Matrix Question: Does inverse commute with transpose? PhL 11.26.09

Here we are, once again, doing 4^{th} grade math and being confused, but what else is new?

1. What is the minor?

minor(M_{ab}) = determinant of (matrix obtained by crossing out row a and row b) = a number

Let's define a matrix called minor(M) which has these matrix elements

$[\text{minor}(M)]_{ab} = \text{minor}(M_{ab})$ = a number

Then it follows that

$[\text{minor}(M^T)]_{ab} = \text{minor}((M^T)_{ab}) = \text{minor}\ (M_{ba}) = [\text{minor}(M)]_{ba} = ([\text{minor}(M)]^T)_{ab}$

Therefore we have shown that

$\text{minor}(M^T) = [\text{minor}(M)]^T$ // a matrix equation

"The transpose of the minor-matrix of M (RHS) is the minor-matrix of the transpose of M (LHS)."
or
"The minor operation commutes with the transpose operation".

2. What is the cofactor?

$\text{cof}(M_{ab}) = (-1)^{a+b}\ \text{minor}(M_{ab})$ = a number

Let's define a matrix called cof(M) which has these matrix elements

$[\text{cof}(M)]_{ab} = \text{cof}(M_{ab}) = (-1)^{a+b}\ [\text{minor}(M)]_{ab}$

Then if follows that

$[\text{cof}(M^T)]_{ab} = \text{cof}((M^T)_{ab}) = (-1)^{a+b}\ [\text{minor}(M^T)]_{ab} = (-1)^{a+b}\ ([\text{minor}(M)]^T)_{ab}$

$= (-1)^{a+b}\ [\text{minor}(M)]_{ba} = (-1)^{b+a}\ [\text{minor}(M)]_{ba} = [\text{cof}(M)]_{ba} = ([\text{cof}(M)]^T)_{ab}$

we have then shown that

$\text{cof}(M^T) = [\text{cof}(M)]^T$ // a matrix equation

" Cofactor operation commutes with T operation "

3. What is the inverse?

$M^{-1} = \text{cof}(M)/\det(M)$

Then we have:

$(M^{T})^{-1} = \text{cof}(M^{T})/\det(M^{T}) = \text{cof}(M^{T})/\det(M) = [\text{cof}(M)]^{T} / \det(M)$

But on the other hand we have

$(M^{-1})^{T} = [\text{cof}(M)]^{T}/\det(M)$

Thus we have proved that

$(M^{T})^{-1} = (M^{-1})^{T}$

which says that "transpose commutes with inverse".