Matrix theorems with expo of matrix PhL 11.20.16

I guess I don't have this stuff written down anywhere, so do it here today.

1. **Theorem 1**: $e^{A}e^{B} = e^{B}e^{A} = e^{A+B}$ provided A and B commute.

Proof: If A and B commute, then clearly A^n and B^m commute, so we have

$$e^{A}e^{B} = (\Sigma_n A^n/n!)\,(\Sigma_m B^m/m!) = (\Sigma_m B^m/m!)(\Sigma_n A^n/n!) = e^{B}e^{A}$$

Next, consider this *simpler* problem of showing for non-matrix x and y

$$e^{x}e^{y} = e^{x+y}$$

We have to show then that

$$(\Sigma_n x^n/n!)\,(\Sigma_m y^m/m!) = \Sigma_n (x+y)^n/n! \; .$$

First write the left side as

$$\text{LHS} = (\Sigma_n x^n/n!)\,(\Sigma_m y^m/m!) = \Sigma_n \Sigma_m \; x^n y^m / (n!m!)$$

On the right side we know that

$$(x+y)^n = \Sigma_{k=0}^{n} (n,k)\, x^{n-k}y^k = \Sigma_{k=0}^{n} \frac{n!}{k!(n-k)!}\, x^{n-k}y^k \qquad (**)$$

So then the right side is this

$$\text{RHS} = \Sigma_n (x+y)^n/n! = \Sigma_n \Sigma_{k=0}^{n} \frac{n!}{k!(n-k)!}\; x^{n-k}y^k /n! = \Sigma_{n=0}^{\infty} \Sigma_{k=0}^{n} \frac{1}{k!(n-k)!}\, x^{n-k}y^k$$

So now we want to show that

$$\text{LHS} = \text{RHS}$$

$$\Sigma_{n=0}^{\infty}\Sigma_{m=0}^{\infty} \; x^n y^m / (n!m!) = \Sigma_{n=0}^{\infty} \Sigma_{k=0}^{n} \frac{1}{k!(n-k)!}\, x^{n-k}y^k \qquad (*)$$

Now forget (*) for a moment and recall this general summation reordering Theorem 3 from integrals and series/ double sum... doc:

$$\Sigma_{m=0}^{\infty} \Sigma_{n=0}^{\infty} \; f_{n,m} = \Sigma_{k=0}^{\infty} \; \Sigma_{m=0}^{k} f_{k-m,m} \qquad m+n=k \qquad \text{Theorem 3} \qquad (1)$$

Now swap order on the left, then rename dummies on the right so that $k \to n$ and $m \to k$

$\Sigma_{n=0}^{\infty} \Sigma_{m=0}^{\infty} f_{n,m} = \Sigma_{n=0}^{\infty} \Sigma_{k=0}^{n} f_{n-k,k}$ m +n = k Theorem 3 (2)

Now let

$f_{n,m} = x^n y^m / (n!m!) \Rightarrow f_{n-k,k} = x^{n-k} y^k / ((n-k)!k!)$

The theorem in form (2) then says

$\Sigma_{n=0}^{\infty} \Sigma_{m=0}^{\infty} x^n y^m / (n!m!) = \Sigma_{n=0}^{\infty} \Sigma_{k=0}^{n} x^{n-k} y^k / ((n-k)!k!)$

But this is exactly equation (*) so we have thus proven that (*) is correct and therefore that $e^x e^y = e^{x+y}$

Now how is this different if x → A and y→ B are matrices? Look at step (**)

$(x+y)^n = \Sigma_{k=0}^{n} (n,k) x^{n-k} y^k$

With matrices this would be for example

$(A+B)^2 = (A+B)(A+B) = A^2 + BA + AB + B^2$

only if A and B commute can you write this as

$A^2 + 2AB + B^2 = \Sigma_{k=0}^{2} (2,k) A^{n-k} B^k$

So to make the above proof work for matrices, they must commute. Then we have

$e^A e^B = e^{A+B}$ iff A and B commute QED

Corollary: $\exp(i [\mathbf{n} + d\mathbf{n}] \bullet \mathbf{J}) \neq \exp(i\, d\mathbf{n} \bullet \mathbf{J}) \exp(i\, \mathbf{n} \bullet \mathbf{J})$ J_i = generators

Write

$A = i\mathbf{n} \bullet \mathbf{J} = i n_i J_i$

$B = i\, d\mathbf{n} \bullet \mathbf{J} = i\, dn_i J_i$

Then

$[A,B] = [i n_i J_i, i\, dn_j J_j] = - n_i dn_j [J_i, J_j] = - n_i dn_j\, i\, \varepsilon_{ijk} J_k \neq 0$

$$[J_i, V_j] = i\varepsilon_{ijk}V_k$$