - \kappa + \rho \\ \frac{1}{2} + \mu + \rho, -\sigma, \sigma, \frac{1}{2} - \mu + \rho \end{matrix} \right. \right)$ $ \arg a  < \pi, \quad \operatorname{Re}(\mu + \rho) > -\frac{1}{2}, \quad \operatorname{Re}(\kappa - \rho - \sigma) > 0$         |
| (9)  | $\int_0^\infty x^{\rho-1} (a+x)^{-\frac{1}{2}} [x^{\frac{1}{2}} + (a+x)^{\frac{1}{2}}]^{2\sigma} e^{-\frac{1}{2}x} M_{\kappa,\mu}(x) dx$ $= \frac{\Gamma(2\mu+1) a^\sigma}{\pi^{\frac{1}{2}} \Gamma(\frac{1}{2} + \kappa + \mu)} G_{34}^{23} \left( a \left  \begin{matrix} 0, \frac{1}{2}, \frac{1}{2} - \kappa + \rho \\ -\sigma, \rho + \mu, \rho - \mu, \sigma \end{matrix} \right. \right)$ $ \arg a  < \pi, \quad \operatorname{Re}(\rho + \mu) > -\frac{1}{2}, \quad \operatorname{Re}(\kappa - \rho - \sigma) > -\frac{1}{2}$   |
| (10) | $\int_0^\infty x^{-\mu-\frac{1}{2}} e^{-\frac{1}{2}x} \sin(2ax^{\frac{1}{2}}) M_{\kappa,\mu}(x) dx$ $= \pi^{\frac{1}{2}} a^{\kappa+\mu-1} \frac{\Gamma(3-2\mu)}{\Gamma(\frac{1}{2} + \kappa + \mu)} \exp\left(-\frac{a^2}{2}\right) \mathbb{W}_{\rho,\sigma}(a^2)$ $a > 0, \quad \operatorname{Re}(\kappa + \mu) > 0, \quad 2\rho = \kappa - 3\mu + 1, \quad 2\sigma = \kappa + \mu - 1$                                                                                                                                                |
| (11) | $\int_0^\infty x^{-\frac{1}{2}} (a+x)^\mu e^{-\frac{1}{2}x} P_\nu^{-2\mu}(1+2x/a) M_{\kappa,\mu}(x) dx$ $= -\frac{\sin(\nu\pi)}{\pi \Gamma(\kappa)} \Gamma(2\mu+1) \Gamma(\kappa - \mu + \nu + \frac{1}{2}) \Gamma(\kappa - \mu - \nu - \frac{1}{2})$ $\times e^{\frac{1}{2}\alpha} \mathbb{W}_{\rho,\sigma}(a) \quad \rho = \frac{1}{2} - \kappa + \mu, \quad \sigma = \frac{1}{2} + \nu$ $ \arg a  < \pi, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re}(\kappa - \mu) >  \operatorname{Re} \nu + \frac{1}{2} $  |
| (12) | $\int_0^\infty x^{-\frac{1}{2}} (a+x)^{-\mu} e^{-\frac{1}{2}x} P_\nu^{-2\mu}(1+2x/a) M_{\kappa,\mu}(x) dx$ $= \frac{\Gamma(2\mu+1) \Gamma(\kappa + \mu + \nu + \frac{1}{2}) \Gamma(\kappa + \mu - \nu - \frac{1}{2})}{\Gamma(\kappa + \mu + \frac{1}{2}) \Gamma(2\mu + \nu + 1) \Gamma(2\mu - \nu)}$ $\times e^{\frac{1}{2}\alpha} \mathbb{W}_{\frac{1}{2}-\kappa-\mu, \frac{1}{2}+\nu}(a)$ $ \arg a  < \pi, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re}(\kappa + \mu) >  \operatorname{Re} \nu + \frac{1}{2} $ |

## Confluent hypergeometric functions (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (13) | $\int_0^\infty x^{-\frac{1}{2}} e^{-\frac{1}{2}x} P_\nu^{-2\mu} [(1+x/a)^{\frac{1}{2}}] M_{\kappa,\mu}(x) dx$ $= \frac{\Gamma(2\mu+1) \Gamma(\kappa+\frac{1}{2}\nu) \Gamma(\kappa-\frac{1}{2}\nu-\frac{1}{2})}{2^{2\mu} a^{\frac{1}{4}} \Gamma(\kappa+\mu+\frac{1}{2}) \Gamma(\mu+\frac{1}{2}\nu+\frac{1}{2}) \Gamma(\mu-\frac{1}{2}\nu)}$ $\times e^{\frac{1}{2}a} W_{\frac{1}{4}-\kappa, \frac{1}{4}+\frac{1}{2}\nu}(a)$ $ \arg a  < \pi, \quad \operatorname{Re} \kappa > \frac{1}{2} \operatorname{Re} \nu - \frac{1}{2}, \quad \operatorname{Re} \kappa > -\frac{1}{2} \operatorname{Re} \nu$              |
| (14) | $\int_0^\infty x^{\mu-\frac{1}{4}\kappa-\frac{1}{2}\nu-\frac{1}{2}} (a+x)^{\frac{1}{2}\nu} e^{-\frac{1}{2}x} Q_{\mu-\kappa+3/2}^\nu (1+2x/a) M_{\kappa,\mu}(x) dx$ $= \frac{e^{\nu\pi i} \Gamma(1+2\mu-\nu) \Gamma(1+2\mu) \Gamma(5/2-\kappa+\mu+\nu)}{2 \Gamma(\frac{1}{2}+\kappa+\mu)}$ $\times a^{\frac{1}{4}(\kappa+2\mu-2\nu+5)} e^{\frac{1}{2}a} W_{\rho,\sigma}(a)$ $ \arg a  < \pi, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} (2\mu-\nu) > -1$ $2\rho = \frac{1}{2} - \kappa - \mu + 2\nu, \quad 2\sigma = \kappa - 3\mu - 3/2$                                               |
| (15) | $\int_0^\infty x^{\frac{1}{2}(\kappa+\mu+\nu)-1} (a+x)^{-\frac{1}{2}} e^{-\frac{1}{2}x} Q_{\kappa-\mu-\nu-1}^{1-\kappa+\mu-\nu} [(1+x/a)^{\frac{1}{2}}] M_{\kappa,\mu}(x) dx$ $= e^{(1-\kappa+\mu-\nu)\pi i} 2^{\mu-\kappa-\nu} a^{\frac{1}{2}(\kappa+\mu-1)} \frac{\Gamma(\frac{1}{2}-\nu) \Gamma(1+2\mu) \Gamma(\kappa+\mu+\nu)}{\Gamma(\kappa+\mu+\frac{1}{2})}$ $\times e^{\frac{1}{2}a} W_{\rho,\sigma}(a)$ $\operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} (\kappa+\mu+\nu) > 0$ $ \arg a  < \pi, \quad \rho = \frac{1}{2} - \kappa - \frac{1}{2}\nu, \quad \sigma = \mu + \frac{1}{2}\nu$ |
| (16) | $\int_0^\infty x^{\nu-\frac{1}{2}} e^{-\frac{1}{2}x} Q_{2\kappa-2\nu-3}^{2\mu-2\nu} [(1+x/a)^{\frac{1}{2}}] M_{\kappa,\mu}(x) dx$ $= e^{2(\mu-\nu)\pi i} 2^{2\mu-2\nu-1} a^{\frac{1}{2}(\kappa+\mu-1)} e^{\frac{1}{2}a}$ $\times \frac{\Gamma(2\mu+1) \Gamma(\nu+1) \Gamma(\kappa+\mu-2\nu-\frac{1}{2})}{\Gamma(\kappa+\mu+\frac{1}{2})} W_{\rho,\sigma}(a)$ $ \arg a  < \pi, \quad \operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re} \nu > -1, \quad \operatorname{Re} (\kappa+\mu-2\nu) > \frac{1}{2}$ $2\rho = 1 - \kappa + \mu - 2\nu, \quad 2\sigma = \kappa - \mu - 2\nu - 2$                 |

## Confluent hypergeometric functions (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (17) | $\int_0^\infty x^{\kappa-3/2} \exp[-\frac{1}{2}(a+1)x] K_\nu(\frac{1}{2}ax) M_{\kappa,\nu}(x) dx$ $= \frac{\pi^{1/2} \Gamma(\kappa) \Gamma(\kappa+2\nu)}{a^{\kappa+\nu} \Gamma(\kappa+\nu+\frac{1}{2})} {}_2F_1(\kappa, \kappa+2\nu; 2\nu+1; -a^{-1})$ $\text{Re } a > 0, \quad \text{Re } \kappa > 0, \quad \text{Re } (\kappa+2\nu) > 0$                                                                                                                                                                                                                        |
| (18) | $\int_0^\infty x^{-1/2} J_\nu(ax^{1/2}) K_{\frac{1}{2}\nu-\mu}(\frac{1}{2}x) M_{\kappa,\mu}(x) dx$ $= \frac{\Gamma(2\mu+1)}{a \Gamma(\kappa+\frac{1}{2}\nu+1)} W_{\frac{1}{2}(\kappa-\mu), \frac{1}{2}\kappa-\frac{1}{4}\nu} \left( \frac{a^2}{2} \right) M_{\frac{1}{2}(\kappa+\mu), \frac{1}{2}\kappa+\frac{1}{4}\nu} \left( \frac{a^2}{2} \right)$ $a > 0, \quad \text{Re } \kappa > -\frac{1}{4}, \quad \text{Re } \mu > -\frac{1}{2}, \quad \text{Re } \nu > -1$                                                                                             |
| (19) | $\int_0^\infty x^{\rho-1} e^{-\frac{1}{2}x} M_{\gamma+\rho, \beta+\rho+\frac{1}{2}}(x) {}_2F_1(a, \beta; \gamma; -\lambda/x) dx$ $= \frac{\Gamma(a+\beta+2\rho) \Gamma(2\beta+2\rho) \Gamma(\gamma)}{\Gamma(\beta) \Gamma(\beta+\gamma+2\rho)} \lambda^{\frac{1}{2}\beta+\rho-\frac{1}{2}} e^{\frac{1}{2}\lambda} W_{\kappa,\mu}(\lambda)$ $ \arg \lambda  < \pi, \quad \text{Re } (\beta+\rho) > 0, \quad \text{Re } (a+\beta+2\rho) > 0, \quad \text{Re } \gamma > 0$ $\kappa = \frac{1}{2} - a - \frac{1}{2}\beta - \rho, \quad \mu = \frac{1}{2}\beta + \rho$ |
| (20) | $\int_0^\infty x^{2\lambda-1} (a+x)^{-\mu-\frac{1}{2}} e^{-\frac{1}{2}x} M_{\kappa,\mu}(a+x) dx$ $= \frac{a^{\lambda-\mu-\frac{1}{2}} \Gamma(2\lambda) \Gamma(2\mu+1) \Gamma(\kappa+\mu-2\lambda+\frac{1}{2})}{\Gamma(\kappa+\mu+\frac{1}{2}) \Gamma(1-2\lambda+2\mu)} M_{\kappa-\lambda, \mu-\lambda}(a)$ $\text{Re } \lambda > 0, \quad \text{Re } (\kappa+\mu-2\lambda) > -\frac{1}{2}$                                                                                                                                                                        |
| (21) | $\int_0^a x^{-\kappa-\lambda-1} (a-x)^{\lambda-1} e^{\frac{1}{2}x} W_{\kappa,\mu}(x) dx$ $= \frac{\Gamma(\lambda) \Gamma(\frac{1}{2}-\kappa-\lambda+\mu) \Gamma(\frac{1}{2}-\kappa-\lambda-\mu)}{a^{\kappa+1} \Gamma(\frac{1}{2}-\kappa+\mu) \Gamma(\frac{1}{2}-\kappa-\mu)} W_{\kappa+\lambda, \mu}(a)$ $\text{Re } \lambda > 0, \quad \text{Re } (\kappa+\lambda) < \frac{1}{2} -  \text{Re } \mu $                                                                                                                                                             |

## Confluent hypergeometric functions (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (22) | $\int_0^\infty W_{\kappa,\mu}(x) \frac{dx}{x} = \frac{\pi^{3/2} 2^\kappa \sec(\mu\pi)}{\Gamma(3/4 - 1/2\kappa + 1/2\mu) \Gamma(3/4 - 1/2\kappa - 1/2\mu)}$ $-1/2 < \operatorname{Re} \mu < 1/2$                                                                                                                                                                                                                                                                |
| (23) | $\int_0^\infty x^{\kappa+2\mu-1} e^{-3x/2} W_{\kappa,\mu}(x) dx = \frac{\Gamma(\kappa + \mu + 1/2) \Gamma[1/4(2\kappa + 6\mu + 5)]}{(\kappa + 3\mu + 1/2) \Gamma[1/4(2\mu - 2\kappa + 3)]}$ $\operatorname{Re}(\kappa + \mu) > -1/2, \quad \operatorname{Re}(\kappa + 3\mu) > -1/2$                                                                                                                                                                            |
| (24) | $\int_0^\infty x^{\rho-1} [x^{1/2} + (a+x)^{1/2}]^{2\sigma} e^{-1/2x} W_{\kappa,\mu}(x) dx$ $= -\pi^{-1/2} \sigma a^\sigma G_{34}^{32} \left( a \left  \begin{matrix} 1/2, 1, 1-\kappa+\rho \\ 1/2+\mu+\rho, 1/2-\mu+\rho, -\sigma, \sigma \end{matrix} \right. \right)$ $ \arg a  < \pi, \quad \operatorname{Re} \rho >  \operatorname{Re} \mu  - 1/2$                                                                                                        |
| (25) | $\int_0^\infty x^{\rho-1} [x^{1/2} + (a+x)^{1/2}]^{2\sigma} e^{1/2x} W_{\kappa,\mu}(x) dx$ $= -\frac{\sigma \pi^{-1/2} a^\sigma}{\Gamma(1/2-\kappa+\mu) \Gamma(1/2-\kappa-\mu)} G_{34}^{33} \left( a \left  \begin{matrix} 1/2, 1, 1+\kappa+\rho \\ 1/2+\mu+\rho, 1/2-\mu+\rho, -\sigma, \sigma \end{matrix} \right. \right)$ $ \arg a  < \pi, \quad \operatorname{Re} \rho >  \operatorname{Re} \mu  - 1/2, \quad \operatorname{Re}(\kappa+\rho+\sigma) < 0$  |
| (26) | $\int_0^\infty x^{\rho-1} (a+x)^{-1/2} [x^{1/2} + (a+x)^{1/2}]^{2\sigma} e^{1/2x} W_{\kappa,\mu}(x) dx$ $= \frac{\pi^{-1/2} a^\sigma}{\Gamma(1/2-\kappa+\mu) \Gamma(1/2-\kappa-\mu)} G_{34}^{33} \left( a \left  \begin{matrix} 0, 1/2, 1/2+\kappa+\rho \\ -\sigma, \rho+\mu, \rho-\mu, \sigma \end{matrix} \right. \right)$ $ \arg a  < \pi, \quad \operatorname{Re} \rho >  \operatorname{Re} \mu  - 1/2, \quad \operatorname{Re}(\kappa+\rho+\sigma) < 1/2$ |
| (27) | $\int_0^\infty x^{\rho-1} (a+x)^{-1/2} [x^{1/2} + (a+x)^{1/2}]^{2\sigma} e^{-1/2x} W_{\kappa,\mu}(x) dx$ $= \pi^{-1/2} a^\sigma G_{34}^{32} \left( a \left  \begin{matrix} 0, 1/2, 1/2-\kappa+\rho \\ -\sigma, \rho+\mu, \rho-\mu, \sigma \end{matrix} \right. \right) \quad  \arg a  < \pi, \operatorname{Re} \rho >  \operatorname{Re} \mu  - 1/2$                                                                                                           |

## Confluent hypergeometric functions (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (28) | $\int_0^\infty x^{\rho-1} \sin(cx^{\frac{1}{2}}) e^{-\frac{1}{2}x} W_{\kappa,\mu}(x) dx = \frac{\Gamma(1+\mu+\rho) \Gamma(1-\mu+\rho)}{\Gamma(3/2-\kappa+\rho)}$ $\times {}_2F_2\left(1+\mu+\rho, 1-\mu+\rho; \frac{3}{2}, \frac{3}{2}-\kappa+\rho; -\frac{c^2}{4}\right)$ $\operatorname{Re} \rho >  \operatorname{Re} \mu  - 1$                                                                                                                                                |
| (29) | $\int_0^\infty x^{\rho-1} \sin(cx^{\frac{1}{2}}) e^{\frac{1}{2}x} W_{\kappa,\mu}(x) dx$ $= \frac{\pi^{\frac{1}{2}}}{\Gamma(\frac{1}{2}-\kappa+\mu) \Gamma(\frac{1}{2}-\kappa-\mu)} G_{23}^{22}\left(\frac{c^2}{4} \middle  \frac{1}{2}+\mu-\rho, \frac{1}{2}-\mu-\rho \atop \frac{1}{2}, -\kappa-\rho, 0\right)$ $c > 0, \quad \operatorname{Re} \rho >  \operatorname{Re} \mu  - 1, \quad \operatorname{Re}(\kappa+\rho) < \frac{1}{2}$                                         |
| (30) | $\int_0^\infty x^{\rho-1} \cos(cx^{\frac{1}{2}}) e^{-\frac{1}{2}x} W_{\kappa,\mu}(x) dx = \frac{\Gamma(\frac{1}{2}+\mu+\rho) \Gamma(\frac{1}{2}-\mu+\rho)}{\Gamma(1-\kappa+\rho)}$ $\times {}_2F_2\left(\frac{1}{2}+\mu+\rho, \frac{1}{2}-\mu+\rho; \frac{1}{2}, 1-\kappa+\rho; -\frac{c^2}{4}\right)$ $\operatorname{Re} \rho >  \operatorname{Re} \mu  - \frac{1}{2}$                                                                                                          |
| (31) | $\int_0^\infty x^{\rho-1} \cos(cx^{\frac{1}{2}}) e^{\frac{1}{2}x} W_{\kappa,\mu}(x) dx = \frac{\pi^{\frac{1}{2}}}{\Gamma(\frac{1}{2}-\kappa+\mu) \Gamma(\frac{1}{2}-\kappa-\mu)}$ $\times G_{23}^{22}\left(\frac{c^2}{4} \middle  \frac{1}{2}+\mu-\rho, \frac{1}{2}-\mu-\rho \atop 0, -\kappa-\rho, \frac{1}{2}\right)$ $c > 0, \quad \operatorname{Re} \rho >  \operatorname{Re} \mu  - \frac{1}{2}, \quad \operatorname{Re}(\kappa+\rho) < \frac{1}{2}$                        |
| (32) | $\int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu} (a+x)^{\frac{1}{2}\mu} e^{-\frac{1}{2}x} P_{\kappa+\nu-3/2}^\mu (1+2x a^{-1}) W_{\kappa,\nu}(x) dx$ $= \frac{\Gamma(1-\mu-2\nu)}{\Gamma(3/2-\kappa-\mu-\nu)} a^{-\frac{1}{2}+\frac{1}{2}\kappa-\frac{1}{2}\nu} e^{\frac{1}{2}a} W_{\rho,\sigma}(a)$ $2\rho = \frac{1}{2} + 2\mu + \nu - \kappa, \quad 2\sigma = \kappa + 3\nu - 3/2$ $ \arg a  < \pi, \quad \operatorname{Re} \mu < 1, \quad \operatorname{Re}(\mu+2\nu) < 1$ |

## Confluent hypergeometric functions (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                             |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (33) | $\int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu} (a+x)^{-\frac{1}{2}\mu} e^{-\frac{1}{2}x} P_{\kappa+\mu+\nu-3/2}^\mu (1+2x a^{-1}) W_{\kappa,\nu}(x) dx$ $= \frac{\Gamma(1-\mu-2\nu)}{\Gamma(3/2-\kappa-\mu-\nu)} a^{-\frac{1}{2}+\frac{1}{2}\kappa-\frac{1}{2}\nu} e^{\frac{1}{2}a} W_{\rho,\sigma}(a)$ $2\rho = \frac{1}{2}-\kappa+\nu, \quad 2\sigma = \kappa+2\mu+3\nu-3/2$ $ \arg a  < \pi, \quad \operatorname{Re} \mu < 1, \quad \operatorname{Re}(\mu+2\nu) < 1$ |
| (34) | $\int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu} e^{-\frac{1}{2}x} P_{2\kappa+\mu+2\nu-3}^\mu [(1+x/a)^{\frac{1}{2}}] W_{\kappa,\nu}(x) dx$ $= \frac{2^\mu \Gamma(1-\mu-2\nu)}{\Gamma(3/2-\kappa-\mu-\nu)} a^{-\frac{1}{2}+\frac{1}{2}\kappa-\frac{1}{2}\nu} e^{\frac{1}{2}a} W_{\rho,\sigma}(a)$ $2\rho = 1-\kappa+\mu+\nu, \quad 2\sigma = \kappa+\mu+3\nu-2$ $ \arg a  < \pi, \quad \operatorname{Re} \mu < 1, \quad \operatorname{Re}(\mu+2\nu) < 1$                  |
| (35) | $\int_0^\infty x^{-\frac{1}{2}-\frac{1}{2}\mu-\nu} (a+x)^{-\frac{1}{2}} e^{-\frac{1}{2}x} P_{2\kappa+\mu+2\nu-2}^\mu [(1+x/a)^{\frac{1}{2}}] W_{\kappa,\nu}(x) dx$ $= \frac{2^\mu \Gamma(1-\mu-2\nu)}{\Gamma(3/2-\kappa-\mu-\nu)} a^{-\frac{1}{2}+\frac{1}{2}\kappa-\frac{1}{2}\nu} e^{\frac{1}{2}a} W_{\rho,\sigma}(a)$ $2\rho = \mu+\nu-\kappa, \quad 2\sigma = \kappa+\mu+3\nu-1$ $ \arg a  < \pi, \quad \operatorname{Re} \mu > 0, \quad \operatorname{Re} \nu > 0$     |
| (36) | $\int_0^\infty x^{-\kappa-3/2} \exp[-\frac{1}{2}(a-1)x] K_\mu(\frac{1}{2}ax) W_{\kappa,\mu}(x) dx$ $= \frac{\pi \Gamma(-\kappa) \Gamma(2\mu-\kappa) \Gamma(-2\mu-\kappa)}{\Gamma(\frac{1}{2}-\kappa) \Gamma(\frac{1}{2}+\mu-\kappa) \Gamma(\frac{1}{2}-\mu-\kappa)} 2^{2\kappa+1} a^{\kappa-\nu}$ $\times {}_2F_1(-\kappa, 2\mu-\kappa; -2\kappa; 1-a^{-1})$ $\operatorname{Re} a > 0, \quad \operatorname{Re} \kappa < 2\operatorname{Re} \mu < -\operatorname{Re} \kappa$ |

**Confluent hypergeometric functions (cont'd)**

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (37) | $\int_0^\infty x^{\rho-1} e^{-\frac{1}{2}x} J_{\lambda+\nu}(ax^{\frac{1}{2}}) J_{\lambda-\nu}(ax^{\frac{1}{2}}) W_{\kappa,\mu}(x) dx$ $= \frac{(\frac{1}{2}a)^{2\lambda} \Gamma(\frac{1}{2}+\lambda+\mu+\rho) \Gamma(\frac{1}{2}+\lambda-\mu+\rho)}{\Gamma(1+\lambda+\nu) \Gamma(1+\lambda-\nu) \Gamma(1+\lambda-\kappa+\rho)}$ $\times {}_4F_4(1+\lambda, \frac{1}{2}+\lambda, \frac{1}{2}+\lambda+\mu+\rho, \frac{1}{2}+\lambda-\mu+\rho;$ $1+\lambda+\nu, 1+\lambda-\nu, 1+2\lambda, 1+\lambda-\kappa+\rho; -a^2)$ $ \operatorname{Re} \mu  < \operatorname{Re}(\lambda + \rho) + \frac{1}{2}$ |
| (38) | $\int_0^\infty x^{\rho-1} e^{-\frac{1}{2}x} I_{\lambda+\nu}(ax^{\frac{1}{2}}) K_{\lambda-\nu}(ax^{\frac{1}{2}}) W_{\kappa,\mu}(x) dx$ $= \frac{1}{2\pi^{\frac{1}{2}}} G_{45}^{24} \left( a^2 \left  \begin{matrix} 0, \frac{1}{2}, \frac{1}{2}+\mu-\rho, \frac{1}{2}-\mu-\rho \\ \lambda, \nu, -\lambda, -\nu, \kappa-\rho \end{matrix} \right. \right)$ $ \operatorname{Re} \mu  < \operatorname{Re}(\lambda + \rho) + \frac{1}{2}, \quad  \operatorname{Re} \mu  < \operatorname{Re}(\nu + \rho) + \frac{1}{2}$                                                                                 |
| (39) | $\int_0^\infty x^{-1} M_{\kappa,\mu}(x) W_{\lambda,\mu}(x) dx = \frac{\Gamma(2\mu+1)}{(\kappa-\lambda) \Gamma(\frac{1}{2}+\mu-\lambda)}$ $\operatorname{Re} \mu > -\frac{1}{2}, \quad \operatorname{Re}(\kappa-\lambda) > 0$                                                                                                                                                                                                                                                                                                                                                                      |
| (40) | $\int_0^\infty x^{-1} W_{\kappa,\mu}(x) W_{\lambda,\mu}(x) dx = \frac{1}{(\kappa-\lambda) \sin(2\mu\pi)} \left[ \frac{1}{\Gamma(\frac{1}{2}-\kappa+\mu) \Gamma(\frac{1}{2}-\lambda-\mu)} \right.$ $\left. - \frac{1}{\Gamma(\frac{1}{2}-\kappa-\mu) \Gamma(\frac{1}{2}-\lambda+\mu)} \right]$ $-\frac{1}{2} < \operatorname{Re} \mu < \frac{1}{2}$                                                                                                                                                                                                                                                |
| (41) | $\int_0^\infty x^{\rho-1} W_{\kappa,\mu}(x) W_{-\kappa,\mu}(x) dx = \frac{\Gamma(\rho+1) \Gamma(\frac{\rho+1}{2}+\mu) \Gamma(\frac{\rho+1}{2}-\mu)}{2\Gamma(1+\frac{1}{2}\rho+\kappa) \Gamma(1+\frac{1}{2}\rho-\kappa)}$ $\operatorname{Re} \rho > 2 \operatorname{Re} \mu  - 1$                                                                                                                                                                                                                                                                                                                  |

## Confluent hypergeometric functions (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (42) | $\int_0^\infty x^{\rho-1} W_{\kappa,\mu}(x) W_{\lambda,\nu}(x) dx$ $= \frac{\Gamma(1+\mu+\nu+\rho) \Gamma(1-\mu+\nu+\rho) \Gamma(-2\nu)}{\Gamma(\frac{1}{2}-\lambda-\nu) \Gamma(3/2-\kappa+\nu+\rho)}$ $\times {}_3F_2(1+\mu+\nu+\rho, 1-\mu+\nu+\rho, 1/2-\lambda+\nu; 1+2\nu, 3/2-\kappa+\nu+\rho; 1)$ $+ \frac{\Gamma(1+\mu-\nu+\rho) \Gamma(1-\mu-\nu+\rho) \Gamma(2\nu)}{\Gamma(1/2-\lambda+\nu) \Gamma(3/2-\kappa-\nu+\rho)}$ $\times {}_3F_2(1+\mu-\nu+\rho, 1-\mu-\nu+\rho, 1/2-\lambda-\nu; 1-2\nu, 3/2-\kappa-\nu+\rho; 1)$ $ \operatorname{Re} \mu  +  \operatorname{Re} \nu  < \operatorname{Re} \rho + 1$ |
| (43) | $\int_0^\infty x^{\rho-1} \exp[-\frac{1}{2}(a+\beta)x] M_{\kappa,\mu}(ax) W_{\lambda,\nu}(\beta x) dx$ $= \frac{\Gamma(1+\mu+\nu+\rho) \Gamma(1+\mu-\nu+\rho)}{\Gamma(3/2-\lambda+\mu+\rho)} a^{\mu+\frac{1}{2}} \beta^{-\mu-\rho-\frac{1}{2}}$ $\times {}_3F_2(1/2+\kappa+\mu, 1+\mu+\nu+\rho, 1+\mu-\nu+\rho; 2\mu+1, 3/2-\lambda+\mu+\rho; -a/\beta)$ $\operatorname{Re} a > 0, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re}(\rho+\mu) >  \operatorname{Re} \nu  - 1$                                                                                                                                 |
| (44) | $\int_0^\infty x^{\rho-1} \exp[\frac{1}{2}(a+\beta)x] W_{\kappa,\mu}(ax) W_{\lambda,\nu}(\beta x) dx$ $= \beta^{-\rho} [\Gamma(\frac{1}{2}-\kappa+\mu) \Gamma(\frac{1}{2}-\kappa-\mu) \Gamma(\frac{1}{2}-\lambda+\nu) \Gamma(\frac{1}{2}-\lambda-\nu)]^{-1}$ $\times G_{33}^{33} \left( \frac{\beta}{a} \left  \begin{matrix} \frac{1}{2}+\mu, \frac{1}{2}-\mu, 1+\lambda+\rho \\ \frac{1}{2}+\nu+\rho, \frac{1}{2}-\nu+\rho, -\kappa \end{matrix} \right. \right)$ $ \operatorname{Re} \mu  +  \operatorname{Re} \nu  < \operatorname{Re} \rho + 1, \quad \operatorname{Re}(\kappa+\lambda+\rho) < 0$                 |



**Confluent hypergeometric functions (cont'd)**

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                            |
|------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (45) | $\int_0^\infty x^{\rho-1} \exp[-\tfrac{1}{2}(\alpha - \beta)x] W_{\kappa, \mu}(ax) W_{\lambda, \nu}(\beta x) dx$ $= \frac{\beta^{-\rho}}{\Gamma(\tfrac{1}{2} - \lambda + \nu) \Gamma(\tfrac{1}{2} - \lambda - \nu)} G_{33}^{23} \left( \frac{\beta}{a} \left  \begin{matrix} \tfrac{1}{2} + \mu, \tfrac{1}{2} - \mu, 1 + \lambda + \rho \\ \tfrac{1}{2} + \nu + \rho, \tfrac{1}{2} - \nu + \rho, \kappa \end{matrix} \right. \right)$ $\operatorname{Re} \alpha > 0, \quad  \operatorname{Re} \mu  +  \operatorname{Re} \nu  < \operatorname{Re} \rho + 1$ |
| (46) | $\int_0^\infty x^{\rho-1} \exp[-\tfrac{1}{2}(\alpha + \beta)x] W_{\kappa, \mu}(ax) W_{\lambda, \nu}(\beta x) dx$ $= \beta^{-\rho} G_{33}^{22} \left( \frac{\beta}{a} \left  \begin{matrix} \tfrac{1}{2} + \mu, \tfrac{1}{2} - \nu, 1 - \lambda + \rho \\ \tfrac{1}{2} + \nu + \rho, \tfrac{1}{2} - \nu + \rho, \kappa \end{matrix} \right. \right)$ $\operatorname{Re}(\alpha + \beta) > 0, \quad  \operatorname{Re} \mu  +  \operatorname{Re} \nu  < \operatorname{Re} \rho + 1$                                                                          |
| (47) | $\int_0^\infty x^{2\lambda-1} (a+x)^{-\mu-\frac{1}{2}} e^{-\frac{1}{2}x} W'_{\kappa, \mu}(a+x) dx$ $= \Gamma(2\lambda) a^{\lambda-\mu-\frac{1}{2}} W'_{\kappa-\lambda, \mu-\lambda}(a) \quad  \arg a  < \pi, \quad \operatorname{Re} \lambda > 0$                                                                                                                                                                                                                                                                                                          |
| (48) | $\int_0^\infty x^{\lambda-1} (a+x)^{\kappa-\lambda-1} e^{-\frac{1}{2}x} W_{\kappa, \mu}(a+x) dx$ $= \Gamma(\lambda) a^{\kappa-1} W_{\kappa-\lambda, \mu}(a) \quad  \arg a  < \pi, \quad \operatorname{Re} \lambda > 0$                                                                                                                                                                                                                                                                                                                                     |
| (49) | $\int_0^\infty x^{\rho-1} (a+x)^{-\sigma} e^{-\frac{1}{2}x} W_{\kappa, \mu}(a+x) dx$ $= \Gamma(\rho) a^{\rho} e^{\frac{1}{2}a} G_{23}^{30} \left( a \left  \begin{matrix} 0, 1 - \kappa - \sigma \\ -\rho, \tfrac{1}{2} + \mu - \sigma, \tfrac{1}{2} - \mu - \sigma \end{matrix} \right. \right)$ $ \arg a  < \pi, \quad \operatorname{Re} \rho > 0$                                                                                                                                                                                                       |
| (50) | $\int_0^\infty x^{2\lambda-1} (a+x)^{-\mu-\frac{1}{2}} e^{\frac{1}{2}x} W_{\kappa, \mu}(a+x) dx$ $= \frac{\Gamma(2\lambda) \Gamma(\tfrac{1}{2} - \kappa + \mu - 2\lambda)}{\Gamma(\tfrac{1}{2} - \kappa + \mu)} a^{\lambda-\mu-\frac{1}{2}} W_{\kappa+\lambda, \mu-\lambda}(a)$ $ \arg a  < \pi, \quad 0 < 2\operatorname{Re} \lambda < \tfrac{1}{2} - \operatorname{Re}(\kappa + \mu)$                                                                                                                                                                    |

## Confluent hypergeometric functions (cont'd)

|                                                                                                                                                                                                                                                                                              |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (51)                                                                                                                                                                                                                                                                                         | $\int_0^\infty x^{\rho-1} (a+x)^{-\sigma} e^{\frac{1}{2}x} W_{\kappa,\mu}(a+x) dx = \frac{\Gamma(\rho) a^\rho e^{-\frac{1}{2}a}}{\Gamma(\frac{1}{2}-\kappa+\mu) \Gamma(\frac{1}{2}-\kappa-\mu)}$ $\times G_{23}^{31} \left( a \left  \begin{matrix} \kappa-\sigma+1, 0 \\ -\rho, \frac{1}{2}+\mu-\sigma, \frac{1}{2}-\mu-\sigma \end{matrix} \right. \right)$ $ \arg a  < \pi, \quad 0 < \operatorname{Re} \rho < \operatorname{Re}(\sigma-\kappa)$                                                                                                                                                                           |
| (52)                                                                                                                                                                                                                                                                                         | $\int_0^\infty x^{\tau-1} (a+x)^{-\lambda} {}_2F_1(\rho, \sigma; \tau; -x/a) e^{\pm \frac{1}{2}x} W_{\kappa,\mu}(a+x) dx$ <p>Express <math>W_{\kappa,\mu}</math> in terms of <math>G</math> and then see under Meijer's <math>G</math>-function.</p>                                                                                                                                                                                                                                                                                                                                                                          |
| <p>Integrals involving products of <math>e^{\pm \frac{1}{2}x} W_{\kappa,\mu}(a+x)</math> with Legendre functions. Express <math>W_{\kappa,\mu}</math> in terms of <math>G</math>, and the Legendre function as a hypergeometric series, then see under Meijer's <math>G</math>-function.</p> |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               |
| (53)                                                                                                                                                                                                                                                                                         | $\int_0^\infty x^{-\frac{1}{2}} \exp \left[ -\frac{1}{2} \left( \frac{x}{a} + \frac{\beta}{x} \right) \right] W_{\kappa,\mu} \left( \frac{x}{a} \right) W_{\kappa,\mu} \left( \frac{\beta}{x} \right) dx$ $= \pi^{\frac{1}{2}} 2^{\frac{1}{2}-2\kappa} (a\beta)^{\frac{1}{4}} \exp \left[ -\left( \frac{\beta}{a} \right)^{\frac{1}{2}} \right] W_{2\kappa-\frac{1}{2}, 2\mu} \left[ 2 \left( \frac{\beta}{a} \right)^{\frac{1}{2}} \right]$ $\operatorname{Re} a > 0, \quad \operatorname{Re} \beta > 0$                                                                                                                     |
| (54)                                                                                                                                                                                                                                                                                         | $\int_0^\infty x^{\rho-1} \exp \left[ -\frac{1}{2} \left( \frac{x}{a} + \frac{\beta}{x} \right) \right] W_{\kappa,\mu} \left( \frac{x}{a} \right) W_{\lambda,\nu} \left( \frac{\beta}{x} \right) dx$ $= \beta^\rho G_{24}^{40} \left( \frac{\beta}{a} \left  \begin{matrix} 1-\kappa, 1-\lambda-\rho \\ \frac{1}{2}+\mu, \frac{1}{2}-\mu, \frac{1}{2}+\nu-\rho, \frac{1}{2}-\nu-\rho \end{matrix} \right. \right)$ $\operatorname{Re} a > 0, \quad \operatorname{Re} \beta > 0$                                                                                                                                               |
| (55)                                                                                                                                                                                                                                                                                         | $\int_0^\infty x^{\rho-1} \exp \left[ \frac{1}{2} \left( \frac{x}{a} - \frac{\beta}{x} \right) \right] W_{\kappa,\mu} \left( \frac{x}{a} \right) W_{\lambda,\nu} \left( \frac{\beta}{x} \right) dx$ $= \frac{\beta^\rho}{\Gamma(\frac{1}{2}-\kappa+\mu) \Gamma(\frac{1}{2}-\kappa-\mu)} G_{24}^{41} \left( \frac{\beta}{a} \left  \begin{matrix} 1+\kappa, 1-\lambda-\rho \\ \frac{1}{2}+\mu, \frac{1}{2}-\mu, \frac{1}{2}+\nu-\rho, \frac{1}{2}-\nu-\rho \end{matrix} \right. \right)$ $ \arg a  < 3\pi/2, \quad \operatorname{Re} \beta > 0, \quad \operatorname{Re}(\kappa+\rho) < - \operatorname{Re} \nu  - \frac{1}{2}$ |

## Confluent hypergeometric functions (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (56) | $\int_0^\infty x^{-1/2} \exp \left[ \frac{1}{2} \left( \frac{x}{a} + \frac{\beta}{x} \right) \right] W_{\kappa, \mu} \left( \frac{x}{a} \right) W_{\kappa, \mu} \left( \frac{\beta}{x} \right) dx$ $= \frac{\pi^{1/2} \Gamma(-\kappa - \mu) \Gamma(-\kappa + \mu) (a\beta)^{1/2}}{2^{1/2+2\kappa} \Gamma(1/2 - \kappa + \mu) \Gamma(1/2 - \kappa - \mu)}$ $\times \exp \left[ \left( \frac{\beta}{a} \right)^{1/2} \right] W_{2\kappa+1/2, 2\mu} \left[ 2 \left( \frac{\beta}{a} \right)^{1/2} \right]$ $ \arg a  < 3\pi/2, \quad  \arg \beta  < 3\pi/2, \quad \operatorname{Re} \kappa < - \operatorname{Re} \mu $                                                                                                               |
| (57) | $\int_0^\infty x^{\rho-1} \exp \left[ \frac{1}{2} \left( \frac{x}{a} + \frac{\beta}{x} \right) \right] W_{\kappa, \mu} \left( \frac{x}{a} \right) W_{\lambda, \nu} \left( \frac{\beta}{x} \right) dx$ $= \frac{\beta^\rho}{\Gamma(1/2 - \kappa + \mu) \Gamma(1/2 - \kappa - \mu) \Gamma(1/2 - \lambda + \nu) \Gamma(1/2 - \lambda - \nu)}$ $\times G_{24}^{42} \left( \frac{\beta}{a} \left  \begin{matrix} 1 + \kappa, 1 + \lambda - \rho \\ 1/2 + \mu, 1/2 - \mu, 1/2 + \nu - \rho, 1/2 - \nu - \rho \end{matrix} \right. \right)$ $ \arg a  < 3\pi/2, \quad  \arg \beta  < 3\pi/2$ $\operatorname{Re}(\lambda - \rho) < 1/2 -  \operatorname{Re} \mu , \quad \operatorname{Re}(\kappa + \rho) < 1/2 -  \operatorname{Re} \nu $ |
| (58) | $\int_{-\infty}^\infty e^{-2\rho x i} \Gamma(1/2 + \nu + ix) \Gamma(1/2 + \nu - ix) M_{ix, \nu}(2a) dx$ $= 2^{1/2-\nu} \pi \beta^{\nu+1/2} (\cosh \rho)^{-2\nu-1} \exp(-a \tanh \rho) \Gamma(2\nu + 1)$ $ \operatorname{Im} \rho  < 1/2\pi, \quad \operatorname{Re} \nu > -1/2$                                                                                                                                                                                                                                                                                                                                                                                                                                                   |
| (59) | $\int_{-\infty}^{\infty} \Gamma(1/2 + \nu + \mu + x) \Gamma(1/2 + \nu + \mu - x) \Gamma(1/2 + \nu - \mu + x) \Gamma(1/2 + \nu - \mu - x)$ $\times M_{\mu+ix, \nu}(a) M_{\mu-ix, \nu}(\beta) dx$ $= \frac{2\pi(a\beta)^{\nu+1/2} [\Gamma(2\nu+1)]^2 \Gamma(2\nu+2\mu+1) \Gamma(2\nu-2\mu+1)}{(a+\beta)^{2\nu+1} \Gamma(4\nu+2)}$ $\times M_{2\mu, 2\nu+1/2}(a+\beta) \quad \operatorname{Re} \nu >  \operatorname{Re} \mu  - 1/2$                                                                                                                                                                                                                                                                                                  |

**Confluent hypergeometric functions (cont'd)**

|                                                                                                                                                                      |                                                                                                                                                                                                                                                                                                                                                                                                         |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (60)                                                                                                                                                                 | $\int_{-\infty}^{\infty} e^{-2\rho x i} \Gamma(\tfrac{1}{2} + \nu + ix) \Gamma(\tfrac{1}{2} + \nu - ix) M_{ix, \nu}(a) M_{ix, \nu}(\beta) dx$ $= \frac{2\pi(a\beta)^{\frac{1}{2}}}{\cosh \rho} \exp[-(a + \beta) \tanh \rho] J_{2\nu} \left( \frac{2a^{\frac{1}{2}} \beta^{\frac{1}{2}}}{\cosh \rho} \right)$ $ \operatorname{Im} \rho  < \tfrac{1}{2}\pi, \quad \operatorname{Re} \nu > -\tfrac{1}{2}$ |
| (61)                                                                                                                                                                 | $\int_{-\infty}^{\infty} \operatorname{sech}(\pi x) W_{ix, 0}(a) W_{-ix, 0}(\beta) dx = 2 \frac{(a\beta)^{\frac{1}{2}}}{a + \beta} \exp[-\tfrac{1}{2}(a + \beta)]$                                                                                                                                                                                                                                      |
| (62)                                                                                                                                                                 | $\int_{-\infty}^{\infty} \Gamma(ix) \Gamma(2\kappa + ix) W_{\kappa + ix, \kappa - \frac{1}{2}}(a) W_{-\kappa - ix, \kappa - \frac{1}{2}}(\beta) dx$ $= 2\pi^{\frac{1}{2}} \Gamma(2\kappa) (a\beta)^{\kappa} (a + \beta)^{\frac{1}{2} - 2\kappa} K_{2\kappa - \frac{1}{2}} \left( \frac{a + \beta}{2} \right)$                                                                                           |
| <p>For numerous integrals with respect to parameters see Buchholz, Herbert, 1953: <i>Die konfluente hypergeometrische Funktion</i>. Springer Verlag, Chapter VI.</p> |                                                                                                                                                                                                                                                                                                                                                                                                         |

**20.4. MacRobert's  $E$ -function**

See also under  $G$ -function.

|     |                                                                                                                                                                                                                                                                                                                                                                                                |
|-----|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (1) | $\int_0^1 x^{\beta-1} (1-x)^{\gamma-\beta-1} E(a_1, \dots, a_p; \rho_1, \dots, \rho_q; xz) dx$ <p>See MacRobert, T.M., 1953: <i>Proc. Glasgow Math. Assoc.</i> 1, 118.</p>                                                                                                                                                                                                                     |
| (2) | $\int_0^1 x^{\beta-1} (1-x)^{\gamma-\beta-1} E(a_1, \dots, a_p; \rho_1, \dots, \rho_q; x^{-m} z) dx$ $= \Gamma(\gamma - \beta) m^{\beta-\gamma} E(a_1, \dots, a_{p+m}; \rho_1, \dots, \rho_{q+m}; z)$ $a_{p+k} = \frac{\beta + k - 1}{m}, \quad \rho_{p+k} = \frac{\gamma + k - 1}{m}, \quad k = 1, \dots, m$ $\operatorname{Re} \gamma > \operatorname{Re} \beta > 0, \quad m = 1, 2, \dots,$ |

MacRobert's  $E$ -function (cont'd)

|     |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      |
|-----|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (3) | $\int_0^\infty x^{\rho-1} (1+x)^{-\sigma} E[a_1, \dots, a_p; \rho_1, \dots, \rho_q; (1+x)z] dx$ $= \Gamma(\rho) E(a_1, \dots, a_p, \sigma - \rho; \rho_1, \dots, \rho_q, \sigma; z) \quad \text{Re } \sigma > \text{Re } \rho > 0$                                                                                                                                                                                                                                                                                                                                                                                   |
| (4) | $\int_0^\infty x^{\beta-1} e^{-x} E(a_1, \dots, a_p; \rho_1, \dots, \rho_q; xz) dx$ $= \pi \csc(\beta\pi) [E(a_1, \dots, a_p; 1 - \beta, \rho_1, \dots, \rho_q; e^{\pm i\pi} z)$ $- z^{-\beta} E(a_1 + \beta, \dots, a_p + \beta; 1 + \beta, \rho_1 + \beta, \dots, \rho_q + \beta; e^{\pm i\pi} z)]$ $p \geq q + 1, \quad \text{Re}(a_r + \beta) > 0, \quad r = 1, \dots, p, \quad  \arg z  < \pi$ <p>For <math>p \leq q</math> the result holds if the integral is convergent.</p>                                                                                                                                 |
| (5) | $\int_0^\infty x^{\beta-1} e^{-x} E(a_1, \dots, a_p; \rho_1, \dots, \rho_q; x^{-m} z) dx$ $= (2\pi)^{\frac{1}{2}-\frac{1}{2}m} m^{\beta-\frac{1}{2}} E(a_1, \dots, a_{p+m}; \rho_1, \dots, \rho_q; m^{-m} z)$ $\text{Re } \beta > 0, \quad m = 1, 2, \dots, \quad a_{p+k} = (\beta + k - 1)/m, \quad k = 1, \dots, m$                                                                                                                                                                                                                                                                                                |
| (6) | $\int_{-1}^1 (1-x)^{-\alpha} (1-x^2)^{-\frac{1}{2}\mu} P_{n-\mu}^\mu(x) E[a_1, \dots, a_p; \rho_1, \dots, \rho_q; (1-x)z] dx$ <p>See MacRobert, T.M., 1953: <i>Proc. Glasgow Math. Assoc.</i> 1, 111-114.</p>                                                                                                                                                                                                                                                                                                                                                                                                        |
| (7) | $\int_0^\infty x^{\beta-1} J_\nu(x) E(a_1, \dots, a_p; \rho_1, \dots, \rho_q; x^{-2m} z) dx$ $= (2\pi)^{-m} (2m)^{\beta-1} \{ \exp[\frac{1}{2}\pi(\beta-\nu-1)i]$ $\times E[a_1, \dots, a_{p+2m}; \rho_1, \dots, \rho_q; (2m)^{-2m} z e^{-m\pi i}]$ $+ \exp[-\frac{1}{2}\pi(\beta-\nu-1)i] E[a_1, \dots, a_{p+2m}; \rho_1, \dots, \rho_q; (2m)^{-2m} z e^{m\pi i}] \}$ $\text{Re}(\beta + \nu) > 0, \quad \text{Re}(2a_r m - \beta) > -3/2, \quad r = 1, \dots, p$ $\alpha_{p+k} = \frac{\beta + \nu + 2k - 2}{2m}, \quad \alpha_{p+m+k} = \frac{\beta - \nu + 2k - 2}{2m}$ $m = 1, 2, \dots, \quad k = 1, \dots, m$ |

MacRobert's *E*-function (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                            |
|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (8)  | $\int_0^\infty x^{\beta-1} K_\nu(x) E(a_1, \dots, a_p; \rho_1, \dots, \rho_q; x^{-2m} z) dx$ $= (2\pi)^{1-m} 2^{\beta-2} m^{\beta-1} E[a_1, \dots, a_{p+2m}; \rho_1, \dots, \rho_q; (2m)^{-2m} z]$ $\alpha_{p+k} = \frac{\beta + \nu + 2k - 2}{2m}, \quad \alpha_{p+m+k} = \frac{\beta - \nu + 2k - 2}{2m}$ $\operatorname{Re} \beta >  \operatorname{Re} \nu , \quad m = 1, 2, \dots, \quad k = 1, \dots, m$                                                                                                              |
| (9)  | $\int_0^\infty x^{\beta-1} e^x K_\nu(x) E(a_1, \dots, a_p; \rho_1, \dots, \rho_q; z/x) dx$ <p>See Ragab, F.M., 1953: <i>Proc. Glasgow Math. Assoc.</i> 1, 192-195.</p>                                                                                                                                                                                                                                                                                                                                                     |
| (10) | $\int_0^\infty x^{\beta-1} e^{-\frac{1}{2}x} W_{\kappa, \mu}(x) E(a_1, \dots, a_p; \rho_1, \dots, \rho_q; x^{-m} z) dx$ $= (2\pi)^{\frac{1}{2}-\frac{1}{2}m} m^{\beta+\kappa-\frac{1}{2}} E(a_1, \dots, a_{p+2m}; \rho_1, \dots, \rho_{q+m}; m^{-m} z)$ $\operatorname{Re} \beta >  \operatorname{Re} \mu  - \frac{1}{2}, \quad m = 1, 2, \dots$ $\alpha_{p+k} = (\beta + k + \mu - \frac{1}{2})/m, \quad \alpha_{p+m+k} = (\beta - \mu + k - \frac{1}{2})/m$ $\rho_{q+k} = (\beta - \kappa + k)/m, \quad k = 1, \dots, m$ |
| (11) | $\int_0^\infty x^{\lambda-1} E(a_1, \dots, a_p; \rho_1, \dots, \rho_q; xy) E(\beta_1, \dots, \beta_r; \sigma_1, \dots, \sigma_s; xz) dx$ $\int_0^\infty x^{\lambda-1} E(a_1, \dots, a_p; \rho_1, \dots, \rho_q; xy) E(\beta_1, \dots, \beta_r; \sigma_1, \dots, \sigma_s; z/x) dx$ <p>See Ragab, F.M., 1953: <i>Proc. Glasgow Math. Assoc.</i> 1, 192-195.</p>                                                                                                                                                             |

20.5. Meijer's  $G$ -function

$$(1) \quad \int_0^1 x^{\rho-1} (1-x)^{\sigma-1} G_{pq}^{mn} \left( ax \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx \\ = \Gamma(\sigma) G_{p+1, q+1}^{m, n+1} \left( a \left| \begin{matrix} 1-\rho, a_1, \dots, a_p \\ b_1, \dots, b_q, 1-\rho-\sigma \end{matrix} \right. \right)$$

First set of conditions of validity:

$$p + q < 2(m + n), \quad |\arg a| < (m + n - \frac{1}{2}p - \frac{1}{2}q)\pi$$

$$\operatorname{Re}(\rho + b_j) > 0, \quad j = 1, \dots, m, \quad \operatorname{Re} \sigma > 0$$

Second set of conditions of validity:

$$p + q \leq 2(m + n), \quad |\arg a| \leq (m + n - \frac{1}{2}p - \frac{1}{2}q)\pi$$

$$\operatorname{Re}(\rho + b_j) > 0, \quad j = 1, \dots, m, \quad \operatorname{Re} \sigma > 0$$

$$\operatorname{Re} \left[ \sum_{j=1}^p a_j - \sum_{j=1}^q b_j + (p - q)(\rho - \frac{1}{2}) \right] > -\frac{1}{2}$$

Third set of conditions of validity:

$$p < q \quad (\text{or } p \leq q \quad \text{and} \quad |a| < 1)$$

$$\operatorname{Re}(\rho + b_j) > 0, \quad j = 1, \dots, m, \quad \operatorname{Re} \sigma > 0$$

$$(2) \quad \int_1^\infty x^{-\rho} (x-1)^{\sigma-1} G_{pq}^{mn} \left( ax \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx \\ = \Gamma(\sigma) G_{p+1, q+1}^{m+1, n} \left( a \left| \begin{matrix} a_1, \dots, a_p, \rho \\ \rho - \sigma, b_1, \dots, b_q \end{matrix} \right. \right)$$

First set of conditions of validity:

$$p + q < 2(m + n), \quad |\arg a| < (m + n - \frac{1}{2}p - \frac{1}{2}q)\pi$$

$$\operatorname{Re}(\rho - \sigma - a_j) > -1, \quad j = 1, \dots, n, \quad \operatorname{Re} \sigma > 0$$

Second set of conditions of validity:

$$p + q \leq 2(m + n), \quad |\arg a| \leq (m + n - \frac{1}{2}p - \frac{1}{2}q)\pi$$

$$\operatorname{Re}(\rho - \sigma - a_j) > -1, \quad j = 1, \dots, n, \quad \operatorname{Re} \sigma > 0$$

$$\operatorname{Re} \left[ \sum_{j=1}^p a_j - \sum_{j=1}^q b_j + (q - p)(\rho - \sigma + \frac{1}{2}) \right] > -\frac{1}{2}$$

Meijer's  $G$ -function (cont'd)

|     |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |
|-----|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (2) | $\int_1^\infty x^{-\rho} (x-1)^{\sigma-1} G_{pq}^{mn} \left( ax \left  \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx$ $= \Gamma(\sigma) G_{p+1, q+1}^{m+1, n} \left( \alpha \left  \begin{matrix} a_1, \dots, a_p, \rho \\ \rho - \sigma, b_1, \dots, b_q \end{matrix} \right. \right)$ <p>Third set of conditions of validity:</p> $q < p \quad (\text{or } q \leq p \quad \text{and} \quad  a  > 1)$ $\operatorname{Re}(\rho - \sigma - a_j) > -1, \quad j = 1, \dots, n, \quad \operatorname{Re} \sigma > 0$                                                                                                                      |
| (3) | $\int_0^\infty x^{\rho-1} G_{pq}^{mn} \left( ax \left  \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx$ $= \frac{\prod_{j=1}^m \Gamma(b_j + \rho) \prod_{j=1}^n \Gamma(1 - a_j - \rho)}{\prod_{j=m+1}^q \Gamma(1 - b_j - \rho) \prod_{j=n+1}^p \Gamma(a_j + \rho)} \alpha^\rho$ $p + q < 2(m + n), \quad  \arg \alpha  < (m + n - \tfrac{1}{2}p - \tfrac{1}{2}q)\pi$ $- \min_{1 \leq j \leq m} \operatorname{Re} b_j < \operatorname{Re} \rho < 1 - \max_{1 \leq j \leq n} \operatorname{Re} a_j$                                                                                                                                      |
| (4) | $\int_0^\infty x^{\rho-1} (x + \beta)^{-\sigma} G_{pq}^{mn} \left( ax \left  \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx$ $= \frac{\beta^{\rho-\sigma}}{\Gamma(\sigma)} G_{p+1, q+1}^{m+1, n+1} \left( \alpha \beta \left  \begin{matrix} 1 - \rho, a_1, \dots, a_p \\ \sigma - \rho, b_1, \dots, b_q \end{matrix} \right. \right)$ <p>First set of conditions of validity:</p> $p + q < 2(m + n), \quad  \arg \alpha  < (m + n - \tfrac{1}{2}p - \tfrac{1}{2}q)\pi, \quad  \arg \beta  < \pi$ $\operatorname{Re}(\rho + b_j) > 0, \quad j = 1, \dots, m, \quad \operatorname{Re}(\rho - \sigma + a_j) < 1, \quad j = 1, \dots, n$ |



Meijer's  $G$ -function (cont'd)

$$(4) \quad \int_0^\infty x^{\rho-1} (x+\beta)^{-\sigma} G_{pq}^{mn} \left( \alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx \\ = \frac{\beta^{\rho-\sigma}}{\Gamma(\sigma)} G_{p+1, q+1}^{m+1, n+1} \left( \alpha \beta \left| \begin{matrix} 1-\rho, a_1, \dots, a_p \\ \sigma-\rho, b_1, \dots, b_q \end{matrix} \right. \right)$$

Second set of conditions of validity:

$$p \leq q, \quad p+q \leq 2(m+n), \quad |\arg \alpha| \leq (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi, \quad |\arg \beta| < \pi$$

$$\operatorname{Re}(\rho + b_j) > 0, \quad j = 1, \dots, m, \quad \operatorname{Re}(\rho - \sigma + a_j) < 1, \quad j = 1, \dots, n$$

$$\operatorname{Re} \left[ \sum_{j=1}^p a_j - \sum_{j=1}^q b_j - (q-p)(\rho - \sigma - \frac{1}{2}) \right] > 1$$

Third set of conditions of validity:

$$p \geq q, \quad p+q \leq 2(m+n), \quad |\arg \alpha| \leq (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi, \quad |\arg \beta| < \pi$$

$$\operatorname{Re}(\rho + b_j) > 0, \quad j = 1, \dots, m, \quad \operatorname{Re}(\rho - \sigma + a_j) < 1, \quad j = 1, \dots, n$$

$$\operatorname{Re} \left[ \sum_{j=1}^p a_j - \sum_{j=1}^q b_j + (p-q)(\rho - \frac{1}{2}) \right] > 1$$

$$(5) \quad \int_0^\infty x^{-\rho} e^{-\beta x} G_{pq}^{mn} \left( \alpha x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx \\ = \beta^{\rho-1} G_{p+1, q}^{m, n+1} \left( \frac{\alpha}{\beta} \left| \begin{matrix} \rho, a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right)$$

$$p+q < 2(m+n), \quad |\arg \alpha| < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi, \quad |\arg \beta| < \frac{1}{2}\pi$$

$$\operatorname{Re}(b_j - \rho) > -1, \quad j = 1, \dots, m$$

$$(6) \quad \int_0^\infty e^{-\beta x} G_{pq}^{mn} \left( \alpha x^2 \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx \\ = \pi^{-\frac{1}{2}} \beta^{-1} G_{p+2, q}^{m, n+2} \left( \frac{4\alpha}{\beta^2} \left| \begin{matrix} 0, \frac{1}{2}, a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right)$$

$$p+q < 2(m+n), \quad |\arg \alpha| < (m+n-\frac{1}{2}p-\frac{1}{2}q)\pi, \quad |\arg \beta| < \frac{1}{2}\pi$$

$$\operatorname{Re} b_j > -\frac{1}{2}, \quad j = 1, \dots, m$$

Meijer's  $G$ -functions (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |
|------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (7)  | $\int_0^\infty \sin(cx) G_{pq}^{mn} \left( ax^2 \left  \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx$ $= \pi^{\frac{1}{2}} c^{-1} G_{p+2,q}^{m,n+1} \left( \frac{4a}{c^2} \left  \begin{matrix} 0, a_1, \dots, a_p, \frac{1}{2} \\ b_1, \dots, b_q \end{matrix} \right. \right)$ <p style="text-align: center;"><math>p+q &lt; 2(m+n), \quad  \arg a  &lt; (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi, \quad c &gt; 0</math><br/> <math>\operatorname{Re} b_j &gt; -1, \quad j = 1, \dots, m, \quad \operatorname{Re} a_j &lt; \frac{1}{2}, \quad j = 1, \dots, n</math></p>                                                                                                                                      |
| (8)  | $\int_0^\infty \cos(cx) G_{pq}^{mn} \left( ax^2 \left  \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx$ $= \pi^{\frac{1}{2}} c^{-1} G_{p+2,q}^{m,n+1} \left( \frac{4a}{c^2} \left  \begin{matrix} \frac{1}{2}, a_1, \dots, a_p, 0 \\ b_1, \dots, b_q \end{matrix} \right. \right)$ <p style="text-align: center;"><math>p+q &lt; 2(m+n), \quad  \arg a  &lt; (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi, \quad c &gt; 0</math><br/> <math>\operatorname{Re} b_j &gt; -\frac{1}{2}, \quad j = 1, \dots, m, \quad \operatorname{Re} a_j &lt; \frac{1}{2}, \quad j = 1, \dots, n</math></p>                                                                                                                            |
| (9)  | $\int_0^\infty x^{-\rho} J_\nu(2x^{\frac{1}{2}}) G_{pq}^{mn} \left( ax \left  \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx$ $= G_{p+2,q}^{m,n+1} \left( a \left  \begin{matrix} \rho - \frac{1}{2}\nu, a_1, \dots, a_p, \rho + \frac{1}{2}\nu \\ b_1, \dots, b_q \end{matrix} \right. \right).$ <p style="text-align: center;"><math>p+q &lt; 2(m+n), \quad  \arg a  &lt; (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi</math><br/> <math>-\frac{3}{4} + \max_{1 \leq j \leq n} \operatorname{Re} a_j &lt; \operatorname{Re} \rho &lt; 1 + \frac{1}{2} \operatorname{Re} \nu + \min_{1 \leq j \leq m} \operatorname{Re} b_j</math></p>                                                                              |
| (10) | $\int_0^\infty x^{-\rho} Y_\nu(2x^{\frac{1}{2}}) G_{pq}^{mn} \left( ax \left  \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx$ $= G_{p+3,q+1}^{m,n+2} \left( a \left  \begin{matrix} \rho - \frac{1}{2}\nu, \rho + \frac{1}{2}\nu, a_1, \dots, a_p, \rho + \frac{1}{2} + \frac{1}{2}\nu \\ b_1, \dots, b_q, \rho + \frac{1}{2} + \frac{1}{2}\nu \end{matrix} \right. \right)$ <p style="text-align: center;"><math>p+q &lt; 2(m+n), \quad  \arg a  &lt; (m+n - \frac{1}{2}p - \frac{1}{2}q)\pi</math><br/> <math>-\frac{3}{4} + \max_{1 \leq j \leq n} \operatorname{Re} a_j &lt; \operatorname{Re} \rho &lt; \min_{1 \leq j \leq m} \operatorname{Re} b_j + \frac{1}{2}  \operatorname{Re} \nu  + 1</math></p> |

Meijer's  $G$ -function (cont'd)

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             |
|------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (11) | $\int_0^\infty x^{-\rho} K_\nu(2x^{1/2}) G_{pq}^{mn} \left( \alpha x \left  \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx$ $= {}_{p+2, q} G_{p+2, q}^{m, n+2} \left( \alpha \left  \begin{matrix} \rho - \frac{1}{2}\nu, \rho + \frac{1}{2}\nu, a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right)$ $p + q < 2(m + n), \quad  \arg \alpha  < (m + n - \frac{1}{2}p - \frac{1}{2}q)\pi$ $\operatorname{Re} \rho < 1 - \frac{1}{2} \operatorname{Re} \nu  + \min_{1 \leq j \leq m} \operatorname{Re} b_j$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                    |
| (12) | $\int_0^\infty x^{-\rho} \mathbf{H}_\nu(2x^{1/2}) G_{pq}^{mn} \left( \alpha x \left  \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx$ $= G_{p+3, q+1}^{m+1, n+1} \left( \alpha \left  \begin{matrix} \rho - \frac{1}{2} - \frac{1}{2}\nu, a_1, \dots, a_p, \rho + \frac{1}{2}\nu, \rho - \frac{1}{2}\nu \\ \rho - \frac{1}{2} - \frac{1}{2}\nu, b_1, \dots, b_q \end{matrix} \right. \right)$ $p + q < 2(m + n), \quad  \arg \alpha  < (m + n - \frac{1}{2}p - \frac{1}{2}q)\pi$ $\max \left( -\frac{3}{4}, \operatorname{Re} \frac{\nu-1}{2} \right) + \max_{1 \leq j \leq n} \operatorname{Re} a_j < \operatorname{Re} \rho < \min_{1 \leq j \leq m} \operatorname{Re} b_j + \frac{1}{2} \operatorname{Re} \nu + \frac{3}{2}$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           |
| (13) | $\int_1^\infty x^{-\rho} (x-1)^{\sigma-1} {}_2F_1(\kappa + \sigma - \rho, \lambda + \sigma - \rho; \sigma; 1-x) G_{pq}^{mn} \left( \alpha x \left  \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) dx$ $= \Gamma(\sigma) G_{p+2, q+2}^{m+2, n} \left( \alpha \left  \begin{matrix} a_1, \dots, a_p, \kappa + \lambda + \sigma - \rho, \rho \\ \kappa, \lambda, b_1, \dots, b_q \end{matrix} \right. \right)$ <p>First set of conditions of validity:</p> $p + q < 2(m + n), \quad  \arg \alpha  < (m + n - \frac{1}{2}p - \frac{1}{2}q)\pi$ $\operatorname{Re} \sigma > 0, \quad \operatorname{Re} \kappa \geq \operatorname{Re} \lambda > \operatorname{Re} a_j - 1, \quad j = 1, \dots, n$ <p>Second set of conditions of validity:</p> $p + q \leq 2(m + n), \quad  \arg \alpha  \leq (m + n - \frac{1}{2}p - \frac{1}{2}q)\pi$ $\operatorname{Re} \sigma > 0, \quad \operatorname{Re} \kappa \geq \operatorname{Re} \lambda > \operatorname{Re} a_j - 1, \quad j = 1, \dots, n$ $\operatorname{Re} \left[ \sum_{j=1}^p a_j - \sum_{j=1}^q b_j + (q-p)(\kappa + \frac{1}{2}) \right] > -\frac{1}{2}$ $\operatorname{Re} \left[ \sum_{j=1}^p a_j - \sum_{j=1}^q b_j + (q-p)(\lambda + \frac{1}{2}) \right] > -\frac{1}{2}$ |

**Meijer's G-function (cont'd)**

|      |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     |
|------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| (14) | $\int_0^\infty G_{pq}^{mn} \left( \alpha x \left  \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) G_{rs}^{kl} \left( \beta x \left  \begin{matrix} c_1, \dots, c_r \\ d_1, \dots, d_s \end{matrix} \right. \right) dx$ $= \alpha^{-1} G_{q+r, p+s}^{k+n, l+m} \left( \frac{\beta}{\alpha} \left  \begin{matrix} -b_1, \dots, -b_m, c_1, \dots, c_r, -b_{m+1}, \dots, -b_q \\ -a_1, \dots, -a_n, d_1, \dots, d_s, -a_{n+1}, \dots, -a_p \end{matrix} \right. \right)$ |
|      | <p>For (five sets of) conditions of validity see Meijer, C.S., 1941:<br/> <i>Nederl. Akad. Wetensch., Proc.</i> 44, 82-92.</p>                                                                                                                                                                                                                                                                                                                                                                      |

## APPENDIX

### NOTATIONS AND DEFINITIONS OF HIGHER TRANSCENDENTAL FUNCTIONS

H.T.F. I refers to volume I, and H.T.F. II to volume II, of *Higher transcendental functions* by the same authors as the present work.

#### Miscellaneous notations

*Ad hoc* notations are explained where they occur. Notations occurring several times on a page are explained at the bottom of the page.

In general, real variables and parameters are denoted by Latin letters, and complex variables and parameters by Greek letters. Exceptions are made to preserve traditional notations (such as  $\gamma$  in chapter XIV). The letters  $m, n$  denote integers mostly.

Re  $z$ , Im  $z$ . Real and imaginary parts of a complex quantity  $z$ .

$|z|$ , arg  $z$ . Modulus and argument (phase) of a complex quantity.

Cauchy Principal Value. If the integrand has a singularity at  $c$ ,  $a < c < b$ , the Cauchy Principal Value of

$$\int_a^b f(x) dx$$

is

$$\oint_a^b f(x) dx = \lim_{\epsilon \rightarrow 0} \left[ \int_a^{c-\epsilon} f(x) dx + \int_{c+\epsilon}^b f(x) dx \right] \quad \epsilon > 0, \quad \epsilon \rightarrow 0.$$

Empty sums are to be interpreted as zero, and empty products as unity.

$\sum_{n=a}^b$ ,  $\prod_{n=a}^b$  are empty if  $b < a$ .

$[x]$  largest integer  $\leq x$ .

$$(a)_\nu = \Gamma(a + \nu) / \Gamma(a)$$

$$(a)_0 = 1$$

$$(a)_n = a(a+1) \cdots (a+n-1) \quad n = 1, 2, \dots$$

$$(a)_n = (-1)^n (1 - a - n)_n \quad n \text{ integer}$$

$$(a)_{-n} = (-1)^n / (1 - a)_n \quad n \text{ integer}$$

Binomial coefficient

$$\binom{a}{\beta} = \frac{\Gamma(a+1)}{\Gamma(\beta+1) \Gamma(a-\beta+1)}.$$

$$\operatorname{sgn} x = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

Euler-Mascheroni constant.

$$C = \lim_{m \rightarrow \infty} \left( \sum_{n=1}^m 1/n - \log m \right) = 0.5772156649 \dots$$

$$\gamma = e^C.$$

Note that in H.T.F. and many other books  $C$  is denoted by  $\gamma$ .

### Orthogonal polynomials

See also H.T.F. II Chapter X and pp. 265-269 of the present volume.

Legendre polynomial

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

Gegenbauer polynomial

$$C_n^\nu(x) = \frac{(-2)^n (\nu)_n}{n! (n+2\nu)_n} (1-x^2)^{\frac{1}{2}-\nu} \frac{d^n}{dx^n} (1-x^2)^{n+\nu-\frac{1}{2}}.$$

Tchebichef polynomials

$$T_n(x) = \cos(n \cos^{-1} x)$$

$$U_n(x) = \frac{\sin[(n+1) \cos^{-1} x]}{\sin(\cos^{-1} x)}.$$

Jacobi polynomial

$$P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^n}{dx^n} [(1-x)^{n+\alpha} (1+x)^{n+\beta}].$$

Laguerre polynomial

$$L_n^\alpha(z) = \frac{e^z z^{-\alpha}}{n!} \frac{d^n}{dz^n} (e^{-z} z^{n+\alpha})$$

$$L_n^0(z) = L_n(z).$$

Hermite polynomials

$$\text{He}_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n} (e^{-\frac{1}{2}x^2})$$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

Charlier polynomial

$$p_n(x; a) = n! a^{-n} L_n^{x-n}(a).$$

### The gamma function and related functions

See also H.T.F. I Chapter I.

Gamma function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \quad \text{Re } z > 0.$$

Logarithmic derivative of the gamma function

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \quad \psi'(z) = \frac{d\psi}{dz}, \quad \text{etc.}$$

Beta function

$$B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}.$$

Euler's dilogarithm

$$L_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} = - \int_0^z \frac{\log(1-z)}{z} dz.$$

Incomplete gamma functions. See under Confluent hypergeometric functions.

Incomplete beta function. See under Hypergeometric functions.

Riemann's zeta function and related functions

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

$$\xi(t) = -\frac{1}{2}(t^2 + \frac{1}{4}) \pi^{-\frac{1}{2}} i t^{-\frac{1}{4}} \Gamma(\frac{1}{2} i t + \frac{1}{4}) \zeta(it + \frac{1}{2})$$

$$\zeta(z, a) = \sum_{n=0}^{\infty} (n+a)^{-z}, \quad \Phi(z, s, v) = \sum_{n=0}^{\infty} \frac{z^n}{(v+n)^s}$$

### Legendre functions

See also H.T.F. I Chapter III. For expressions of products of Legendre functions as hypergeometric series see Meijer, C.S., 1936: *Math. Ann.* 112, 469-489 and *Proc. Nederl. Akad. Wetensch.* 39, 394-403 and 519-527; 1938: *Nieuw Arch. Wiskunde* (2) 19, 207-234.

$$P_{\nu}^{\mu}(z) = \frac{1}{\Gamma(1-\mu)} \left( \frac{z+1}{z-1} \right)^{\frac{1}{2}\mu} {}_2F_1(-\nu, \nu+1; 1-\mu; \frac{1}{2} - \frac{1}{2}z)$$

$$Q_{\nu}^{\mu}(z) = \frac{e^{\mu\pi i} \pi^{\frac{1}{2}} \Gamma(\mu+\nu+1)}{2^{\nu+1} \Gamma(\nu+3/2)} z^{-\mu-\nu-1} (z^2-1)^{\frac{1}{2}\mu} \\ \times {}_2F_1\left(\frac{\mu+\nu+1}{2}, \frac{\mu+\nu+2}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right)$$

$z$  in the complex plane cut along the real axis from  $-1$  to  $1$ .

$$P_{\nu}^{\mu}(x) = \frac{1}{\Gamma(1-\mu)} \left( \frac{1+x}{1-x} \right)^{\frac{1}{2}\mu} {}_2F_1(-\nu, \nu+1; 1-\mu; \frac{1}{2} - \frac{1}{2}x) \quad -1 < x < 1$$

$$Q_{\nu}^{\mu}(x) = \frac{1}{2} e^{-i\mu\pi} [e^{-\frac{1}{2}\mu\pi i} Q_{\nu}^{\mu}(x+i0) + e^{\frac{1}{2}\mu\pi i} Q_{\nu}^{\mu}(x-i0)] \quad -1 < x < 1$$

$$P_{\nu}(z) = P_{\nu}^0(z), \quad Q_{\nu}(z) = Q_{\nu}^0(z).$$

### Bessel functions and related functions

See also H.T.F. II Chapter VII, and pp. 331-332 of the present volume.

Bessel functions

$$J_{\nu}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{1}{2}z)^{\nu+2m}}{m! \Gamma(\nu+m+1)}$$

$$Y_{\nu}(z) = \operatorname{cosec} \nu\pi [J_{\nu}(z) \cos \nu\pi - J_{-\nu}(z)]$$



$$H_{\nu}^{(1)}(z) = J_{\nu}(z) + i Y_{\nu}(z)$$

$$H_{\nu}^{(2)}(z) = J_{\nu}(z) - i Y_{\nu}(z)$$

$$Ji_{\nu}(x) = \int_{\infty}^x J_{\nu}(t) \frac{dt}{t}.$$

Modified Bessel functions

$$I_{\nu}(z) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2}z)^{\nu+2m}}{m! \Gamma(\nu+m+1)}$$

$$K_{\nu}(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_{\nu}(z)}{\sin \nu \pi}.$$

Kelvin's and related functions

$$\text{ber}_{\nu}(z) + i \text{bei}_{\nu}(z) = J_{\nu}(ze^{\frac{3}{4}\pi i})$$

$$\text{ber}_{\nu}(z) - i \text{bei}_{\nu}(z) = J_{\nu}(ze^{-\frac{3}{4}\pi i})$$

$$\text{ker}_{\nu}(z) + i \text{kei}_{\nu}(z) = K_{\nu}(ze^{\frac{1}{4}\pi i})$$

$$\text{ker}_{\nu}(z) - i \text{kei}_{\nu}(z) = K_{\nu}(ze^{-\frac{1}{4}\pi i})$$

$$\text{ber}(z) = \text{ber}_0(z), \quad \text{bei}(z) = \text{bei}_0(z),$$

$$\text{ker}(z) = \text{ker}_0(z), \quad \text{kei}(z) = \text{kei}_0(z).$$

Note that the definition of  $\text{ker}_{\nu}(z)$  and  $\text{kei}_{\nu}(z)$  differs from that given in H.T.F. II sec. 7.2.3.

$$X_{\nu}^{(b)}(z) = \text{ber}_{\nu}^2(z) + \text{bei}_{\nu}^2(z)$$

$$V_{\nu}^{(b)}(z) = [\text{ber}_{\nu}'(z)]^2 + [\text{bei}_{\nu}'(z)]^2$$

$$W_{\nu}^{(b)}(z) = \text{ber}_{\nu}(z) \text{bei}_{\nu}'(z) - \text{bei}_{\nu}(z) \text{ber}_{\nu}'(z)$$

$$\frac{1}{2} Z_{\nu}^{(b)}(z) = \text{ber}_{\nu}(z) \text{bei}_{\nu}'(z) + \text{bei}_{\nu}(z) \text{ber}_{\nu}'(z).$$

Neumann polynomials

$$O_0(x) = \frac{1}{x}; \quad O_n(x) = \frac{1}{4} \sum_{m=0}^{\leq \frac{1}{2}n} \frac{n(n-m-1)!}{m! (\frac{1}{2}x)^{n-2m+1}} \quad n = 1, 2, \dots$$

$$O_{-n}(x) = (-1)^n O_n(x), \quad n = 1, 2, \dots$$

Anger-Weber functions

$$\mathbf{J}_\nu(z) = \pi^{-1} \int_0^\pi \cos(\nu\theta - z \sin \theta) d\theta$$

$$\mathbf{E}_\nu(z) = \pi^{-1} \int_0^\pi \sin(\nu\theta - z \sin \theta) d\theta.$$

Struve's functions

$$\begin{aligned} \mathbf{H}_\nu(z) &= \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{\nu+2m+1}}{\Gamma(m+3/2) \Gamma(\nu+m+3/2)} \\ &= \frac{(z/2)^{\nu+1}}{\Gamma(3/2) \Gamma(\nu+3/2)} {}_1F_2\left(1; 3/2, \nu+3/2; -z^2/4\right) \\ &= 2^{1-\nu} \pi^{-1/2} [\Gamma(\nu+1/2)]^{-1} s_{\nu,\nu}(z). \end{aligned}$$

$$\mathbf{L}_\nu(z) = e^{-1/2(\nu+1)\pi i} \mathbf{H}_\nu(ze^{1/2 i \pi}).$$

Lommel's functions

$$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)} {}_1F_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; -\frac{z^2}{4}\right)$$

$$\begin{aligned} S_{\mu,\nu}(z) &= s_{\mu,\nu}(z) + 2^{\mu-1} \Gamma\left(\frac{\mu-\nu+1}{2}\right) \Gamma\left(\frac{\mu+\nu+1}{2}\right) \\ &\quad \times \left[ \sin\left(\frac{\mu-\nu}{2}\pi\right) J_\nu(z) - \cos\left(\frac{\mu-\nu}{2}\pi\right) Y_\nu(z) \right]. \end{aligned}$$

Lommel's functions of two variables

$$U_\nu(w, z) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{w}{z}\right)^{\nu+2m} J_{\nu+2m}(z)$$

$$V_\nu(w, z) = \cos\left(\frac{w}{2} + \frac{z^2}{2w} + \frac{\nu\pi}{2}\right) + U_{2-\nu}(w, z).$$

**Hypergeometric functions**

See also H.T.F. I Chapters II, IV.

Generalized hypergeometric series

$${}_mF_n(a_1, \dots, a_m; \gamma_1, \dots, \gamma_n; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_m)_k}{(\gamma_1)_k \dots (\gamma_n)_k} \frac{z^k}{k!}.$$

${}_2F_1(a, b; c; z)$  is Gauss' hypergeometric series and is often (for instance in H.T.F. I Chapter II) denoted by  $F(a, b; c; z)$ .

${}_1F_1(a; c; z)$  is Kummer's confluent hypergeometric series and is sometimes (for instance in H.T.F. I Chapter VI) denoted by  $\Phi(a; c; z)$ .

${}_mF_n(a_1, \dots, a_m; \gamma_1, \dots, \gamma_n; z)$  is sometimes written as

$${}_mF_n \left[ \begin{matrix} a_1, \dots, a_m; \\ \gamma_1, \dots, \gamma_n \end{matrix} ; z \right].$$

Incomplete beta function

$$B_x(p, q) = \int_0^x t^{p-1} (1-t)^{q-1} dt = p^{-1} x^p {}_2F_1(p, 1-q; p+1; x).$$

$$I_x(p, q) = \frac{B_x(p, q)}{B_1(p, q)}$$

$$S_n(b_1, b_2, b_3, b_4; z) = \sum_{h=1}^n \frac{\prod_{j=1}^n{}' \Gamma(b_j - b_h)}{\prod_{j=n+1}^4 \Gamma(1 + b_h - b_j)} z^{1+2b_h}$$

$$\times {}_0F_3(1 + b_h - b_1, \dots, *, \dots, 1 + b_h - b_4; (-1)^n z^2)$$

The prime in  $\Pi'$  and the asterisk in  ${}_0F_3$  mean that the term containing  $b_h - b_h$  is to be omitted. For  $n = 1$  the product  $\Pi$  in the numerator, for  $n = 4$  that in the denominator is to be replaced by unity.

**Confluent hypergeometric functions**

See also H.T.F. I Chapter VI and H.T.F. II Chapters VIII and IX. See also under Hypergeometric functions, Orthogonal polynomials,  $E$ -function,  $G$ -function.

Whittaker's functions

$$M_{\kappa, \mu}(z) = z^{1/2 + \mu} e^{-1/2 z} {}_1F_1(1/2 + \mu - \kappa; 2\mu + 1; z)$$

$$W_{\kappa, \mu}(z) = \frac{\Gamma(-2\mu) M_{\kappa, \mu}(z)}{\Gamma(1/2 - \mu - \kappa)} + \frac{\Gamma(2\mu) M_{\kappa, -\mu}(z)}{\Gamma(1/2 + \mu - \kappa)}.$$

Parabolic cylinder functions

$$D_\nu(z) = 2^{1/2 \nu + 1/4} z^{-1/2} W_{1/2 \nu + 1/4, 1/4}(1/2 z^2)$$

$$D_n(z) = (-1)^n e^{1/4 z^2} \frac{d^n}{dz^n} (e^{-1/2 z^2}).$$

Bateman's function

$$k_{2\nu}(z) = \frac{1}{\Gamma(\nu + 1)} W_{\nu, 1/2}(2z).$$

The exponential integral and related functions

$$-\text{Ei}(-x) = E_1(x) = \int_x^\infty e^{-t} \frac{dt}{t} = \Gamma(0, x) \quad -\pi < \arg x < \pi$$

$$\text{Ei}^+(x) = \text{Ei}(x + i0), \quad \text{Ei}^-(x) = \text{Ei}(x - i0) \quad x > 0$$

$$\overline{\text{Ei}}(x) = 1/2 [\text{Ei}^+(x) + \text{Ei}^-(x)] \quad x > 0.$$

The last function is denoted by  $E^*(x)$  in H.T.F. II sec. 9.7.

$$\text{li}(z) = \int_0^z \frac{dt}{\log t} = \text{Ei}(\log z)$$

$$\text{si}(x) = - \int_x^\infty \frac{\sin t}{t} dt = \frac{1}{2i} [\text{Ei}(ix) - \text{Ei}(-ix)]$$

$$\text{Si}(x) = \int_0^x \frac{\sin t}{t} dt = 1/2 \pi + \text{si}(x)$$

$$\text{Ci}(x) = - \int_x^\infty \frac{\cos t}{t} dt = -\text{ci}(x) = 1/2 [\text{Ei}(ix) + \text{Ei}(-ix)]$$

## Error functions and related functions

$$\operatorname{Erf}(x) = 2\pi^{-1/2} \int_0^x e^{-t^2} dt = \frac{2x}{\sqrt{\pi}} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x^2\right)$$

$$\operatorname{Erfc}(x) = 2\pi^{-1/2} \int_x^\infty e^{-t^2} dt = 1 - \operatorname{Erf}(x) = (\pi x)^{-1/2} e^{-1/2 x^2} W_{-1/4, 1/4}(x^2).$$

These functions differ by the factor  $2\pi^{-1/2}$  from the functions introduced in H.T.F., II sec. 9.9.

$$C(x) = 2^{-1/2} \pi^{-1/2} \int_0^x t^{-1/2} \cos t dt$$

$$S(x) = 2^{-1/2} \pi^{-1/2} \int_0^x t^{-1/2} \sin t dt.$$

## Incomplete gamma functions

$$\gamma(a, x) = \int_0^x e^{-t} t^{a-1} dt = a^{-1} x^a {}_1F_1(a; a+1; -x)$$

$$\begin{aligned} \Gamma(a, x) &= \int_x^\infty e^{-t} t^{a-1} dt = \Gamma(a) - \gamma(a, x) \\ &= x^{1/2(a-1)} e^{-1/2 x} W_{1/2(a-1), 1/2 a}(x). \end{aligned}$$

## Particular cases of Whittaker's functions

$$M_{-1/4, 1/4}(x) = 1/2 \pi^{-1/2} x^{1/4} e^{1/2 x} \operatorname{Erf}(x^{1/2})$$

$$M_{n+1/4, -1/4}(x) = \frac{(-2)^{-n}}{(1/2)_n} x^{1/4} e^{-1/2 x} \operatorname{He}_{2n}[(2x)^{1/2}]$$

$$M_{n+1/4, 1/4}(x) = \frac{(-1)^n 2^{-n-1/2}}{(3/2)_n} x^{1/4} e^{-1/2 x} \operatorname{He}_{2n+1}[(2x)^{1/2}]$$

$$M_{0, \mu}(x) = 2^{2\mu} \Gamma(\mu+1) x^{1/2} I_\mu(1/2 x)$$

$$M_{0, \mu}(\pm ix) = 2^{2\mu} \Gamma(\mu + 1) e^{\pm \frac{1}{2}\mu\pi i} x^{\frac{1}{2}} J_{\mu}(\frac{1}{2}x)$$

$$M_{\mu+\frac{1}{2}, \mu}(x) = x^{\mu+\frac{1}{2}} e^{-\frac{1}{2}x}$$

$$M_{-\mu-\frac{1}{2}, \mu}(x) = x^{\mu+\frac{1}{2}} e^{\frac{1}{2}x}$$

$$M_{\mu-\frac{1}{2}, \mu}(x) = 2\mu x^{\frac{1}{2}-\mu} e^{\frac{1}{2}x} \gamma(2\mu, x)$$

$$M_{\mu+n+\frac{1}{2}, \mu}(x) = \frac{n!}{(2\mu+1)_n} x^{\mu+\frac{1}{2}} e^{-\frac{1}{2}x} L_n^{2\mu}(x)$$

$$W_{-\frac{1}{2}, 0}(x) = -x^{\frac{1}{2}} e^{\frac{1}{2}x} \text{Ei}(-x)$$

$$W_{-\frac{1}{4}, \pm\frac{1}{4}}(x) = \pi^{\frac{1}{2}} x^{\frac{1}{4}} e^{\frac{1}{2}x} \text{Erfc}(x^{\frac{1}{2}})$$

$$W_{\frac{1}{2}n+\frac{1}{4}, \pm\frac{1}{4}}(x) = 2^{-\frac{1}{2}n} x^{\frac{1}{4}} e^{-\frac{1}{2}x} \text{He}_n[(2x)^{\frac{1}{2}}]$$

$$W_{\mu, \pm\frac{1}{4}}(x) = 2^{-\mu} (2x)^{\frac{1}{4}} D_{2\mu-\frac{1}{2}}[(2x)^{\frac{1}{2}}]$$

$$W_{0, \mu}(x) = \left(\frac{x}{\pi}\right)^{\frac{1}{2}} K_{\mu}\left(\frac{x}{2}\right)$$

$$W_{0, \mu}(ix) = \frac{(\pi x)^{\frac{1}{2}}}{2} \exp[-(\frac{1}{2}\nu + \frac{1}{4})\pi i] H_{\mu}^{(2)}\left(\frac{x}{2}\right)$$

$$W_{0, \mu}(-ix) = \frac{(\pi x)^{\frac{1}{2}}}{2} \exp[(\frac{1}{2}\nu + \frac{1}{4})\pi i] H_{\mu}^{(1)}\left(\frac{x}{2}\right)$$

$$W_{\mu+\frac{1}{2}, \pm\mu}(x) = x^{\mu+\frac{1}{2}} e^{-\frac{1}{2}x}$$

$$W_{\mu-\frac{1}{2}, \pm\mu}(x) = x^{\frac{1}{2}-\mu} e^{\frac{1}{2}x} \Gamma(2\mu, x)$$

$$W_{\mu+n+\frac{1}{2}, \pm\mu}(x) = (-1)^n n! x^{\mu+\frac{1}{2}} e^{-\frac{1}{2}x} L_n^{2\mu}(x)$$

**MacRobert's  $E$ -function**

See also H.T.F. I Chapter V.

If  $p \geq q + 1$ ,

$$E(p; \alpha_r; q; \rho_s; x) = \sum_{r=1}^p \frac{\prod_{s=1}^q \Gamma(\alpha_s - \alpha_r)}{\prod_{t=1}^q \Gamma(\rho_t - \alpha_r)} \Gamma(\alpha_r) x^{\alpha_r} \\ \times {}_{q+1}F_{p-1}(\alpha_r, \alpha_r - \rho_1 + 1, \dots, \alpha_r - \rho_q + 1; \alpha_r - \alpha_1 + 1, \dots, *, \dots, \alpha_r - \alpha_p + 1; \\ (-1)^{p-q} x).$$

where  $|x| < 1$  when  $p = q + 1$ .

If  $p \leq q + 1$ ,

$$E(p; \alpha_r; q; \rho_s; x) = \frac{\prod_{r=1}^p \Gamma(\alpha_r)}{\prod_{s=1}^q \Gamma(\rho_s)} {}_pF_q(\alpha_1, \dots, \alpha_p; \rho_1, \dots, \rho_q; -1/x)$$

where  $x \neq 0$  and  $|x| > 1$  if  $p = q + 1$ . If  $p > q + 1$ , the last relation gives the asymptotic expansion of the  $E$ -function for large  $x$ .

$$E(\alpha; : x) = \Gamma(\alpha) (1 + x^{-1})^{-\alpha}$$

$$E(\nu + 1; : x) = x^{\frac{1}{2}\nu} J_\nu(2x^{-\frac{1}{2}})$$

$$E(\frac{1}{2} + \nu, \frac{1}{2} - \nu; : 2x) = \sec(\nu\pi) (2\pi x)^{\frac{1}{2}} e^x K_\nu(x)$$

$$E(\alpha, \beta; : x) = \Gamma(\alpha) \Gamma(\beta) x^{-k} e^{\frac{1}{2}x} W_{k,m}(x)$$

$$k = \frac{1}{2}(1 - \alpha - \beta), \quad m = \frac{1}{2}(\alpha - \beta)$$

$$\begin{aligned}
 E \left( a, \beta, \frac{a+\beta}{2}, \frac{a+\beta+1}{2} : a+\beta : \frac{x^2}{4} \right) \\
 = \pi^{\frac{1}{2}} \Gamma(a) \Gamma(\beta) \left( \frac{x}{2} \right)^{-2k} W_{k,m}(ix) W_{k,m}(-ix) \\
 k = \frac{1}{2}(1 - \alpha - \beta), \quad m = \frac{1}{2}(\alpha - \beta)
 \end{aligned}$$

### Meijer's $G$ -function

See also H.T.F. I Chapter V.

$$\begin{aligned}
 G_{p,q}^{m,n} \left( x \left| \begin{array}{c} a_1, \dots, a_p \\ b_1, \dots, b_q \end{array} \right. \right) \\
 = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} x^s ds
 \end{aligned}$$

where  $L$  is a path separating the poles of  $\Gamma(b_1 - s) \dots \Gamma(b_m - s)$  from those of  $\Gamma(1 - a_1 + s) \dots \Gamma(1 - a_n + s)$ . For a more detailed definition see H.T.F. I sec. 5.3.

Formulas involving the  $G$ -function may be used as key formulas from which many integrals with Bessel functions, Legendre functions, and other higher transcendental functions follow by specializing parameters. The following two lists give expressions of certain special  $G$ -functions in terms of well-known higher transcendental functions, and, conversely, expressions for higher transcendental functions in terms of  $G$ -functions. The list is not complete. See also H.T.F. I sec. 5.6.

Particular cases of the  $G$ -function

$$\begin{aligned}
 G_{02}^{10}(x|a, b) &= x^{\frac{1}{2}(a+b)} J_{a-b}(2x^{\frac{1}{2}}) \\
 G_{02}^{20}(x|a, b) &= 2x^{\frac{1}{2}(a+b)} K_{a-b}(2x^{\frac{1}{2}}) \\
 G_{12}^{11} \left( x \left| \begin{array}{c} \frac{1}{2} \\ b, -b \end{array} \right. \right) &= \pi^{\frac{1}{2}} e^{-\frac{1}{2}x} I_b(\frac{1}{2}x)
 \end{aligned}$$



$$G_{12}^{11} \left( x \left| \begin{matrix} a \\ b, c \end{matrix} \right. \right) = \frac{\Gamma(1-a+b)}{\Gamma(1+b-c)} x^b {}_1F_1(1-a+b; 1+b-c; -x)$$

$$G_{12}^{20} \left( x \left| \begin{matrix} \frac{1}{2} \\ b, -b \end{matrix} \right. \right) = \pi^{-1/2} e^{-1/2 x} K_b(1/2 x)$$

$$G_{12}^{20} \left( x \left| \begin{matrix} a \\ b, c \end{matrix} \right. \right) = x^{1/2(b+c-1)} e^{-1/2 x} W_{k, m}(x) \\ k = 1/2(1+b+c) - a, \quad m = 1/2 b - 1/2 c$$

$$G_{12}^{21} \left( x \left| \begin{matrix} \frac{1}{2} \\ b, -b \end{matrix} \right. \right) = \frac{\pi^{1/2}}{\cos b\pi} e^{1/2 x} K_b(1/2 x)$$

$$G_{12}^{21} \left( x \left| \begin{matrix} a \\ b, c \end{matrix} \right. \right) = \Gamma(b-a+1) \Gamma(c-a+1) x^{1/2(b+c-1)} e^{1/2 x} W_{k, m}(x) \\ k = a - 1/2(b+c+1), \quad m = 1/2 b - 1/2 c$$

$$G_{04}^{10}(x|a, b, 2b-a, b+1/2) = \pi^{-1/2} x^b I_{2(a-b)}(2^{3/2} x^{1/4}) J_{2(a-b)}(2^{3/2} x^{1/4})$$

$$G_{04}^{10}(x|a, a+1/2, b, 2a-b) = 1/2 \pi^{-1/2} \sec(b-a) \pi \\ \times x^a [J_{2(a-b)}(2^{3/2} x^{1/4}) I_{2(b-a)}(2^{3/2} x^{1/4}) \\ + I_{2(a-b)}(2^{3/2} x^{1/4}) J_{2(b-a)}(2^{3/2} x^{1/4})]$$

$$G_{04}^{10}(x|a+1/2, a, b, 2a-b) = 1/2 \pi^{-1/2} [\sin(a-b) \pi]^{-1} \\ \times x^a [J_{2(a-b)}(2^{3/2} x^{1/4}) I_{2(b-a)}(2^{3/2} x^{1/4}) \\ - I_{2(a-b)}(2^{3/2} x^{1/4}) J_{2(b-a)}(2^{3/2} x^{1/4})]$$

$$G_{04}^{20}(x|a, a+1/2, b, b+1/2) = x^{1/2(a+b)} J_{2(a-b)}(4x^{1/4})$$

$$G_{04}^{20}(x|a, -a, 0, 1/2) = -\pi^{1/2} (\sin 2a\pi)^{-1} [J_{2a}(ze^{\pi i/4}) J_{2a}(ze^{-\pi i/4}) \\ - J_{-2a}(ze^{\pi i/4}) J_{-2a}(ze^{-\pi i/4})] \\ z = 2^{3/2} x^{1/4}$$

$$G_{04}^{20}(x|0, \frac{1}{2}, a, -a) = \pi^{\frac{1}{2}} i^{-1} (\sin 2a\pi)^{-1} \\ \times [e^{2a\pi i} J_{2a}(ze^{-\pi i/4}) J_{-2a}(ze^{\pi i/4}) - e^{-2a\pi i} J_{2a}(ze^{\pi i/4}) \\ \times J_{-2a}(ze^{-\pi i/4})] \quad z = 2^{3/2} x^{1/4}$$

$$G_{04}^{30}(x|3a - \frac{1}{2}, a, -a - \frac{1}{2}, a - \frac{1}{2}) = 2\pi^{\frac{1}{2}} (\cos 2a\pi)^{-1} \\ \times x^{a-1/2} K_{4a}(2^{3/2} x^{1/4}) [J_{4a}(2^{3/2} x^{1/4}) + J_{-4a}(2^{3/2} x^{1/4})]$$

$$G_{04}^{30}(x|0, a - \frac{1}{2}, -a - \frac{1}{2}, -\frac{1}{2}) = 4\pi^{\frac{1}{2}} x^{-\frac{1}{2}} \\ \times K_{2a}(2^{3/2} x^{1/4}) [J_{2a}(2^{3/2} x^{1/4}) \cos a\pi - Y_{2a}(2^{3/2} x^{1/4}) \sin a\pi]$$

$$G_{04}^{30}(x|-\frac{1}{2}, a - \frac{1}{2}, -a - \frac{1}{2}, 0) = -4\pi^{\frac{1}{2}} x^{-\frac{1}{2}} \\ \times K_{2a}(2^{3/2} x^{1/4}) [J_{2a}(2^{3/2} x^{1/4}) \sin a\pi + Y_{2a}(2^{3/2} x^{1/4}) \cos a\pi]$$

$$G_{04}^{30}(x|a, b + \frac{1}{2}, b, 2b - a) = \pi^{\frac{1}{2}} 2^{\frac{1}{2}} x^b K_{2(a-b)}(2^{3/2} x^{1/4}) \\ \times J_{2(a-b)}(2^{3/2} x^{1/4})$$

$$G_{04}^{40}(x|a, a + \frac{1}{2}, b, b + \frac{1}{2}) = 4\pi x^{\frac{1}{2}(a+b)} K_{2(a-b)}(4x^{\frac{1}{4}})$$

$$G_{04}^{40}(x|a, a + \frac{1}{2}, b, 2a - b) = 2^3 \pi^{\frac{1}{2}} x^a \\ \times K_{2(b-a)}(2^{3/2} x^{1/4} e^{\pi i/4}) K_{2(b-a)}(2^{3/2} x^{1/4} e^{-\pi i/4})$$

$$G_{04}^{n0}(x|a, b, c, d) = x^{-\frac{1}{2}} S_n(a, b, c, d; x^{\frac{1}{2}}) \quad n = 1, 2, 3, 4$$

$$G_{13}^{11}\left(x \left| \begin{array}{c} \frac{1}{2} \\ a, 0, -a \end{array} \right. \right) = \pi^{\frac{1}{2}} J_a^2(x^{\frac{1}{2}})$$

$$G_{13}^{11}\left(x \left| \begin{array}{c} \frac{1}{2} \\ 0, a, -a \end{array} \right. \right) = \pi^{\frac{1}{2}} J_a(x^{\frac{1}{2}}) J_{-a}(x^{\frac{1}{2}})$$

$$G_{13}^{11}\left(x \left| \begin{array}{c} a \\ a, b, a - \frac{1}{2} \end{array} \right. \right) = x^{\frac{1}{2}a + \frac{1}{2}b - \frac{1}{4}} \mathbf{H}_{a-b-\frac{1}{2}}(2x^{\frac{1}{2}})$$

$$G_{13}^{20}\left(x \left| \begin{array}{c} a - \frac{1}{2} \\ a, b, a - \frac{1}{2} \end{array} \right. \right) = x^{\frac{1}{2}(a+b)} Y_{b-a}(2x^{\frac{1}{2}})$$

$$G_{13}^{20} \left( x \left| \begin{array}{c} a + \frac{1}{2} \\ b, a, 2a - b \end{array} \right. \right) = -\pi^{\frac{1}{2}} x^a J_{b-a}(x^{\frac{1}{2}}) Y_{b-a}(x^{\frac{1}{2}})$$

$$G_{13}^{20} \left( x \left| \begin{array}{c} \frac{1}{2} \\ a, -a, 0 \end{array} \right. \right) = \pi^{\frac{1}{2}} 2^{-1} (\sin a \pi)^{-1} [J_{-a}^2(x^{\frac{1}{2}}) - J_a^2(x^{\frac{1}{2}})]$$

$$G_{13}^{21} \left( x \left| \begin{array}{c} \frac{1}{2} \\ a, 0, -a \end{array} \right. \right) = 2\pi^{\frac{1}{2}} I_a(x^{\frac{1}{2}}) K_a(x^{\frac{1}{2}})$$

$$G_{13}^{21} \left( x \left| \begin{array}{c} \frac{1}{2} \\ a, -a, 0 \end{array} \right. \right) = \pi^{3/2} (\sin 2a \pi)^{-1} [I_{-a}^2(x^{\frac{1}{2}}) - I_a^2(x^{\frac{1}{2}})]$$

$$G_{13}^{21} \left( x \left| \begin{array}{c} a + \frac{1}{2} \\ a + \frac{1}{2}, b, a \end{array} \right. \right) = \frac{\pi x^{\frac{1}{2}(a+b)}}{\cos(a-b)\pi} [I_{b-a}(2x^{\frac{1}{2}}) - \mathbf{L}_{a-b}(2x^{\frac{1}{2}})]$$

$$G_{13}^{21} \left( x \left| \begin{array}{c} a + \frac{1}{2} \\ a, a + \frac{1}{2}, b \end{array} \right. \right) = \pi x^{\frac{1}{2}(a+b)} [I_{a-b}(2x^{\frac{1}{2}}) - \mathbf{L}_{a-b}(2x^{\frac{1}{2}})]$$

$$G_{13}^{30} \left( x \left| \begin{array}{c} a + \frac{1}{2} \\ a + b, a - b, a \end{array} \right. \right) = 2\pi^{-\frac{1}{2}} x^a K_b^2(x^{\frac{1}{2}})$$

$$G_{13}^{31} \left( x \left| \begin{array}{c} a + \frac{1}{2} \\ a + \frac{1}{2}, -a, a \end{array} \right. \right) = \frac{\pi^2}{\cos 2a \pi} [\mathbf{H}_{2a}(2x^{\frac{1}{2}}) - Y_{2a}(2x^{\frac{1}{2}})]$$

$$G_{13}^{31} \left( x \left| \begin{array}{c} a \\ a, b, -b \end{array} \right. \right) = 2^{-2a+2} \Gamma(1-a-b) \Gamma(1-a+b) S_{2a-1, 2b}(2x^{\frac{1}{2}})$$

$$G_{13}^{31} \left( x \left| \begin{array}{c} a + \frac{1}{2} \\ b, 2a - b, a \end{array} \right. \right) = \pi^{5/2} 2^{-1} [\cos(b-a)\pi]^{-1} \\ \times x^a H_{b-a}^{(1)}(x^{\frac{1}{2}}) H_{b-a}^{(2)}(x^{\frac{1}{2}})$$

$$G_{22}^{12} \left( x \left| \begin{array}{c} -c_1, -c_2 \\ a-1, -b \end{array} \right. \right) = \frac{\Gamma(a+c_1) \Gamma(a+c_2)}{\Gamma(a+b)} \\ \times x^{a-1} {}_2F_1(a+c_1, a+c_2; a+b; -x)$$

$$G_{24}^{12} \left( x \left| \begin{array}{c} a + \frac{1}{2}, a \\ b+a, a-c, a+c, a-b \end{array} \right. \right) = \pi^{\frac{1}{2}} x^a J_{b+c}(x^{\frac{1}{2}}) J_{b-c}(x^{\frac{1}{2}})$$

$$G_{24}^{22} \left( x \left| \begin{array}{c} a, a + \frac{1}{2} \\ b, c, 2a - c, 2a - b \end{array} \right. \right) = 2\pi^{\frac{1}{2}} x^a I_{b+c-2a}(x^{\frac{1}{2}}) K_{b-c}(x^{\frac{1}{2}})$$

$$G_{24}^{30} \left( x \left| \begin{array}{c} 0, \frac{1}{2} \\ a, b, -b, -a \end{array} \right. \right) = i 2^{-2} \pi^{\frac{1}{2}} \\ \times [H_{a-b}^{(1)}(x^{\frac{1}{2}}) H_{a+b}^{(1)}(x^{\frac{1}{2}}) - H_{a-b}^{(2)}(x^{\frac{1}{2}}) H_{a+b}^{(2)}(x^{\frac{1}{2}})]$$

$$G_{24}^{31} \left( x \left| \begin{array}{c} \frac{1}{2} + a, \frac{1}{2} - a \\ 0, \frac{1}{2}, b, -b \end{array} \right. \right) = \frac{\pi^{\frac{1}{2}} \Gamma(\frac{1}{2} - a + b) x^{-\frac{1}{2}}}{\Gamma(1 + 2a)} W_{a,b}(2x^{\frac{1}{2}}) M_{-a,b}(2x^{\frac{1}{2}})$$

$$G_{24}^{40} \left( x \left| \begin{array}{c} \frac{1}{2} + a, \frac{1}{2} - a \\ 0, \frac{1}{2}, b, -b \end{array} \right. \right) = \pi^{\frac{1}{2}} x^{-\frac{1}{2}} W_{a,b}(2x^{\frac{1}{2}}) W_{-a,b}(2x^{\frac{1}{2}})$$

$$G_{24}^{40} \left( x \left| \begin{array}{c} a, a + \frac{1}{2} \\ b + c, b - c, b + \frac{1}{2} + c, b + \frac{1}{2} - c \end{array} \right. \right) \\ = \pi^{\frac{1}{2}} 2^{-k} x^{b-\frac{1}{4}} e^{-x^{\frac{1}{2}}} W_{k,2c}(2x^{\frac{1}{2}}) \quad k = \frac{1}{2} + 2b - 2c$$

$$G_{24}^{40} \left( x \left| \begin{array}{c} a, a + \frac{1}{2} \\ a + b, a + c, a - c, a - b \end{array} \right. \right) = 2\pi^{-\frac{1}{2}} x^a K_{b+c}(x^{\frac{1}{2}}) K_{b-c}(x^{\frac{1}{2}})$$

$$G_{24}^{41} \left( x \left| \begin{array}{c} 0, \frac{1}{2} \\ a, b, -b, -a \end{array} \right. \right) = \frac{-2^{-2} \pi^{5/2}}{i \sin a\pi \sin b\pi} \\ \times [e^{-b\pi i} H_{a-b}^{(1)}(x^{\frac{1}{2}}) H_{a+b}^{(2)}(x^{\frac{1}{2}}) - e^{b\pi i} H_{a+b}^{(1)}(x^{\frac{1}{2}}) H_{a-b}^{(2)}(x^{\frac{1}{2}})]$$

$$G_{24}^{41} \left( x \left| \begin{array}{c} \frac{1}{2}, 0 \\ a, b, -b, -a \end{array} \right. \right) = \frac{2^{-2} \pi^{5/2}}{\cos a\pi \cos b\pi} \\ \times [e^{-b\pi i} H_{a-b}^{(1)}(x^{\frac{1}{2}}) H_{a+b}^{(2)}(x^{\frac{1}{2}}) + e^{b\pi i} H_{a+b}^{(1)}(x^{\frac{1}{2}}) H_{a-b}^{(2)}(x^{\frac{1}{2}})]$$

$$G_{24}^{41} \left( x \left| \begin{array}{c} \frac{1}{2} + a, \frac{1}{2} - a \\ 0, \frac{1}{2}, b, -b \end{array} \right. \right) = x^{-\frac{1}{2}} \pi^{\frac{1}{2}} \Gamma(\frac{1}{2} + b - a) \Gamma(\frac{1}{2} - b - a) \\ \times W_{a,b}(2ix^{\frac{1}{2}}) W_{a,b}(-2ix^{\frac{1}{2}})$$

$$G_{24}^{42} \left( x \left| \begin{matrix} a, a + \frac{1}{2} \\ b + c, b - c, b + \frac{1}{2} + c, b + \frac{1}{2} - c \end{matrix} \right. \right) \\ = 2^{k+1} \pi^{3/2} \Gamma(1 - 2a + 2b + 2c) \Gamma(1 - 2a + 2b - 2c) \\ \times x^{b-\frac{1}{4}} e^{x^{\frac{1}{2}}} W_{k, 2c}(2x^{\frac{1}{2}}) \quad k = 2a - 2b - \frac{1}{2}$$

$$G_{44}^{14} \left( x \left| \begin{matrix} a-1, -c_1, -c_2, -c_3 \\ -b_1, -b_2, -b_3, -b_4 \end{matrix} \right. \right) = \frac{\prod_{h=1}^4 \Gamma(a + b_h)}{\prod_{h=1}^3 \Gamma(a + c_h)} x^{a-1} \\ \times {}_4F_3(a + b_1, a + b_2, a + b_3, a + b_4; a + c_1, a + c_2, a + c_3; -x)$$

$$G_{pq}^{1p} \left( x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) = \frac{\prod_{j=1}^p \Gamma(1 + b_1 - a_j)}{\prod_{j=2}^q \Gamma(1 + b_1 - b_j)} x^{b_1} \\ \times {}_pF_{q-1}(1 + b_1 - a_1, \dots, 1 + b_1 - a_p; \\ 1 + b_1 - b_2, \dots, 1 + b_1 - b_q; -x) \quad p \leq q$$

$$G_{pq}^{1n} \left( x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) = \frac{\prod_{j=1}^n \Gamma(1 + b_1 - a_j) x^{b_1}}{\prod_{j=2}^q \Gamma(1 + b_1 - b_j) \prod_{j=n+1}^p \Gamma(a_j - b_1)} \\ \times {}_pF_{q-1}(1 + b_1 - a_1, \dots, 1 + b_1 - a_p; \\ 1 + b_1 - b_2, \dots, 1 + b_1 - b_q; -x) \quad p \leq q$$

$$G_{pq}^{q1} \left( x \left| \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \right. \right) \\ \times x^{a_1-1} E(1 - a_1 + b_1, \dots, 1 - a_1 + b_q; 1 - a_1 + a_2, \dots, 1 - a_1 + a_p; x)$$

Functions expressible in terms of the  $G$ -function

$$x^\mu J_\nu(x) = 2^\mu G_{02}^{10} \left( \frac{1}{4} x^2 \middle| \frac{1}{2} \nu + \frac{1}{2} \mu, \frac{1}{2} \mu - \frac{1}{2} \nu \right)$$

$$x^\mu J_\nu(x) = 4^\mu G_{04}^{20} \left( 4^{-4} x^4 \middle| \frac{1}{4} \nu + \frac{1}{4} \mu, \frac{1}{4} \nu + \frac{1}{4} \mu + \frac{1}{2}, \frac{1}{4} \mu - \frac{1}{4} \nu, \frac{1}{2} + \frac{1}{4} \mu - \frac{1}{4} \nu \right)$$

$$x^\mu Y_\nu(x) = 2^\mu G_{13}^{20} \left( \frac{1}{4} x^2 \middle| \begin{matrix} \frac{1}{2} \mu - \frac{1}{2} \nu - \frac{1}{2} \\ \frac{1}{2} \mu - \frac{1}{2} \nu, \frac{1}{2} \mu + \frac{1}{2} \nu, \frac{1}{2} \mu - \frac{1}{2} \nu - \frac{1}{2} \end{matrix} \right)$$

$$x^\mu K_\nu(x) = 2^{\mu-1} G_{02}^{20} \left( \frac{1}{4} x^2 \middle| \frac{1}{2} \mu + \frac{1}{2} \nu, \frac{1}{2} \mu - \frac{1}{2} \nu \right)$$

$$x^\mu K_\nu(x) = 4^{\mu-1} \pi^{-1} \times G_{04}^{40} \left( 4^{-4} x^4 \middle| \frac{1}{4} \nu + \frac{1}{4} \mu, \frac{1}{2} + \frac{1}{4} \nu + \frac{1}{4} \mu, -\frac{1}{4} \nu + \frac{1}{4} \mu, \frac{1}{2} - \frac{1}{4} \nu + \frac{1}{4} \mu \right)$$

$$e^{-x} I_\nu(x) = \pi^{-\frac{1}{2}} G_{12}^{11} \left( 2x \middle| \begin{matrix} \frac{1}{2} \\ \nu, -\nu \end{matrix} \right)$$

$$e^{-x} K_\nu(x) = \pi^{\frac{1}{2}} G_{12}^{20} \left( 2x \middle| \begin{matrix} \frac{1}{2} \\ \nu, -\nu \end{matrix} \right)$$

$$e^x K_\nu(x) = \pi^{-\frac{1}{2}} \cos \nu \pi G_{12}^{21} \left( 2x \middle| \begin{matrix} \frac{1}{2} \\ \nu, -\nu \end{matrix} \right)$$

$$x^\mu \mathbf{H}_\nu(x) = 2^\mu G_{13}^{11} \left( \frac{1}{4} x^2 \middle| \begin{matrix} \frac{1}{2} + \frac{1}{2} \nu + \frac{1}{2} \mu \\ \frac{1}{2} + \frac{1}{2} \nu + \frac{1}{2} \mu, \frac{1}{2} \mu - \frac{1}{2} \nu, \frac{1}{2} \mu + \frac{1}{2} \nu \end{matrix} \right)$$

$$\mathbf{H}_\nu(x) - Y_\nu(x) = \pi^{-2} \cos \nu \pi G_{13}^{31} \left( \frac{1}{4} x^2 \middle| \begin{matrix} \frac{1}{2} + \frac{1}{2} \nu \\ \frac{1}{2} + \frac{1}{2} \nu, -\frac{1}{2} \nu, \frac{1}{2} \nu \end{matrix} \right)$$

$$x^\mu [I_\nu(x) - \mathbf{L}_\nu(x)] = \pi^{-1} 2^\mu G_{13}^{21} \left( \frac{1}{4} x^2 \middle| \begin{matrix} \frac{1}{2} \mu + \frac{1}{2} \nu + \frac{1}{2} \\ \frac{1}{2} \mu + \frac{1}{2} \nu, \frac{1}{2} \mu + \frac{1}{2} \nu + \frac{1}{2}, \frac{1}{2} \mu - \frac{1}{2} \nu \end{matrix} \right)$$

$$x^\mu [I_{-\nu}(x) - L_\nu(x)] \\ = \pi^{-1} 2^\mu \cos \nu\pi G_{13}^{21} \left( x^2 \left| \begin{array}{c} \frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\mu \\ \frac{1}{2} + \frac{1}{2}\nu + \frac{1}{2}\mu, \frac{1}{2}\mu - \frac{1}{2}\nu, \frac{1}{2}\mu + \frac{1}{2}\nu \end{array} \right. \right)$$

$$S_{\mu, \nu}(x) = 2^{\mu-1} \frac{1}{\Gamma(\frac{1}{2} - \frac{1}{2}\mu - \frac{1}{2}\nu) \Gamma(\frac{1}{2} - \frac{1}{2}\mu + \frac{1}{2}\nu)} \\ \times G_{13}^{31} \left( x^2 \left| \begin{array}{c} \frac{1}{2} + \frac{1}{2}\mu \\ \frac{1}{2} + \frac{1}{2}\mu, \frac{1}{2}\nu, -\frac{1}{2}\nu \end{array} \right. \right)$$

$$J_\nu^2(x) = \pi^{-\frac{1}{2}} G_{13}^{11} \left( x^2 \left| \begin{array}{c} \frac{1}{2} \\ \nu, 0, -\nu \end{array} \right. \right)$$

$$J_\nu(x) J_{-\nu}(x) = \pi^{-\frac{1}{2}} G_{13}^{11} \left( x^2 \left| \begin{array}{c} \frac{1}{2} \\ 0, \nu, -\nu \end{array} \right. \right)$$

$$x^\sigma J_\mu(x) J_\nu(x) \\ = \pi^{-\frac{1}{2}} G_{24}^{12} \left[ x^2 \left| \begin{array}{c} \frac{1}{2} + \frac{1}{2}\sigma, \frac{1}{2}\sigma \\ \frac{1}{2}(\mu + \nu + \sigma), \frac{1}{2}(\nu + \sigma - \mu), \frac{1}{2}(\mu + \sigma - \nu), \frac{1}{2}(\sigma - \mu - \nu) \end{array} \right. \right]$$

$$x^\mu I_\nu(x) J_\nu(x) = \pi^{\frac{1}{2}} 2^{3\mu/2} G_{04}^{10} \left( \frac{x^4}{64} \left| \begin{array}{c} \frac{1}{4}\mu + \frac{1}{2}\nu, \frac{1}{4}\mu - \frac{1}{2}\nu, \frac{1}{4}\mu, \frac{1}{4}\mu + \frac{1}{2} \end{array} \right. \right)$$

$$I_\nu(x) J_{-\nu}(x) = \pi^{\frac{1}{2}} \cos(\frac{1}{2}\nu\pi) G_{04}^{10} \left( \frac{x^4}{64} \left| \begin{array}{c} 0, \frac{1}{2}, \frac{1}{2}\nu, -\frac{1}{2}\nu \end{array} \right. \right) \\ - \pi^{\frac{1}{2}} \sin(\frac{1}{2}\nu\pi) G_{04}^{10} \left( \frac{x^4}{64} \left| \begin{array}{c} \frac{1}{2}, 0, \frac{1}{2}\nu, -\frac{1}{2}\nu \end{array} \right. \right)$$

$$x^\mu J_\nu(x) Y_\nu(x) = -\pi^{-\frac{1}{2}} G_{13}^{20} \left( x^2 \left| \begin{array}{c} \frac{1}{2} + \frac{1}{2}\mu \\ \nu + \frac{1}{2}\mu, \frac{1}{2}\mu, \frac{1}{2}\mu - \nu \end{array} \right. \right)$$

$$I_\nu(x) K_\nu(x) = 2^{-1} \pi^{-\frac{1}{2}} G_{13}^{21} \left( x^2 \left| \begin{array}{c} \frac{1}{2} \\ \nu, 0, -\nu \end{array} \right. \right)$$

$$x^\mu K_\nu(x) J_\nu(x) = \pi^{-1/2} 2^{3\mu/2-1/2} \\ \times G_{04}^{30} \left( \frac{1}{64} x^4 \left| \begin{matrix} 1/4\mu + 1/2\nu, 1/4\mu + 1/2, 1/4\mu, 1/4\mu - 1/2\nu \end{matrix} \right. \right)$$

$$x^\sigma I_\nu(x) K_\mu(x) = 2^{-1} \pi^{-1/2} \\ \times G_{24}^{22} \left[ x^2 \left| \begin{matrix} 1/2\sigma, 1/2\sigma + 1/2 \\ 1/2(\nu + \mu + \sigma), 1/2(\nu + \sigma - \mu), 1/2(\mu + \sigma - \nu), 1/2(\sigma - \nu - \mu) \end{matrix} \right. \right]$$

$$x^\mu H_\nu^{(1)}(x) H_\nu^{(2)}(x) = \pi^{-5/2} 2 \cos \nu\pi \\ \times G_{13}^{31} \left( x^2 \left| \begin{matrix} 1/2 + 1/2\mu \\ 1/2\mu + \nu, 1/2\mu - \nu, 1/2\mu \end{matrix} \right. \right)$$

$$x^\mu K_\nu^2(x) = 2^{-1} \pi^{1/2} G_{13}^{30} \left( x^2 \left| \begin{matrix} 1/2 + 1/2\mu \\ \nu + 1/2\mu, -\nu + 1/2\mu, 1/2\mu \end{matrix} \right. \right)$$

$$x^\sigma K_\nu(x) K_\mu(x) = 2^{-1} \pi^{1/2} \\ \times G_{24}^{40} \left[ x^2 \left| \begin{matrix} 1/2\sigma, 1/2\sigma + 1/2 \\ 1/2(\nu + \mu + \sigma), 1/2(\nu + \sigma - \mu), 1/2(\mu + \sigma - \nu), 1/2(\sigma - \nu - \mu) \end{matrix} \right. \right]$$

$$x^{2\mu} K_{2\nu}(xe^{\pi i/4}) K_{2\nu}(xe^{-\pi i/4}) = 2^{3\mu-3} \pi^{-1/2} \\ \times G_{04}^{40} \left( \frac{1}{64} x^4 \left| \begin{matrix} 1/2\mu, 1/2\mu + 1/2, 1/2\mu + \nu, 1/2\mu - \nu \end{matrix} \right. \right)$$

$$x^l e^{-1/2x} M_{k,m}(x) = \frac{\Gamma(2m+1)}{\Gamma(1/2+k+m)} G_{12}^{11} \left( x \left| \begin{matrix} 1-k+l \\ 1/2+l+m, 1/2+l-m \end{matrix} \right. \right)$$

$$x^l e^{-1/2x} W_{k,m}(x) = G_{12}^{20} \left( x \left| \begin{matrix} l-k+1 \\ m+l+1/2, l-m+1/2 \end{matrix} \right. \right)$$

$$x^l e^{1/2x} W_{k,m}(x) = \frac{1}{\Gamma(1/2+m-k) \Gamma(1/2-m-k)} G_{12}^{21} \left( x \left| \begin{matrix} k+l+1 \\ l-m+1/2, m+l+1/2 \end{matrix} \right. \right)$$



$$e^{-\frac{1}{2}x} W_{k,m}(x) = \pi^{-\frac{1}{2}} x^{\frac{1}{2}} 2^{k-\frac{1}{2}}$$

$$\times G_{24}^{40} \left( 2^{-2} x^2 \left| \begin{array}{c} \frac{1}{4} - \frac{1}{2}k, \frac{3}{4} - \frac{1}{2}k \\ \frac{1}{2} + \frac{1}{2}m, \frac{1}{2} - \frac{1}{2}m, \frac{1}{2}m, -\frac{1}{2}m \end{array} \right. \right)$$

$$e^x W_{k,m}(2x) = \frac{x^{\frac{1}{2}} 2^{-(k+1)} \pi^{-3/2}}{\Gamma(\frac{1}{2} + m - k) \Gamma(\frac{1}{2} - m - k)}$$

$$\times G_{24}^{42} \left( x^2 \left| \begin{array}{c} \frac{1}{4} + \frac{1}{2}k, \frac{3}{4} + \frac{1}{2}k \\ \frac{1}{2}m, \frac{1}{2} + \frac{1}{2}m, -\frac{1}{2}m, \frac{1}{2} - \frac{1}{2}m \end{array} \right. \right)$$

$$W_{k,m}(x) M_{-k,m}(x) = \frac{\pi^{-\frac{1}{2}} \Gamma(1 + 2m)}{\Gamma(\frac{1}{2} - k + m)} G_{24}^{31} \left( \frac{1}{4} x^2 \left| \begin{array}{c} 1 + k, 1 - k \\ \frac{1}{2}, 1, \frac{1}{2} + m, \frac{1}{2} - m \end{array} \right. \right)$$

$$x^l W_{k,m}(2ix) W_{k,m}(-2ix) = \frac{x \pi^{-\frac{1}{2}}}{\Gamma(\frac{1}{2} + m - k) \Gamma(\frac{1}{2} - m - k)}$$

$$\times G_{24}^{41} \left( x^2 \left| \begin{array}{c} \frac{1}{2} + \frac{1}{2}l + k, \frac{1}{2} + \frac{1}{2}l - k \\ \frac{1}{2}l, \frac{1}{2} + \frac{1}{2}l, \frac{1}{2}l + m, \frac{1}{2}l - m \end{array} \right. \right)$$

$$W_{k,m}(x) W_{-k,m}(x)$$

$$= \pi^{-\frac{1}{2}} G_{24}^{40} \left( \frac{1}{4} x^2 \left| \begin{array}{c} k + 1, -k + 1 \\ \frac{1}{2}, 1, m + \frac{1}{2}, -m + \frac{1}{2} \end{array} \right. \right)$$

$${}_2F_1(a, b; c; -x) = \frac{\Gamma(c) x}{\Gamma(a) \Gamma(b)} G_{22}^{12} \left( x \left| \begin{array}{c} -a, -b \\ -1, -c \end{array} \right. \right)$$

$${}_4F_3(a, b, c, d; e, f, l; -x) = \frac{\Gamma(e) \Gamma(f) \Gamma(l)}{\Gamma(a) \Gamma(b) \Gamma(c) \Gamma(d)} x$$

$$\times G_{44}^{14} \left( x \left| \begin{array}{c} -a, -b, -c, -d \\ -1, -e, -f, -l \end{array} \right. \right)$$

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; -x) = \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{j=1}^p \Gamma(a_j)} \times x G_{p, q+1}^{1, p} \left( x \left| \begin{matrix} -a_1, \dots, -a_p \\ -1, -b_1, \dots, -b_q \end{matrix} \right. \right) \quad p \leq q + 1$$

$$E(p; a_r; q; \beta_s; x) = G_{q+1, p}^{p, 1} \left( x \left| \begin{matrix} 1, \beta_1, \dots, \beta_q \\ a_1, \dots, a_p \end{matrix} \right. \right)$$

For further special functions expressible in terms of the  $G$ -function, in particular for combinations of Legendre functions, and also combinations of generalized hypergeometric series, see C.S. Meijer, *Nederl. Akad. Wetensch., Proc.* 43 (1940), 198-210 and 366-378; 44 (1941), 82-92, 186-194, 298-307, 435-451, 590-605, 1062-1070; 49 (1946), 227-235, 344-356, 457-469, 632-641, 765-772, 936-943, 1063-1072, 1164-1175; 55 (1952), 369-379, 483-487; 56 (1953), 43-49, 187-193.

#### Hypergeometric series of several variables

See also H.T.F. I Chapter V.

Hypergeometric series of two variables. In all double sums  $m$  and  $n$  run for 0 to  $\infty$ .

$$F_1(a; \beta, \beta'; \gamma; x, y) = \sum \frac{(a)_{m+n} (\beta)_m (\beta')_n}{(\gamma)_{m+n} m! n!} x^m y^n$$

$$F_2(a; \beta, \beta'; \gamma, \gamma'; x, y) = \sum \frac{(a)_{m+n} (\beta)_m (\beta')_n}{(\gamma)_m (\gamma')_n m! n!} x^m y^n$$

$$F_3(a, a'; \beta, \beta'; \gamma; x, y) = \sum \frac{(a)_m (a')_n (\beta)_m (\beta')_n}{(\gamma)_{m+n} m! n!} x^m y^n$$

$$F_4(a, \beta; \gamma, \gamma'; x, y) = \sum \frac{(a)_{m+n} (\beta)_{m+n}}{(\gamma)_m (\gamma')_n m! n!} x^m y^n$$

$$\Phi_1(a, \beta, \gamma; x, y) = \sum \frac{(a)_{m+n} (\beta)_m}{(\gamma)_{m+n} m! n!} x^m y^n$$

$$\Phi_2(\beta, \beta', \gamma; x, y) = \sum \frac{(\beta)_m (\beta')_n}{(\gamma)_{m+n} m! n!} x^m y^n$$

$$\Phi_3(\beta, \gamma; x, y) = \sum \frac{(\beta)_m}{(\gamma)_{m+n} m! n!} x^m y^n$$

$$\Psi_1(a, \beta, \gamma, \gamma'; x, y) = \sum \frac{(a)_{m+n} (\beta)_m}{(\gamma)_m (\gamma')_n m! n!} x^m y^n$$

$$\Psi_2(a, \gamma, \gamma'; x, y) = \sum \frac{(a)_{m+n}}{(\gamma)_m (\gamma')_n m! n!} x^m y^n$$

$$\Xi_1(a, a', \beta, \gamma; x, y) = \sum \frac{(a)_m (a')_n (\beta)_m}{(\gamma)_{m+n} m! n!} x^m y^n$$

$$\Xi_2(a, \beta, \gamma; x, y) = \sum \frac{(a)_m (\beta)_m}{(\gamma)_{m+n} m! n!} x^m y^n$$

For other hypergeometric series of two variables see H.T.F. I sec. 5.7.1.

Hypergeometric series of several variables. All summations run from 0 to  $\infty$ .

$$F_A(a; \beta_1, \dots, \beta_n; \gamma_1, \dots, \gamma_n; z_1, \dots, z_n)$$

$$= \sum \frac{(a)_{m_1+\dots+m_n} (\beta_1)_{m_1} \dots (\beta_n)_{m_n}}{(\gamma_1)_{m_1} \dots (\gamma_n)_{m_n} m_1! \dots m_n!} z_1^{m_1} \dots z_n^{m_n}$$

$$\Phi_2(\beta_1, \dots, \beta_n; \gamma; z_1, \dots, z_n) = \sum \frac{(\beta_1)_{m_1} \dots (\beta_n)_{m_n}}{(\gamma)_{m_1+\dots+m_n} m_1! \dots m_n!} z_1^{m_1} \dots z_n^{m_n}$$

$$\Psi_2(a; \gamma_1, \dots, \gamma_n; z_1, \dots, z_n) = \sum \frac{(a)_{m_1+\dots+m_n}}{(\gamma_1)_{m_1} \dots (\gamma_n)_{m_n} m_1! \dots m_n!} z_1^{m_1} \dots z_n^{m_n}$$

**Elliptic functions and integrals**

See also H.T.F. II Chapter XIII.

Complete elliptic integrals

$$K(k) = \int_0^{\frac{1}{2}\pi} (1 - k^2 \sin^2 \phi)^{-\frac{1}{2}} d\phi = \frac{1}{2}\pi {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right)$$

$$E(k) = \int_0^{\frac{1}{2}\pi} (1 - k^2 \sin^2 \phi)^{\frac{1}{2}} d\phi = \frac{1}{2}\pi {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; k^2\right)$$

Theta functions

$$\theta_0(v|\tau) = (-i\tau)^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} e^{-i\pi(v - \frac{1}{2} + n)^2/\tau}$$

$$\theta_1(v|\tau) = (-i\tau)^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} (-1)^n e^{-i\pi(v - \frac{1}{2} + n)^2/\tau}$$

$$\theta_2(v|\tau) = (-i\tau)^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} (-1)^n e^{-i\pi(v + n)^2/\tau}$$

$$\theta_3(v|\tau) = (-i\tau)^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} e^{-i\pi(v + n)^2/\tau}$$

$$\theta_4(v|\tau) = \theta_0(v|\tau).$$

The series given here are connected with the definitions given in H.T.F. II equations 13.19(10) to (13) by means of Jacobi's imaginary transformation, see H.T.F. II equations 13.22(8).

Modified theta functions

$$\hat{\theta}_0(v|\tau) = (-i\tau)^{-\frac{1}{2}} \left[ \sum_{n=0}^{\infty} e^{-i\pi(v + \frac{1}{2} + n)^2/\tau} - \sum_{n=-1}^{-\infty} e^{-i\pi(v + \frac{1}{2} + n)^2/\tau} \right]$$

$$\hat{\theta}_2(v|\tau) = (-i\tau)^{-\frac{1}{2}} \left[ \sum_{n=0}^{\infty} (-1)^n e^{-i\pi(v + n)^2/\tau} - \sum_{n=-1}^{-\infty} (-1)^n e^{-i\pi(v + n)^2/\tau} \right]$$

$$\hat{\theta}_3(v|\tau) = (-i\tau)^{-\frac{1}{2}} \left[ \sum_{n=0}^{\infty} e^{-i\pi(v + n)^2/\tau} - \sum_{n=-1}^{-\infty} e^{-i\pi(v + n)^2/\tau} \right].$$

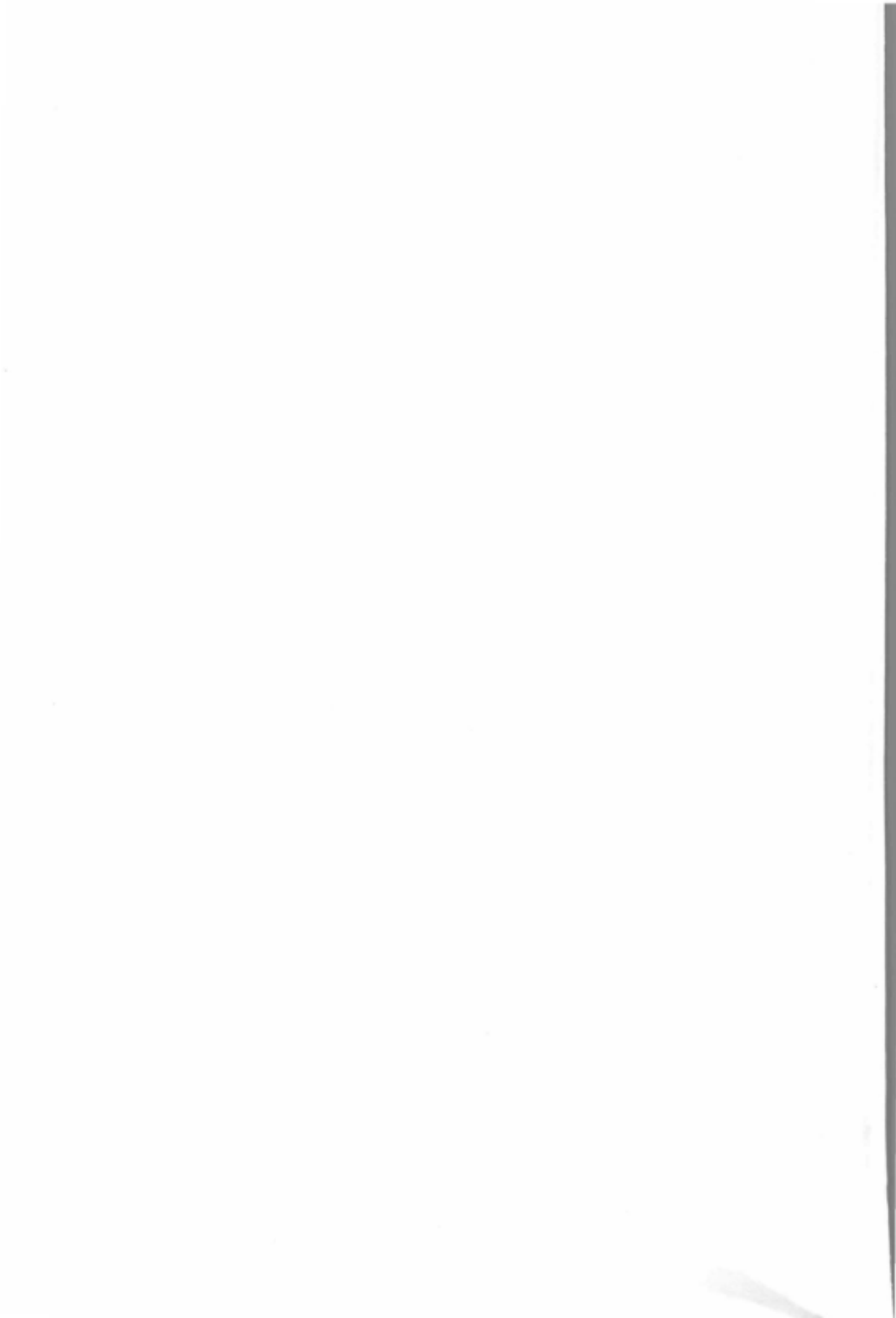
**Miscellaneous functions**

See also H.T.F. III Chapter XVIII.

$$\mu(x, a) = \int_0^{\infty} \frac{x^s s^a}{\Gamma(s+1)} ds$$

$$\nu(x) = \int_0^{\infty} \frac{x^s}{\Gamma(s+1)} ds$$

$$\nu(x, a) = \int_0^{\infty} \frac{x^{s+a}}{\Gamma(s+a+1)} ds = \int_a^{\infty} \frac{x^s}{\Gamma(s+1)} ds$$



## INDEX OF NOTATIONS

### B

$\text{bei}(x)$ ,  $\text{bei}_\nu(x)$ ,  $\text{ber}(x)$ ,  $\text{ber}_\nu(x)$   
Kelvin's functions, 427

### C

Cauchy Principal Value, 423  
 $\text{ci}(x)$ ,  $\text{Ci}(x)$  Cosine integrals, 430  
 $C(x)$  Fresnel integral, 431  
 $C_\nu^\nu(x)$  Gegenbauer polynomial, 424

### D

$D_n(z)$ ,  $D_\nu(z)$  Parabolic cylinder functions, 430

### E

$E(k)$  Complete elliptic integral, 446  
 $E(p; \alpha_r; q; \rho_s; x)$  MacRobert's  
E-function, 429  
 $\text{Ei}(-x)$ ,  $\text{Ei}^+(x)$ ,  $\text{Ei}^-(x)$ ,  $\overline{\text{Ei}}(x)$   
Exponential integrals, 430  
 $\text{Erf}(x)$ ,  $\text{Erfc}(x)$  Error functions, 431  
 $\text{E}_\nu(z)$  Weber's function, 428

### F

$F(a, b; c; z)$ ,  ${}_mF_n(a_1, \dots, a_m; \gamma_1, \dots, \gamma_n; z)$   
Hypergeometric series, 429  
 $F_1(\dots; x, y)$ ,  $\dots$ ,  $F_4(\dots; x, y)$  Hypergeometric series of two variables, 444

$F_A(\dots; z_1, \dots, z_n)$

Lauricella's series, 445

$\mathfrak{F}_c$ ,  $\mathfrak{F}_e$ ,  $\mathfrak{F}_s$  Fourier transforms, xi

### G

$G_{pq}^{mn}(x)$  Meijer's G-function, 434

### H

H.T.F. 423  
 $H_n(x)$ ,  $\text{He}_n(x)$  Hermite polynomials, 425  
 $H_\nu^{(1)}(z)$ ,  $H_\nu^{(2)}(z)$  Bessel functions of the third kind, 427  
 $\mathbf{H}_\nu(z)$  Struve's function, 428  
 $\mathfrak{H}_\nu$  Hankel transform, 3

### I

$I_\nu(z)$  modified Bessel function of the first kind, 427  
 $I_x(p, q)$  Incomplete beta function, 429

### J

$J_\nu(z)$  Bessel function of the first kind, 426  
 $Ji_\nu(x)$  Bessel integral function, 427  
 $\mathbf{J}_\nu(z)$  Anger's function, 428

## K

- $k_{2\nu}(z)$  Bateman's function, 430  
 $\text{kei}(x)$ ,  $\text{kei}_\nu(x)$ ,  $\text{ker}(x)$ ,  $\text{ker}_\nu(x)$   
 Modified Kelvin functions, 427  
 $K(k)$  Complete elliptic integral, 446  
 $K_\nu(z)$  modified Bessel function of the third kind, 427  
 $\mathfrak{K}_\nu$  K-transform, 121

## L

- $\text{li}(z)$  Logarithmic integral, 430  
 $L_2(z)$  Euler's dilogarithm, 425  
 $L_n(z)$ ,  $L_n^\alpha(z)$  Laguerre polynomials, 425  
 $\mathbf{L}_\nu(z)$  Modified Struve function, 428  
 $\mathfrak{L}$  Laplace transform, xi

## M

- $M_{\kappa,\mu}(z)$  Whittaker's confluent hypergeometric function, 430  
 $\mathfrak{M}$  Mellin transform, xi

## O

- $O_n(x)$  Neumann's polynomial, 427

## P

- $p_n(x; a)$  Charlier polynomial, 425  
 $P_n(x)$  Legendre polynomial, 424  
 $P_n^{(\alpha, \beta)}(x)$  Jacobi polynomials, 424  
 $P_\nu(z)$ ,  $P_\nu^\mu(z)$ ,  $P_\nu^\mu(x)$  Legendre functions of the first kind, 426

## Q

- $Q_\nu(z)$ ,  $Q_\nu^\mu(z)$ ,  $Q_\nu^\mu(x)$  Legendre functions of the second kind, 426

## R

- $\mathfrak{R}_\mu$  Fractional integral, 181

## S

- $\text{si}(x)$ ,  $\text{Si}(x)$  Sine integrals, 430  
 $s_{\mu,\nu}(z)$ ,  $S_{\mu,\nu}(z)$  Lommel's functions, 428  
 $S(x)$  Fresnel integral, 431  
 $S_n(b_1, \dots, b_4; z)$ , 429  
 $\mathfrak{S}$ ,  $\mathfrak{S}_\rho$  Stieltjes transforms, 213

## T

- $T_n(x)$  Tchebichef polynomial, 424

## U

- $U_n(x)$  Tchebichef polynomial, 424  
 $U_\nu(w, z)$  Lommel's function of two variables, 428

## V

- $V_\nu^{(b)}(z)$ , 427  
 $V_\nu(w, z)$  Lommel's function of two variables, 428

## W

- $W_{\kappa,\mu}(z)$  Whittaker's confluent hypergeometric function, 430  
 $W_\nu^{(b)}(z)$ , 427  
 $\mathfrak{W}_\mu$  Fractional integral, 181

## X

- $X_\nu^{(b)}(z)$ , 427



## Y

$Y_\nu(z)$  Bessel function of the second kind, 426

$\mathfrak{Y}_\nu$  Y-transforms, 95

## Z

$Z_\nu^{(b)}$ , 427

## GREEK LETTERS

$B(x, y)$  Beta function, 425

$B_x(p, q)$  Incomplete beta function, 429

$\Gamma(z)$  Gamma function, 425

$\gamma(\alpha, x)$ ,  $\Gamma(\alpha, x)$  Incomplete gamma functions, 431

$\zeta(s)$ ,  $\zeta(z, a)$  Zeta functions, 426

$\theta_0(v|r), \dots, \theta_4(v|r), \hat{\theta}_0(v|r), \dots, \hat{\theta}_3(v|r)$   
Theta functions, 446

$\mu(x, a)$ , 447

$\nu(x)$ ,  $\nu(x, a)$ , 447

$\Phi(z, s, v)$ , 426

$\Phi(a; c; z)$  Confluent hypergeometric series, 429

$\Phi_1(\dots; x, y), \dots, \Phi_3(\dots; x, y)$  Confluent hypergeometric series of two variables, 444 ff.

$\Phi_2(\dots; z_1, \dots, z_n)$  Confluent hypergeometric series of  $n$  variables, 445

$\psi(z)$  Logarithmic derivative of the gamma function, 425

$\Psi_1(\dots; x, y), \Psi_2(\dots; x, y)$  Confluent hypergeometric series of two variables, 445

$\Psi_2(\dots; z_1, \dots, z_n)$  Confluent hypergeometric series of  $n$  variables, 445

$\xi(t)$ , 426

$\Xi_1(\dots; x, y), \Xi_2(\dots; x, y)$   
Confluent hypergeometric series of two variables, 445

## MISCELLANEOUS NOTATIONS

$\binom{\alpha}{\beta}$  binomial coefficient, 424

$(a)_\nu = \Gamma(a + \nu)/\Gamma(a)$ , 423 ff.

$C, \gamma$  Euler-Mascheroni constant, 424

$\operatorname{sgn} x$ , 424

$[x]$  largest integer  $\leq x$

$\operatorname{Re} z$  real part of  $z$  (complex)

$\operatorname{Im} z$  imaginary part of  $z$  (complex)

$|z|$  modulus of  $z$  (complex)

$\arg z$  argument (or phase) of  $z$  (complex)

$\oint$  Cauchy Principal Value, 423