Matrix Theorems Addendum PhL 9.9.11

This subject has cooled off for a while, I have my fat "matrix binder" which is a collection of matrix theorems, but every once in a while I come across something I did not include there. So I will add those new theorems in this addendum document.

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A. Some cofactor-related theorems.

1. Expressions for the cofactor and determinant

This item appears in matrix binder section 2 under the section 10.3 Minors and Cofactors (M&M).

I show that the following is one way to write a cofactor

$$\text{cof}(a_{pq}) = \varepsilon_{ij.....q.....\ell}\, a_{1i}\, a_{2j} \,..... a_{p-1x}\, \cancel{(a_{py})}\, a_{p+1z} \,..... \qquad \text{// in general}$$

Remember that the "cofactor of a matrix element" is a signed minor = a signed determinant = a number (not a matrix). In the above line, q and p are some fixed numbers. On the ε, q appears in the pth position.

Stealing from my tensor.doc Appendix A, here is a more precise way to write the above

$$\text{cof}(a_{pq}) = \varepsilon_{i_1 i_2 i_3 \ldots [\cancel{i_p} \to q] \ldots i_N}\; a_{1i_1}\, a_{2i_2} \,..... \cancel{a_{pi_p}} ..... a_{Ni_N}$$

The factor a_{pi_p} is *missing* from the set of factors (it is crossed out), so there are only N-1 a factors, and the index that would normally be called i_p has been replaced with the *number* q. I am not proving this right here, but a reasonable derivation is given by me in the above mentioned notes. Notice how p appears on the RHS only implicitly: it is the missing a factor, and index i_p replaced by constant q.

Now here is one way you can *verify* that the above must be correct. Multiply both sides by a_{pq} and do an implied sum on q. On the right, shove this factor into the missing a position, so we have

$$a_{pq}\, \text{cof}(a_{pq}) = \varepsilon_{i_1 i_2 i_3 \ldots [\cancel{i_p} \to q] \ldots i_N}\; a_{1i_1}\, a_{2i_2} \,.... a_{pq} ..... a_{Ni_N}$$

Now just on the RHS, rename the summation index q to be i_p (since any dummy summation name is allowed). We then have

$$a_{pq}\,\mathrm{cof}(a_{pq}) = \varepsilon_{i_1 i_2 i_3 \ldots\, i_p \ldots i_N}\; a_{1 i_1}\, a_{2 i_2} \;\ldots.\; a_{p i_p} \ldots\ldots\ldots a_{N i_N}$$

But now summation index i_p is in its exact right position, and factor $a_{p i_p}$ is in its exact right position, so we recognize the RHS as being just det(a) and we have shown that

$$\Sigma_q\, a_{pq}\,\mathrm{cof}(a_{pq}) = \det(a)$$

and the fascinating fact is that this result is true for any integer p in the range 1...N ! The index p is the row we are moving across to make this form of the determinant. The formula is also true if you

If you accept the above expression for cofactor, then this is a proof that det(a) can be written in this manner.

As I then argue in those notes, the sum down columns variation is easily obtained by replacing $a \rightarrow a^T$ in the above formula

$$\Sigma_q\, a^T_{pq}\,\mathrm{cof}(a^T_{pq}) = \det(a^T)$$

which we rewrite as

$$\Sigma_q\, a_{qp}\,\mathrm{cof}(a_{qp}) = \det(a)$$

and now we are summing on the first indices instead of the second on the LHS.

Now suppose we take our opening line above apply it to a^T

$$\mathrm{cof}(a^T_{pq}) = \varepsilon_{i_1 i_2 i_3 \ldots [\cancel{i_p} \rightarrow q] \ldots i_N}\; a^T_{1 i_1}\, a^T_{2 i_2} \;\ldots\ldots\; \cancel{a^T_{p i_p}} \ldots\ldots\ldots a^T_{N i_N}$$

Then we find that

$$\mathrm{cof}(a_{qp}) = \varepsilon_{i_1 i_2 i_3 \ldots [\cancel{i_p} \rightarrow q] \ldots i_N}\; a_{i_1 1}\, a_{i_2 2} \;\ldots\ldots\; \cancel{a_{i_p p}} \ldots\ldots\ldots a_{i_N N}$$

where now we have swapped the indices on all the a factors everywhere and removed all T's. So this is another way to write the cofactor. Then of course the det(a) formula in terms of cof is the same

Summary of all results:

- The cofactor can be written two ways

$$\mathrm{cof}(a_{pq}) = \varepsilon_{i_1 i_2 i_3 \ldots [\cancel{i_p} \rightarrow q] \ldots i_N}\; a_{1 i_1}\, a_{2 i_2} \;\ldots\ldots\; \cancel{a_{p i_p}} \ldots\ldots\ldots a_{N i_N}$$

$$\mathrm{cof}(a_{qp}) = \varepsilon_{i_1 i_2 i_3 \ldots [\cancel{i_p} \rightarrow q] \ldots i_N}\; a_{i_1 1}\, a_{i_2 2} \;\ldots\ldots\; \cancel{a_{i_p p}} \ldots\ldots\ldots a_{i_N N}$$

• The determinant evaluation rule can be written in terms of the cofactor in two ways

$\Sigma_{q}\, a_{pq}\, \mathrm{cof}(a_{pq}) = \det(a)$ for any p sum on second indices

$\Sigma_{q}\, a_{qp}\, \mathrm{cof}(a_{qp}) = \det(a)$ for any p sum on first indices

2. An even fancier formula for cofactor

(a) We have established this fact so far, where $[i_P \to Q]$ is in the Pth subscript position on ε,

$$\mathrm{cof}(A_{PQ}) = \varepsilon_{i_1 i_2 i_3 \ldots [i_P \to Q] \ldots i_N}\; A_{1 i_1}\, A_{2 i_2} \;\ldots\ldots\; \cancel{A_{P i_P}} \ldots\ldots\ldots\; A_{N i_N}$$

If we move $[i_P \to Q]$ to the first position on ε, we pick up a factor $(-1)^{P-1}$ from doing P swaps,

$$= (-1)^{P-1}\, \varepsilon_{Q i_1 i_2 i_3 \ldots i_P \ldots i_N}\, A_{1 i_1}\, A_{2 i_2} \;\ldots\ldots\; \cancel{A_{P i_P}} \ldots\ldots\ldots\; A_{N i_N}$$

(b) Let {a,b,c...x} be a list of N-1 elements which are taken from the set of N integers {1,2,3....N} such that every number in the list {a,b,c...x} is different. Let P be the omitted number.

(c) Let {a,b,c,....P....x} be a permutation of the integers {1,2.3.....N} where P is as described above. We place P in the Pth position in this list {a,b,c,....P....x}.

(d) Consider the following object

$$F^{PQ}{}_{abc..x} \equiv (-1)^{P-1} \varepsilon_{Q i_1 i_2 i_3 \ldots i_P \ldots i_N}\, A_{a i_1}\, A_{b i_2} \;\ldots\ldots\; \cancel{A_{P i_P}} \ldots\ldots\ldots\; A_{x i_N}$$

where the A factors (including the struck-out one) bear the labels {a,b,c,....P....x} as shown.

Observation #1: Although we said all the items in the list {a,b,c...x} are different, if we were to allow two of them to be the same, the RHS above vanishes from the usual symmetry rule of contracting a symmetric object (like $A_{a i_1}\, A_{b i_2}$) against an antisymmetric object like the ε. Thus the observation is that the quantity $F^{PQ}{}_{abc..x}$ vanishes if two subscripts are the same.

Observation #2: If we swap any pair of subscripts on $F^{PQ}{}_{abc..x}$, the RHS changes sign, which is easy to show.

Observation #3. $F^{PQ}{}_{123..P..N} = \mathrm{cof}(A_{PQ})$ from (a) above

Therefore, we can express any $F^{PQ}{}_{abc..x}$ as a sign times $F^{PQ}{}_{123..P..N}$. The correct sign is given by using the ε tensor in this manner

$$F^{PQ}{}_{abc..x} = \varepsilon_{abc..P..x}\, F^{PQ}{}_{123..P..N}$$

where ε has N indices, and P appears in the Pth position. As an example, if N=4 and P=3,

$$F^{3Q}_{abc} = \varepsilon_{ab3c} F^{3Q}_{124}$$

and if we set abc = 124, we get

$$F^{3Q}_{124} = \varepsilon_{1234} F^{3Q}_{124} = F^{3Q}_{124}$$

which is the desired result. Moving the P to the first position, we then have

$$F^{PQ}_{abc..x} = (-1)^{P-1}\varepsilon_{Pabc..x} F^{PQ}_{123..\not{P}..N}$$

$$= (-1)^{P-1}\varepsilon_{Pabc..x} \text{cof}(A_{PQ})$$

Therefore we have shown that

$$(-1)^{P-1} \varepsilon_{Qi_1i_2i_3...\not{i}_P...i_N} A_{ai_1} A_{bi_2} ..... A_{xi_N} = (-1)^{P-1}\varepsilon_{Pabc..x} \text{cof}(A_{PQ})$$

or

$$\varepsilon_{Qi_1i_2i_3...\not{i}_P...i_N} A_{ai_1} A_{bi_2} ..... A_{xi_N} = \varepsilon_{Pabc..x} \text{cof}(A_{PQ})$$

Basically if the labels 1,2,3... ~~P~~..N in our cofactor structure have been permuted into a,b,c....x, this formula tells us how to unpermute them.

(e) At this point, we can rename the N-1 summation indices in this way

$$\{i_1i_2,... \not{i}_P ..i_N\} \rightarrow \{a',b',c'...x'\}$$

so the above becomes

$$\varepsilon_{Qa'b'c'...x'} A_{aa'} A_{bb'} ... A_{xx'} = \varepsilon_{Pabc..x} \text{cof}(A_{PQ})$$

As noted earlier, if any two indices in {a,b,c...x} are the same, the above equation is true because both sides vanish. Assuming that all the indices in {a,b,c...x} are different, then, we can replace the right hand side with a sum as follows

$$\varepsilon_{Pabc..x} \text{cof}(A_{PQ}) = \Sigma_S \varepsilon_{Sabc..x} \text{cof}(A_{SQ})$$

This is true since only the term S=P makes a contribution, all other terms in the sum have two indices being the same. So we arrive then at this final result which we state this way

If in the list of N-1 letters {a,b,c...x} each letter takes on some value from the set {1,2,3..N}, then the following is true:

$$\varepsilon_{Qa'b'c'...x'} A_{aa'} A_{bb'} ... A_{xx'} = \varepsilon_{Sabc..x} \text{cof}(A_{SQ})$$

where now we have implied sums on all repeated indices including S.

(f) We can restate the main results above replacing $A \rightarrow A^{T}$. We then find:

$$\text{cof}(A_{QP}) = \varepsilon_{i_1 i_2 i_3 \ldots [i_P \rightarrow Q] \ldots i_N} A_{i_1 1} A_{i_2 2} \ldots\ldots \cancel{A_{i_P P}} \ldots\ldots\ldots A_{i_N N}$$

$$= (-1)^{P-1} \varepsilon_{Q i_1 i_2 i_3 \ldots \cancel{i_P} \ldots i_N} A_{i_1 1} A_{i_2 2} \ldots\ldots \cancel{A_{i_P P}} \ldots\ldots\ldots A_{i_N N}$$

$$\varepsilon_{Q i_1 i_2 i_3 \ldots \cancel{i_P} \ldots i_N} A_{i_1 a} A_{i_2 b} \ldots\ldots \cancel{A_{i_P P}} \ldots\ldots\ldots A_{i_N x} = \varepsilon_{P a b \ldots x} \text{cof}(A_{QP})$$

$$\varepsilon_{Q a' b' c' \ldots x'} A_{a' a} A_{b' b} \ldots A_{x' x} = \varepsilon_{S a b \ldots x} \text{cof}(A_{QS})$$

3. The famous matrix inversion theorem and cofactor transpose identity

$$[a^{-1}]_{pq} = \text{cof}(a_{qp})/\det(a)$$

where again $\text{cof}(a_{qp})$ is a number. We could write this also as

$$[a^{-1}]_{pq} = \text{cof}(a^{T}{}_{pq})/\det(a)$$

One can define a cofactor *matrix* this way

$$[\text{cof}(a^{T})]_{pq} \equiv \text{cof}(a^{T}{}_{pq})$$

and then the above matrix inversion theorem would read

$$a^{-1} = \text{cof}(a^{T})/\det(a)$$

Then here is a theorem about this cofactor matrix

$$\text{cof}(a^{T}) = [\text{cof}(a)]^{T} \qquad \text{// both sides are matrices}$$

Proof:

$$\text{LHS}_{pq} = [\text{cof}(a^{T})]_{pq} = \text{cof}(a^{T}{}_{pq}) = \text{cof}(a_{qp})$$

$$\text{RHS}_{pq} = [\text{cof}(a)]^{T}{}_{pq} = [\text{cof}(a)]_{qp} = [\text{cof}(b^{T})]_{qp} \qquad b \equiv a^{T}$$

$$= \text{cof}(b^{T}{}_{qp}) = \text{cof}(a_{qp}) \qquad \text{QED}$$

4. Theorem 1: cof(R_{ab}) = R_{ab} for rotation in N dimensions

For an NxN rotation matrix R = R^T and det(R) = 1, we claim the following is true

$$\text{cof}(R_{ab}) = R_{ab}$$

The thing on the left is a linear combination of N-1 R_{ij} factors, the right is just a single element.

Proof:

(1) Since $R^T R = 1$, we know det(R) = ±1, let assume +1 and we are doing a rotation.

(2) We also know that $R^{-1} = R^T$ and therefore $R^{-1,T} = R$

(3) Therefore: [note that $\det(R^{-1}) = \det(R^T) = \det(R) = 1$]

$$R_{ab} = R^T_{ba} = (R^{-1})_{ba} = \text{cof}(R^T_{ba}) / \det(R^{-1}) = \text{cof}(R_{ab})$$ QED

If we set det(R) = σ = ± 1, we can restate item (3) as

(3a) Therefore: [note that $\det(R^{-1}) = \det(R^T) = \det(R) = \sigma$]

$$R_{ab} = R^T_{ba} = (R^{-1})_{ba} = \text{cof}(R^T_{ba}) / \det(R^{-1}) = \text{cof}(R_{ab})/\sigma = \sigma\, \text{cof}(R_{ab})$$

and then

$$R_{ab} = \sigma\, \text{cof}(R_{ab})$$

5. Application of the cofactor expressions to generalized cross products.

Consider the cross product of N-1 vectors, each of which is an N-vector,

$$Q_a = \varepsilon_{abc\ldots x} B_b C_c D_d \ldots\ldots X_x$$

Suppose all the RHS vectors transform according to a linear transformation

$$B'_a = R_{aa'} B_{a'}$$

Then we can define a "transformed object" as (implicit sums on all indices except a)

$$Q'_a = \varepsilon_{abc\ldots x} B'_b C'_c D'_d \ldots\ldots X'_x$$

$$= (\varepsilon_{abc\ldots x} R_{bb'} R_{cc'} R_{dd'} \ldots R_{xx'})\ B_{b'} C_{c'} D_{d'} \ldots\ldots X_{x'}$$

In Section 2 (f) we showed that

$$\varepsilon_{Qa'b'c'\ldots x'} A_{a'a} A_{b'b} \ldots A_{x'x} = \varepsilon_{Sabc..x} \, \mathrm{cof}(A_{QS})$$

We now make a set of changes to this result. First, take A → R and swap lower case primes with non-primes to get

$$\varepsilon_{Qabc\ldots x} R_{aa'} R_{bb'} \ldots R_{xx'} = \varepsilon_{Sa'b'c'..x'} \, \mathrm{cof}(R_{QS})$$

Now increment all the lower case letters by one letter (we will still use x to represent the last letter)

$$\varepsilon_{Qbcd\ldots x} R_{bb'} R_{cc'} \ldots R_{xx'} = \varepsilon_{Sb'c'..x'} \, \mathrm{cof}(R_{QS})$$

Now replace Q by a, and S by a'

$$\varepsilon_{abcd\ldots x} R_{bb'} R_{cc'} \ldots R_{xx'} = \varepsilon_{a'b'c'..x'} \, \mathrm{cof}(R_{aa'})$$

We have now shown that

$$Q'_a = (\varepsilon_{abc\ldots x} R_{bb'} R_{cc'} R_{dd'} \ldots R_{xx'}) \; B_{b'} C_{c'} D_{d'} \ldots\ldots X_{x'}$$

$$= (\varepsilon_{a'b'c'..x'} \, \mathrm{cof}(R_{aa'})) \; B_{b'} C_{c'} D_{d'} \ldots\ldots X_{x'}$$

$$= \mathrm{cof}(R_{aa'}) \;\; \varepsilon_{a'b'c'..x'} B_{b'} C_{c'} D_{d'} \ldots\ldots X_{x'}$$

$$= \mathrm{cof}(R_{aa'}) \;\; \varepsilon_{a'bc..x} B_b C_c D_d \ldots\ldots X_x$$

$$= \mathrm{cof}(R_{aa'}) \, Q_{a'}$$

This then shows how a generalized cross product transforms under a linear transformation. Q transforms under a linear transformation $M_{aa'} = \mathrm{cof}(R_{aa'})$, but in general M is not the same as R.

If the linear transformation R satisfies $RR^T = 1$ (which says $\det R = \sigma = \pm 1$), then from Section 4 above we know that $\mathrm{cof}(R_{aa'}) = \sigma R_{aa'}$ and then we have

$$Q'_a = \sigma R_{aa'} Q_{a'}$$

and finally if R is a true rotation with det(R) = 1, we get

$$Q'_a = R_{aa'} Q_{a'}$$

This says that if R is a rotation matrix, then the generalized cross product of N-1 vectors itself transforms as a vector (all in N dimensions). Moreover, in this case we have

$$\varepsilon_{abcd...x} R_{bb'} R_{cc'} \ldots R_{xx'} = \varepsilon_{a'b'c'..x'} \, \mathrm{cof}(R_{aa'})$$

$$= \varepsilon_{a'b'c'..x'} [\, \sigma R_{aa'}]$$

$$= \sigma \, \varepsilon_{a'b'c'..x'} R_{aa'}$$

For N = 3 this says for example

$$\varepsilon_{abc} R_{bb'} R_{cc'} = \sigma \, \varepsilon_{a'b'c'} R_{aa'}$$

and when R is a rotation with detR = σ = 1, this says

$$\varepsilon_{abc} R_{bb'} R_{cc'} = \varepsilon_{a'b'c'} R_{aa'}$$

This obscure result sometimes comes in handy! Since this is true for any rotation R, it is also true for the rotation R^T so we can write

$$\varepsilon_{abc} R^T_{bb'} R^T_{cc'} = \varepsilon_{a'b'c'} R^T_{aa'}$$

or

$$\varepsilon_{abc} R_{b'b} R_{c'c} = \varepsilon_{a'b'c'} R_{a'a}$$

Note added 11.13.11. Staying on "developmental notation", consider again

$$Q'_a = \mathrm{cof}(R_{aa'}) \, Q_{a'}$$

Write

$$\mathrm{cof}(R_{aa'}) = (R^{-1})_{a'a} \det(R) = S_{a'a} \det(R) = S^T_{aa'} \det(R)$$

then we can write the result as

$$Q'_a = \det(R) \, S^T_{aa'} Q_{a'} = J^{-1} \, S^T_{aa'} Q_{a'}$$

But Q_a is a contravariant vector by intension, so should have R and not S^T here, and J is the wrong power.

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Try repeating the above converting to Standard Notation,

5A. Application of the cofactor expressions to generalized cross products (Std Notation)

Consider the cross product of N-1 vectors, each of which is an N-vector,

$$Q_a = \varepsilon_{abc\ldots x}\, B^b C^c D^d \ldots\ldots X^x$$

Suppose all the RHS vectors transform according to a linear transformation

$$B'^a = R^a{}_{a'} B^{a'}$$

Then we can define a "transformed object" as (implicit sums on all indices except a)

$$Q'_a = \varepsilon'_{abc\ldots x}\, B'^b C'^c D'^d \ldots\ldots X'^x = \varepsilon_{abc\ldots x}\, B'^b C'^c D'^d \ldots\ldots X'^x$$

$$= (\varepsilon_{abc\ldots x}\, R^b{}_{b'} R^c{}_{c'} R^d{}_{d'} \ldots R^x{}_{x'})\ B^{b'} C^{c'} D^{d'} \ldots\ldots X^{x'}$$

In Section 2 (f) we showed that

$$\varepsilon_{Qa'b'c'\ldots x'}\, A^{a'}{}_a\, A^{b'}{}_b \ldots A^{x'}{}_x = \varepsilon_{Sabc..x}\, \mathrm{cof}(A^Q{}_S)$$

We now make a set of changes to this result. First, take A → R and swap lower case primes with non-primes to get

$$\varepsilon_{Qabc\ldots x}\, R^a{}_{a'}\, R^b{}_{b'} \ldots R^x{}_{x'} = \varepsilon_{Sa'b'c'..x'}\, \mathrm{cof}(R^Q{}_S)$$

Now increment all the lower case letters by one letter (we will still use x to represent the last letter)

$$\varepsilon_{Qbcd\ldots x}\, R^b{}_{b'}\, R^c{}_{c'} \ldots R^x{}_{x'} = \varepsilon_{Sb'c'..x'}\, \mathrm{cof}(R^Q{}_S)$$

Now replace Q by a, and S by a'

$$\varepsilon_{abcd\ldots x}\, R^b{}_{b'}\, R^c{}_{c'} \ldots R^x{}_{x'} = \varepsilon_{a'b'c'..x'}\, \mathrm{cof}(R^a{}_{a'})$$

We have now shown that

$$Q'_a = (\varepsilon_{abc\ldots x}\, R^b{}_{b'} R^c{}_{c'} R^d{}_{d'} \ldots R^x{}_{x'})\ B^{b'} C^{c'} D^{d'} \ldots\ldots X^{x'}$$

$$= (\varepsilon_{a'b'c'..x'}\, \mathrm{cof}(R_{aa'}))\ B^{b'} C^{c'} D^{d'} \ldots\ldots X^{x'}$$

$$= \mathrm{cof}(R^a{}_{a'})\ \ \varepsilon_{a'b'c'..x'}\, B^{b'} C^{c'} D^{d'} \ldots\ldots X^{x'}$$

$$= \mathrm{cof}(R_{aa'})\ \ \varepsilon_{a'bc..x} B^b C^c D^d \ldots\ldots X^x$$

$$= \mathrm{cof}(R_{aa'})\, Q_{a'}$$

$$= J^{-1}\, S^T{}_{aa'} Q_{a'} \qquad \text{// see earlier}$$

This says that Q_a transforms as a vector density with weight +1. So this is consistent with my work elsewhere in the tensor doc support docs. You get the same result with either approach:

$\varepsilon'_{abc\ldots x} = \varepsilon_{abc\ldots x}$ // used in the above

$\varepsilon'_{abc..x} = J^{-1}\ S_{a'}{}^{a}\, S_{b'}{}^{b} \ldots\ldots S_{x'}{}^{x}\, \varepsilon_{a'b'c'\ldots x'}$

B. Traceless Matrix Theorem #1:

Claim: (implied sums on all repeated indices)

$\varepsilon_{jmn}\, S_{ij} S_{nm} + \varepsilon_{ijk} S_{mj} S_{km} = 0$ for i = 1,2,3 where $S_{kk} = 0$

Proof: This is in mws of this name in the matrix folder. First, define eps in the usual way. Then

```
S[3,3] := -sum(S[a,a],a=1..3-1);
```

$$S_{3,3} := -S_{1,1} - S_{2,2}$$

```
T1 := Sum(Sum(Sum(  eps(j,m,n)* S[i,j]*S[n,m],j=1..3),m=1..3),n=1..3);
```

$$T1 := \sum_{n=1}^{3}\left(\sum_{m=1}^{3}\left(\sum_{j=1}^{3} \mathrm{eps}(j,m,n)\, S_{i,j}\, S_{n,m}\right)\right)$$

```
T2 := Sum(Sum(Sum(  eps(i,j,k)* S[m,j]*S[k,m],j=1..3),m=1..3),k=1..3);
```

$$T2 := \sum_{k=1}^{3}\left(\sum_{m=1}^{3}\left(\sum_{j=1}^{3} \mathrm{eps}(i,j,k)\, S_{m,j}\, S_{k,m}\right)\right)$$

You can see that these are right. Now here are the three evaluations of the LHS:

```
i := 1; T1; T2; T1+T2: simplify(%);
```

$$i := 1$$

$$S_{1,3} S_{2,1} - S_{1,1} S_{2,3} - S_{1,2} S_{3,1} + S_{1,1} S_{3,2}$$

$$-S_{1,3} S_{2,1} - S_{2,3} S_{2,2} - (-S_{1,1} - S_{2,2}) S_{2,3} + S_{1,2} S_{3,1} + S_{2,2} S_{3,2} + S_{3,2} (-S_{1,1} - S_{2,2})$$

$$0$$

```
i := 2; T1; T2; T1+T2: simplify(%);
```

$$i := 2$$

$$S_{1,3} S_{2,1} - S_{1,1} S_{2,3} - S_{1,2} S_{3,1} + S_{1,1} S_{3,2}$$

$$-S_{1,3} S_{2,1} - S_{2,3} S_{2,2} - (-S_{1,1} - S_{2,2}) S_{2,3} + S_{1,2} S_{3,1} + S_{2,2} S_{3,2} + S_{3,2} (-S_{1,1} - S_{2,2})$$

$$0$$

```
i := 3; T1; T2; T1+T2: simplify(%);
```

$$i := 3$$

$$S_{1,3} S_{2,1} - S_{1,1} S_{2,3} - S_{1,2} S_{3,1} + S_{1,1} S_{3,2}$$

$$-S_{1,3} S_{2,1} - S_{2,3} S_{2,2} - (-S_{1,1} - S_{2,2}) S_{2,3} + S_{1,2} S_{3,1} + S_{2,2} S_{3,2} + S_{3,2} (-S_{1,1} - S_{2,2})$$

$$0$$

In each case the result is 0, QED. This is a "proof by direct evaluation of all cases".