

ON SOME DUAL INTEGRAL EQUATIONS

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We develop an elementary method to solve the dual integral equations of the type

$$\int_0^{\infty} h_1(xt) \phi(t) dt = f(x), \quad 0 < x < 1$$

$$\int_0^{\infty} h_2(xt) \phi(t) dt = g(x), \quad x < 1$$

where h_1 and h_2 are Bessel functions of the first and the second kind. First we consider the dual equations with $g(x)=0$ and determine an appropriate form of the unknown function ϕ in terms of an arbitrary function. Then this arbitrary function is evaluated so that ϕ satisfies the equations. Similarly the second set of dual equation with $f(x)=0$ is solved and combining the solutions of these two sets the solution of the original set of dual equations is obtained.

1. INTRODUCTION

In this paper we develop an elementary method for solving certain dual integral equations of the type

$$\left. \begin{aligned} \int_0^{\infty} h_1(xt) \phi(t) dt &= f(x), \quad 0 < x < 1 \\ \int_0^{\infty} h_2(xt) \phi(t) dt &= g(x), \quad 1 < x < \infty. \end{aligned} \right\} \quad \dots(1.1)$$

The method consists of decomposing the system of eqns. (1.1) into two sets of dual integral equations. We derive the solution of the first set, which is the system of eqns. (1.1) with $g(x) = 0$, and then find the solution of the second set of equations,

the system (1.1) with $f(x) = 0$. Combining the solutions of these two sets gives us the solution of (1.1). To solve each of the two above mentioned sets of integral equations, an important part of our technique is to establish a suitable form of the function ϕ , in terms of an appropriately defined function. We then find the value of this unknown function, so that ϕ satisfies both the integral equations of that set, thus establishing its solution.

We make extensive use of the Erdelyi-Kober operators and their properties. This type of approach for solving dual integral equations has been used widely in a variety of methods (Gordon 1954, Noble 1958), though not expressed in the same notations in all the cases. A good account of such methods can be found in Erdelyi and Sneddon (1962) and in Sneddon (1966, pp. 84-96).

To illustrate our technique, we shall first obtain solutions of some simple pairs of dual integral equations. Then we shall apply the procedure to solve dual integral equations of Titchmarsh's type and integral equations whose kernels involve Y - and H -functions, where Y is the usual Bessel function of second kind and H is Struve's function. As is the usually the case, the analysis used in solving the dual integral equations in this paper is purely formal.

2. SOME SIMPLE PAIRS OF DUAL INTEGRAL EQUATIONS

(i) We first consider the dual integral equations of Beltrami's type,

$$\int_0^{\infty} t^{-1} \phi(t) J_0(xt) dt = f(x), \quad 0 < x < 1 \quad \dots(2.1a)$$

$$\int_0^{\infty} \phi(t) J_0(xt) dt = 0, \quad 1 < x < \infty \quad \dots(2.1b)$$

which arise in the problem of the electrified disk. We shall write (2.1b) as

$$\int_0^{\infty} t^{-1/2} \phi(t) J_0(xt) (xt)^{1/2} dt = \alpha(x) H(1-x)$$

where $0 < x < \infty$, $\alpha(x)$ an arbitrary function and H being the Heaviside function. Then by the Hankel inversion,

$$t^{-1/2} \phi(t) = \int_0^1 \alpha(x) J_0(xt) (xt)^{1/2} dx.$$

If we let (Erdelyi *et al.* 1954a, p. 11)

$$J_0(xt) = \frac{2}{\pi} \int_0^x \frac{\cos(ut)}{\sqrt{x^2 - u^2}} du$$

then

$$\begin{aligned} t^{-1/2} \phi(t) &= \frac{2}{\pi} \int_0^1 \alpha(x) (xt)^{1/2} dx \int_0^x \frac{\cos(ut)}{\sqrt{x^2 - u^2}} du \\ &= \frac{2}{\pi} t^{1/2} \int_0^1 \cos(ut) du \int_u^1 \frac{x^{1/2} \alpha(x)}{\sqrt{x^2 - u^2}} dx \end{aligned}$$

or,

$$\phi(t) = t \int_0^1 \beta(u) \cos(ut) du \quad \dots(2.2)$$

where

$$\beta(u) = \frac{2}{\pi} \int_u^1 \frac{x^{1/2} \alpha(x)}{\sqrt{x^2 - u^2}} dx.$$

To find an appropriate value of $\beta(u)$, so that ϕ satisfies (2.1a) as well, substitute ϕ in (2.1a) to obtain

$$\begin{aligned} f(x) &= \int_0^\infty J_0(xt) dt \int_0^1 \beta(u) \cos(ut) du, \quad 0 < x < 1 \\ &= \int_0^1 \beta(u) du \int_0^\infty \cos(ut) J_0(xt) dt \\ &= \int_0^x \frac{\beta(u)}{\sqrt{x^2 - u^2}} du, \quad (\text{Erdelyi } et al. 1954a, p. 43) \end{aligned}$$

whence (Sneddon 1966, p. 41)

$$\beta(u) = \frac{2}{\pi} \frac{d}{du} \int_0^u \frac{xf(x)}{\sqrt{u^2 - x^2}} dx.$$

Thus from (2.2)

$$\phi(t) = \frac{2}{\pi} t \int_0^1 \cos(ut) du \frac{d}{du} \int_0^u \frac{xf(x)}{\sqrt{u^2-x^2}} dx$$

which now satisfies both (2.1a) and (2.1b), giving us the solution of the system (2.1). A more familiar form of this solution is obtained by integrating u -integral by parts and simplifying, that is

$$\phi(t) = \frac{2}{\pi} t \left\{ \cos t \int_0^1 \frac{xf(x)}{\sqrt{1-x^2}} dx + t \int_0^1 \frac{v dv}{\sqrt{1-v^2}} \int_0^1 u \sin(ut) \cdot f(uv) du \right\}.$$

If $f(x) = 1$, $0 < x < 1$, the solution of the corresponding dual integral eqns. (2.1), becomes

$$\phi(t) = \frac{2}{\pi} \sin t$$

another well-known solution of this particular case.

(ii) Let

$$\left. \begin{aligned} \int_0^\infty \phi(t) \cos(xt) dt &= f(x), \quad 0 < x < 1 \\ \int_0^\infty \phi(t) \sin(xt) dt &= g(x), \quad 1 < x < \infty. \end{aligned} \right\} \quad \dots(2.3)$$

The dual integral equations of this type, with trigonometrical kernels, arise frequently in the analysis of mixed boundary value problems in the plane; such as crack problems in the two-dimensional theory of elasticity.

We shall consider two cases of (2.3).

Case a — First consider

$$\int_0^\infty \phi(t) \cos(xt) dt = f(x), \quad 0 < x < 1 \quad \dots(2.4a)$$

$$\int_0^\infty \phi(t) \sin(xt) dt = 0, \quad 1 < x < \infty. \quad \dots(2.4b)$$

Equation (2.4b) can be written as

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} \phi(t) \sin(xt) dt = \alpha(x) H(1-x), \quad 0 < x < \infty$$

for some $\alpha(x)$, H being the Heaviside function. By the usual sine-transformation

$$\phi(t) = \sqrt{\frac{2}{\pi}} \int_0^1 \alpha(x) \sin(xt) dx.$$

Now if we let, (Erdelyi *et al.* 1954b, p. 7)

$$\sin(xt) = t \int_0^x \frac{u J_0(ut)}{\sqrt{x^2 - u^2}} du.$$

Then

$$\begin{aligned} \phi(t) &= \sqrt{\frac{2}{\pi}} t \int_0^1 \alpha(x) dx \int_0^x \frac{u J_0(ut)}{\sqrt{x^2 - u^2}} du \\ &= \sqrt{\frac{2}{\pi}} t \int_0^1 u J_0(ut) du \int_u^1 \frac{\alpha(x)}{\sqrt{x^2 - u^2}} dx \end{aligned}$$

or

$$\phi(t) = t \int_0^1 \beta(u) J_0(ut) du \quad \dots(2.5)$$

where

$$\beta(u) = \sqrt{\frac{2}{\pi}} u \int_u^1 \frac{\alpha(x)}{\sqrt{x^2 - u^2}} dx.$$

No we shall write (2.4a) as

$$f(x) = \frac{d}{dx} \int_0^{\infty} t^{-1} \phi(t) \sin(xt) dt, \quad 0 < x < 1.$$

Substituting for ϕ from (2.5), we obtain

$$\begin{aligned} f(x) &= \frac{d}{dx} \int_0^{\infty} \sin(xt) dt \int_0^1 \beta(u) J_0(ut) du \\ &= \frac{d}{dx} \int_0^1 \beta(u) du \int_0^{\infty} \sin(xt) J_0(ut) dt \\ &= \frac{d}{dx} \int_0^1 \frac{\beta(u)}{\sqrt{x^2 - u^2}} du, \quad 0 < x < 1 \end{aligned}$$

due to (Erdelyi 1954, p. 10) whence (Sneddon 1966, p. 41)

$$\beta(u) = \frac{2}{\pi} u \int_0^u \frac{f(x)}{\sqrt{u^2 - x^2}} dx.$$

Hence from eqn. (2.5)

$$\phi(t) = \frac{2}{\pi} t \int_0^1 u J_0(ut) du \int_0^u \frac{f(x)}{\sqrt{u^2 - x^2}} dx \quad \dots(2.6)$$

which gives us the solution of the integral equations (2.4).

Case b — Let

$$\int_0^{\infty} \phi(t) \cos(xt) dt = 0, \quad 0 < x < 1 \quad \dots(2.7a)$$

$$\int_0^{\infty} \phi(t) \sin(xt) dt = g(x), \quad 1 < x < \infty. \quad \dots(2.7b)$$

Once again, from (2.7a), using the cosine transformation

$$\phi(t) = \sqrt{\frac{2}{\pi}} \int_1^{\infty} \alpha(x) \cos(xt) dx$$

for an appropriately defined function $\alpha(x)$. We use the representation (Erdelyi *et al.* 1954b, p. 7)

$$\cos (xt) = t \int_x^{\infty} \frac{u J_0 (ut)}{\sqrt{u^2 - x^2}} du$$

in the above expression and change the order of integration to obtain

$$\phi (t) = t \int_1^{\infty} \beta (u) J_0 (ut) du \quad \dots(2.8)$$

where

$$\beta (u) = \sqrt{\frac{2}{\pi}} \int_1^u \frac{\alpha (x)}{\sqrt{u^2 - x^2}} dx.$$

Now eqn. (2.7b) is equivalent to

$$g (x) = - \frac{d}{dx} \int_0^{\infty} t^{-1} \phi (t) \cos (xt) dt, \quad 1 < x < \infty$$

and substituting for ϕ from (2.8), get

$$\begin{aligned} g (x) &= - \frac{d}{dx} \int_0^{\infty} \cos (xt) dt \int_1^{\infty} \beta (u) J_0 (ut) du \\ &= - \frac{d}{dx} \int_1^{\infty} \beta (u) du \int_0^{\infty} \cos (xt) J_0 (ut) dt \\ &= - \frac{d}{dx} \int_x^{\infty} \frac{\beta (u)}{\sqrt{u^2 - x^2}} du, \quad 1 < x < \infty. \end{aligned}$$

Then by inversion (Sneddon 1966, p. 42)

$$\beta (u) = \frac{2}{\pi} u \int_u^{\infty} \frac{g (x)}{\sqrt{x^2 - u^2}} dx.$$

Hence, from (2.8)

$$\phi (t) = \frac{2}{\pi} t \int_1^{\infty} u J_0 (ut) du \int_u^{\infty} \frac{g (x)}{\sqrt{x^2 - u^2}} dx \quad \dots(2.9)$$

gives the solution of eqns. (27). Now combining (2.6) and (2.9), we obtain

$$\begin{aligned}\phi(t) = & \frac{2}{\pi} t \int_0^1 u J_0(ut) du \int_0^u \frac{f(x) dx}{\sqrt{u^2 - x^2}} \\ & + \frac{2}{\pi} t \int_1^\infty u J_0(ut) du \int_u^\infty \frac{g(x)}{\sqrt{x^2 - u^2}} dx\end{aligned}\quad \dots(2.10)$$

which is the solution of the dual integral equations (2.3).

The solutions of a variety of dual integral equations with trigonometrical kernels can be deduced from the solution (2.10); we shall mention only one important case below.

Write (2.4a) as

$$\begin{aligned}\int_0^\infty t^{-1} \phi(t) \sin(xt) dt &= \int_0^x f(x) dx, \quad 0 < x < 1 \\ &= F(x), \quad 0 < x < 1\end{aligned}$$

then the pair of integral equations (2.4) become

$$\begin{aligned}\int_0^\infty t^{-1} \phi(t) \sin(xt) dt &= F(x), \quad 0 < x < 1 \\ \int_0^\infty \phi(t) \sin(xt) dt &= 0, \quad 1 < x < \infty.\end{aligned}$$

The solution of this pair of equations is then derived from (2.6) to give (Sneddon 1966, p. 106)

$$\phi(t) = \frac{2}{\pi} t \int_0^1 u J_0(ut) du \int_0^u \frac{F'(x)}{\sqrt{u^2 - x^2}} dx.$$

If we put $F(x) = x$, then

$$\phi(t) = J_1(x)$$

giving the solution of the dual integral equations

$$\int_0^{\infty} t^{-1} \phi(t) \sin(xt) dt = x, \quad 0 < x < 1$$

$$\int_0^{\infty} \phi(t) \sin(xt) dt = 0, \quad 1 < x < \infty.$$

3. DUAL INTEGRAL EQUATIONS OF TITCHMARSH TYPE

Consider

$$\int_0^{\infty} t^{-2\alpha} \phi(t) J_{\mu}(xt) dt = f(x), \quad 0 < x < 1$$

$$\int_0^{\infty} t^{-2\beta} \phi(t) J_{\nu}(xt) dt = g(x), \quad 1 < x < \infty. \quad \dots(3.1)$$

Case a—Let

$$\int_0^{\infty} t^{-2\alpha} \phi(t) J_{\mu}(xt) dt = f(x), \quad 0 < x < 1 \quad \dots(3.2a)$$

$$\int_0^{\infty} t^{-2\beta} \phi(t) J_{\nu}(xt) dt = 0, \quad 1 < x < \infty. \quad \dots(3.2b)$$

Write (3.2b) as

$$\int_0^{\infty} t^{-2\beta-1/2} \phi(t) J_{\nu}(xt) (xt)^{1/2} dt = \alpha(x) H(1-x), \quad \nu > -\frac{1}{2}$$

and by Hankel inversion,

$$t^{-2\beta-1/2} \phi(t) = \int_0^1 \alpha(x) J_{\nu}(xt) (xt)^{1/2} dx, \quad \nu > -\frac{1}{2}.$$

Now using expression

$$J_{\nu}(xt) = \frac{2^{\eta-\nu+1}}{\Gamma(\nu-\eta)} x^{-\nu} t^{\nu-\eta} \int_0^x u^{\eta+1} (x^2-u^2)^{\nu-\eta-1} J_{\eta}(ut) du$$

where $\eta_1 = \frac{1}{2}\mu + \frac{1}{2}\nu - \alpha + \beta$ and $-1 < \eta_1 < \nu$, in the above integral and then changing the order of integration we obtain

$$t^{-2\mu-1/2} \phi(t) = \frac{2^{\eta-\nu+1}}{\Gamma(\nu-\eta_1)} t^{-\nu-\eta_1+1/2} \int_0^1 u^{\eta+1} J_0(ut) du \\ \times \int_u^1 x^{1/2-\nu} \alpha(x) (x^2-u^2)^{\nu-\eta-1} dx$$

or

$$\phi(t) = t^{-2\mu+\nu-\eta+1} \int_0^1 \beta(u) J_\eta(ut) du \quad \dots(3.3)$$

say.

Substituting in (3.2a),

$$f(x) = \int_0^\infty t^{\eta-\nu+1} J_\mu(xt) dt \int_0^1 \beta(u) J_\eta(ut) du, \quad 0 < x < 1 \\ = \int_0^1 \beta(u) du \int_0^\infty t^{\eta-\nu+1} J_\mu(xt) J_\eta(ut) dt \\ = \frac{2^{\eta-\mu+1}}{\Gamma(\mu-\eta_1)} x^{-\mu} \int_0^x \beta(u) u^\eta (x^2-u^2)^{\mu-\eta-1} du, \quad 0 < x < 1$$

where

$$\eta_1 = \frac{1}{2}\mu + \frac{1}{2}\nu - \alpha + \beta, \quad -1 < \eta_1 < \mu, \text{ and } -1 < \eta_1 < \nu. \quad \dots(3.4)$$

Or,

$$f(x) = 2^{\eta-\mu} I_{\eta-1/2\mu, \mu-\eta} [x^{\mu-\eta-1} \beta(x)]$$

using Erdelyi-Kober fractional integral operator I , defined in Sneddon (1966, p. 48). Then, by the usual inversion

$$\beta(x) = 2^{\mu-\eta} x^{\eta-\mu+1} I_{\eta-1/2\mu, \mu-\eta}^{-1} [f(x)] \\ = 2^{\mu-\eta} x^{\eta-\mu+1} I_{1/2\mu, \eta-\mu} [f(x)].$$

Hence from (3.3)

$$\begin{aligned}\phi(t) &= 2^{\mu-\eta} t^{2\beta+\eta-1} \int_0^1 u^{\eta-\mu+1} J_{\eta}(ut) I_{1/2\mu, \eta-\mu}[f] du, \\ &\equiv \phi_1(t)\end{aligned}\quad \dots(3.5)$$

which is the solution of the dual integral equations (3.2). When the conditions (3.4) are not satisfied, (3.5) still gives a solution of the system (3.2), but the operator I has to be interpreted appropriately.

Case b — Let

$$\int_0^{\infty} t^{-2\alpha} \phi(t) J_{\mu}(xt) dt = 0, \quad 0 < x < 1 \quad \dots(3.6a)$$

$$\int_0^{\infty} t^{-2\beta} \phi(t) J_{\nu}(xt) dt = g(x), \quad 1 < x < \infty. \quad \dots(3.6b)$$

As before, by the Hankel inversion and for an appropriate $\alpha(x)$, (3.6a) gives

$$t^{-2\alpha-1/2} \phi(t) = \int_1^{\infty} \alpha(x) J_{\mu}(xt) (xt)^{1/2} dx, \quad \mu > -\frac{1}{2}.$$

If we use the representation (Erdelyi *et al.* 1954b, p. 25).

$$J_{\mu}(xt) = \frac{2^{\mu-\eta+1}}{\Gamma(\eta-\mu)} x^{\mu} t^{\eta-\mu} \int_x^{\infty} u^{1-\eta} (u^2-x^2)^{\eta-\mu-1} J_{\eta}(ut) du$$

where $\eta = \frac{1}{2}\mu + \frac{1}{2}\nu - \alpha + \beta$ and $\mu < \eta < 2\mu + 3/2$, in the above integral and change the order of integration, we then obtain

$$\begin{aligned}\phi(t) &= \frac{2^{\mu-\eta+1}}{\Gamma(\eta-\mu)} t^{2\alpha+\eta-1} \int_1^{\infty} u^{1-\eta} J_{\eta}(ut) du \int_1^u \alpha(x) x^{1/2+\mu} \\ &\quad \times (u^2-x^2)^{\eta-\mu-1} dx \\ &= t^{2\alpha+\eta-1} \int_1^{\infty} \beta(u) J_{\eta}(ut) du, \quad \text{say.}\end{aligned}\quad \dots(3.7)$$

Substituting in (3.6b),

$$\begin{aligned} g(x) &= \int_0^{\infty} t^{2\alpha-2\beta+\eta-\mu+1} J_{\nu}(xt) dt \int_1^{\infty} \beta(u) J_{\eta}(ut) du, \quad x > 1 \\ &= \int_1^{\infty} \beta(u) du \int_0^{\infty} t^{\nu-\eta+1} J_{\nu}(xt) J_{\eta}(ut) dt \\ &= \frac{2^{\nu-\eta+1}}{\Gamma(\eta-\nu)} x^{\nu} \int_x^{\infty} \beta(u) u^{-\eta} (u^2-x^2)^{\eta-\nu-1} du \end{aligned}$$

where $-1 < \nu < \eta$ and $\mu < \eta < 2\mu + 3/2$. Or using the Erdelyi-Kober operator K (Sneddon 1966, p. 50)

$$g(x) = 2^{\nu-\eta} K_{1/2\nu, \eta-\nu} [x^{\eta-\nu-1} \beta(x)]$$

which on inverting gives

$$\beta(x) = 2^{\eta-\nu} x^{\nu-\eta+1} K_{\eta-1/2\nu, \nu-\eta} [g(x)].$$

Hence from (3.7),

$$\begin{aligned} \phi(t) &= 2^{\eta-\nu} t^{2\alpha+\eta-\mu+1} \int_1^{\infty} u^{\nu-\eta+1} J_{\eta}(ut) K_{\eta-1/2\nu, \nu-\eta} [g] du \quad \dots(3.8) \\ &\equiv \phi_2(t) \end{aligned}$$

giving a solution of the system (3.6). Now combining (3.5) and (3.8), we have

$$\phi(t) = \phi_1(t) + \phi_2(t) \quad \dots(3.9)$$

the solution of the dual integral equations (3.1) of Titchmarsh type.

In particular if $\eta - \mu > 0$ and $\nu - \eta > 0$, i.e. $\mu - \nu < 2(\beta - \alpha)$, then (3.9) becomes

$$\begin{aligned} \phi(t) &= \frac{2^{\mu-\eta+1}}{\Gamma(\eta-\mu)} t^{2\alpha+\eta-\mu+1} \int_0^1 x^{1-\eta} J_{\eta}(xt) dx \int_0^x (x^2-u^2)^{\eta-\mu-1} u^{\mu+1} f(u) du \\ &\quad + \frac{2^{\eta-\nu+1}}{\Gamma(\nu-\eta)} t^{2\alpha+\eta-\mu+1} \int_1^{\infty} x^{1+\eta} J_{\eta}(xt) dx \int_x^{\infty} (u^2-x^2)^{\nu-\eta-1} u^{1-\nu} g(u) du. \end{aligned}$$

Also if $\mu = \nu$, then

$$\begin{aligned} \phi(t) = & 2^{\alpha-\beta} t^{\alpha+\beta+1} \left\{ \int_0^1 x^{\beta-\alpha+1} J_{\mu+\beta-\alpha}(xt) I_{1/2\mu, \beta-\alpha}[f] dx \right. \\ & \left. + \int_1^\infty x^{\alpha-\beta+1} J_{\mu+\beta-\alpha}(xt) K_{1/2\mu+\beta-\alpha, \alpha-\beta}[g] dx \right\} \end{aligned} \quad \dots(3.10)$$

is the solution of the system

$$\left. \begin{aligned} \int_0^\infty t^{-2\alpha} \phi(t) J_\mu(xt) dt &= f(x), \quad 0 < x < 1 \\ \int_0^\infty t^{-2\beta} \phi(t) J_\mu(xt) dt &= g(x), \quad 1 < x < \infty. \end{aligned} \right\} \quad \dots(3.11)$$

Further, if $0 < \beta - \alpha < 1$, then (3.10) gives

$$\begin{aligned} \phi(t) = & \frac{2^{1+\alpha-\beta}}{\Gamma(\beta-\alpha)} t^{\beta+\alpha+1} \int_0^1 x^{\alpha-\beta-\mu-1} J_{\mu+\beta-\alpha}(xt) dx \\ & \times \int_0^x u^{\mu+1} (x^2-u^2)^{\beta-\alpha-1} f(u) du \\ & - \frac{2^{1+\alpha-\beta}}{\Gamma(1+\alpha-\beta)} t^{\beta+\alpha+1} \int_1^\infty x^{\mu+\beta-\alpha} J_{\mu+\beta-\alpha}(xt) dx \frac{d}{dx} \\ & \times \int_x^\infty u^{1-\mu} (u^2-x^2)^{\alpha-\beta} g(u) du \end{aligned}$$

a well known solution of the system (3.11).

4. DUAL INTEGRAL EQUATIONS INVOLVING Y-FUNCTIONS

First, we define operators $I_{a,\alpha}(a, x: 2)$ and $K_{a,\alpha}(x, b: 2)$ which are simple generalization of Erdelyi-Kober operators, by the following equations.

$$I_{\eta;\alpha}(a, x; 2)[f] = \begin{cases} \frac{2}{\Gamma(\alpha)} x^{-2(\eta+\alpha)} \int_a^x (x^2-t^2)^{\alpha-1} t^{2\eta+1} f(t) dt, & \alpha > 0 \\ x^{-(2\eta+2\alpha+1)} \frac{d}{dx} \int_a^x (x^2-t^2)^{\alpha-1} t^{2\eta+1} f(t) dt, & -1 < \alpha < 0 \end{cases}$$

and

$$K_{\eta;\alpha}(x, b; 2)[f] = \begin{cases} \frac{2}{\Gamma(\alpha)} x^{2\eta} \int_x^b (t^2-x^2)^{\alpha-1} t^{1-2\alpha-2\eta} f(t) dt, & \alpha > 0 \\ -\frac{x^{2\eta-1}}{\Gamma(1+\alpha)} \frac{d}{dx} \int_x^b (t^2-x^2)^{\alpha} t^{1-2\alpha-2\eta} f(t) dt, & -1 < \alpha < 0. \end{cases}$$

Note that

$$I_{\eta;\alpha}(0, x; 2) = I_{\eta;\alpha} \quad \text{and} \quad K_{\eta;\alpha}(x, \infty; 2) = K_{\eta;\alpha}.$$

It can be easily shown that

$$I_{\eta;\alpha}^{-1}(a, x; 2) = I_{\eta;\alpha,-\alpha}(a, x; 2)$$

and

$$K_{\eta;\alpha}^{-1}(x, b; 2) = K_{\eta+\alpha,-\alpha}(x, b; 2)$$

defining the inverse operators.

Now consider the dual integral equations

$$\int_0^\infty t^{-2\alpha} \phi(t) Y_\mu(xt) dt = f(x), \quad 0 < x < 1 \quad \dots(4.1)$$

$$\int_0^\infty t^{-2\beta} \phi(t) Y_\nu(xt) dt = g(x), \quad 1 < x < \infty$$

where $2(\alpha - \beta) = \nu - \mu$, $|\mu| < \frac{1}{2}$, and $|\nu| < \frac{1}{2}$.

Case a — Let

$$\int_0^\infty t^{-2\alpha} \phi(t) Y_\mu(xt) dt = f(x), \quad 0 < x < 1 \quad \dots(4.2a)$$

$$\int_0^{\infty} t^{-2\beta} \phi(t) Y_\nu(xt) dt = 0, \quad 1 < x < \infty. \quad \dots(4.2b)$$

We write (4.2b) as

$$\int_0^{\infty} t^{-2\beta-1/2} \phi(t) Y_\nu(xt) (xt)^{1/2} dt = \alpha(x) H(1-x)$$

and by the usual inversion,

$$t^{-2\beta-1/2} \phi(t) = \int_0^1 \alpha(u) H_\nu(ut) (ut)^{1/2} du, \quad |v| < \frac{1}{2} \quad \dots(4.3)$$

where H_ν is Struve's function of order ν . Now substituting the value of ϕ from (4.3) into (4.2a) and simplifying, we obtain

$$\begin{aligned} f(x) &= \int_0^1 \alpha(u) u^{1/2} du \int_0^{\infty} t^{2\beta-2\alpha+1} Y_\mu(xt) H_\nu(ut) dt, \quad 0 < x < 1 \\ &= \int_0^1 \alpha(u) u^{1/2} du \int_u^{\infty} t^{\mu-\nu+1} Y_\mu(xt) H_\nu(ut) dt \end{aligned}$$

since $2(\alpha - \beta) = \nu - \mu$. The t -integral can be evaluated (Erdelyi *et al.* 1954b, p. 114) to give

$$f(x) = \frac{2^{\mu-\nu+1}}{\Gamma(\nu-\mu)} x^\mu \int_x^1 \alpha(u) u^{1/2-\nu} (u^2-x^2)^{\nu-\mu-1} du, \quad 0 < x < 1$$

where $\nu - \mu > 0$. Or,

$$f(x) = 2^{\mu-\nu} K_{1/2\mu, \nu-\mu}(x, 1:2) [u^{\nu-\mu-1/2} \alpha(u)]$$

whence,

$$\alpha(x) = 2^{\nu-\mu} x^{\mu-\nu+1/2} K_{\nu-1/2\mu, \mu-\nu}(x, 1:2) [f].$$

Substituting in (4.3), gives us the solution

$$\phi(t) = 2^{\nu-\mu} t^{2\beta+1} \int_0^1 u^{\mu-\nu+1} H_\nu(ut) K_{\nu-1/2\mu, \mu-\nu}(u, 1:2) [f] du \quad \dots(4.4)$$

of the system (4.2). If $\nu - \mu < 0$, then the operator K is to be interpreted accordingly.

Case b — Let

$$\int_0^{\infty} t^{-2\alpha} \phi(t) Y_\mu(xt) dt = 0, \quad 0 < x < 1 \quad \dots(4.5a)$$

$$\int_0^{\infty} t^{-2\beta} \phi(t) Y_{\nu}(xt) dt g(x), \quad 1 < x < \infty \quad \dots(4.5b)$$

where $2(\alpha - \beta) = \nu - \mu$, $|\nu| < \frac{1}{2}$ and $|\mu| < \frac{1}{2}$. Once again we invert (4.5a), for an appropriately defined function $\alpha(x)$, to get

$$\phi(t) = t^{2\alpha+1} \int_1^{\infty} \alpha(u) H_{\mu}(ut) u^{1/2} du, \quad |\mu| < \frac{1}{2}. \quad \dots(4.6)$$

Substituting in (4.5b) and simplifying, we obtain

$$\begin{aligned} g(x) &= \int_1^{\infty} (\alpha(u)) u^{1/2} du \int_0^{\infty} t^{\nu-\mu+1} Y_{\nu}(xt) H_{\mu}(ut) dt, \quad x > 1 \\ &= \frac{2^{\nu-\mu+1}}{\Gamma(\mu-\nu)} x^{\nu} \int_x^{\infty} \alpha(u) u^{1/2-\mu} (u^2 - x^2)^{\mu-\nu-1} du, \quad \mu - \nu > 0 \\ &= 2^{\nu-\mu} K_{1/2\nu, \mu-\nu} [u^{\mu-\nu-1/2} \alpha(u)] \end{aligned}$$

whence

$$\alpha(x) = 2^{\mu-\nu} x^{\nu-\mu+1/2} K_{\mu-1/2\nu, \nu-\mu} [g].$$

Then from (4.6)

$$\phi(t) = 2^{\mu-\nu} t^{2\alpha+1} \int_1^{\infty} u^{\nu-\mu+1} H_{\mu}(ut) K_{\mu-1/2\nu, \nu-\mu} [g] du \quad \dots(4.7)$$

giving the solution of the dual integral equations (4.5). This is also the solution even if $\nu - \mu > 0$.

Now combining (4.4) and (4.7), we have

$$\begin{aligned} \phi(t) &= 2^{\nu-\mu} t^{2\beta+1} \int_0^1 u^{\mu-\nu+1} H_{\nu}(ut) K_{\nu-1/2\mu, \mu-\nu}(u, 1:2) [f] du \\ &\quad + 2^{\mu-\nu} t^{2\alpha+1} \int_1^{\infty} u^{\nu-\mu+1} H_{\mu}(ut) K_{\mu-1/2\nu, \nu-\mu} [g] du \end{aligned} \quad (4.8)$$

the solution of the system of dual integral equations (4.1). In particular if $0 < \nu - \mu < 1$, then (4.8) becomes

$$\begin{aligned}\phi(t) = & - \frac{2^{v-\mu}}{\Gamma(1+\mu-v)} t^{2\beta+1} \int_0^1 u^v H_v(ut) du \frac{d}{du} \\ & \times \int_u^1 (x^2 - u^2)^{\mu-v} x^{1-\mu} f(x) dx \\ & + \frac{2^{\mu-v}}{\Gamma(\mu-v)} \int_1^\infty u^{1+\mu} H_\mu(ut) du \int_u^\infty (x^2 - u^2)^{v-\mu-1} x^{1-v} g(x) dx.\end{aligned}$$

Also if $-1 < v - \mu < 0$, then

$$\begin{aligned}\phi(t) = & \frac{2^{1-v-\mu}}{\Gamma(\mu-v)} t^{2\beta+1} \int_0^1 u^{v+1} H_v(ut) du \int_u^1 x^{1-\mu} (x^2 - u^2)^{\mu-v-1} f(x) dx \\ & + \frac{2^{\mu-v}}{\Gamma(1+v-\mu)} t^{2\alpha+1} \int_1^\infty u^\mu H_\mu(ut) du \frac{d}{du} \\ & \times \int_u^\infty x^{1-v} (x^2 - u^2)^{v-\mu} g(x) dx.\end{aligned}$$

If we let $f(x) = x^\mu$ and $g(x) = 0$, then

$$\phi(t) = t^{2\alpha} H_{\mu+1}(t), \quad |v - \mu| < 1$$

giving the solution of the system (4.2).

5. DUAL INTEGRAL EQUATIONS WITH H -FUNCTIONS

Consider

$$\begin{aligned}\int_0^\infty t^{-2\alpha} \phi(t) H_\mu(xt) dt &= f(x), \quad 0 < x < 1 \\ \int_0^\infty t^{-2\beta} \phi(t) H_v(xt) dt &= f(x), \quad 1 < x < \infty\end{aligned} \quad \dots(5.1)$$

where $2(\beta - \alpha) = v - \mu$, $|v| < \frac{1}{2}$ and $|\mu| < \frac{1}{2}$.

For deriving the solution of this system, we proceed exactly as before and establish first the solution of the system with $g(x) = 0$, which is

$$\phi(t) = 2^{\mu-\nu} t^{2\mu+1} \int_0^1 u^{\nu-\mu+1} Y_\nu(ut) I_{1/2\mu, \nu-\mu} [f] du$$

$$\equiv \phi_1(t)$$

where I is the Erdelyi-Kober operator. Then we let $f(x) \equiv 0$ in (5.1) and find the solution of the system, which is

$$\phi(t) = 2^{\nu-\mu} t^{2\mu+1} \int_1^\infty u^{\mu-\nu+1} Y_\mu(ut) I_{1/2\nu, \mu-\nu}^{(1, \mu-2)} [f] du$$

$$\equiv \phi_2(t).$$

where $I(1, x: 2)$ is the generalized Erdelyi-Kober operator.

Hence

$$\phi(t) = \phi_1(t) + \phi_2(t)$$

is the solution of the dual integral equations (5.1)

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