Matrix Notes PhL 1.11.08 last update: 3.23.09

This section of notes used to live in one of my green math binders, but I moved it to be with all my other matrix stuff in The Matrix Binder. Here, I will comment on the contents of each page (or set of pages) appearing in this section. Some material touches on the subject of Lie Algebras and Lie Groups and scattering theory and other applications, but the main focus is on the *matrix aspect* of whatever we are talking about.

In earlier times I was not accustomed to putting dates on my pages, too bad. For sure, all these notes are 1980 and earlier (≥ 27 years ago!), since I stopped doing math when I departed the physics world. Some notes are from the Utah physics era 9/77-5/80, but I think most are from the Berkeley era 9/70-9/77. The last two sections however are 1991 when I was doing Galois Fields work.

It is interesting to see how I struggled with different methods of dealing with the typesetting problem. Most notes were written with my very fine black fountain pen, both text and equations. But I tried using Flairs a few times, and sometimes pencil. I had the green Olivetti manual typewriter, so I tried using it from time to time; but I would have to remove the paper from the machine to hand-enter each equation, not very nice. The Mac Plus introduced some capabilities in 1984, and by 1991 or so I was doing pretty fancy stuff using a BTS work Mac, such as in my several "books" written during idle times, all within Microsoft Word. Eventually things migrated to the PC (my first home PC appeared 1997 after I departed BTS), and over the years I have added many things to normal.dot and also to symbols.doc.

The nice thing about math is that nothing ever goes out of date! These things always were and always will be true and probably useful.

I might use the numbers below when I refer to these pages from elsewhere.

1. Theorem on Determinants. If an off-diagonal quadrant vanishes, then the other off-diagonal quadrant plays no role in the determinant:

$$\begin{vmatrix} A & B \\ 0 & C \end{vmatrix} = \det(A)\det(C) = \begin{vmatrix} A & 0 \\ B & C \end{vmatrix}$$

I give a proof first for n=6, then general dimension, then a short symbolic proof.

2. The determinant expansion theorem. For a discrete matrix, the theorem says this:

$$\det(1+A) = 1 + \frac{1}{1!}\,\Sigma_{\mathtt{i}}\, A_{\mathtt{ii}} + \frac{1}{2!}\Sigma_{\mathtt{ij}} \begin{vmatrix} Aii & Aij \\ Aji & Ajj \end{vmatrix} + \frac{1}{3!}\Sigma_{\mathtt{ijk}} \begin{vmatrix} Aii & Aij & Aik \\ Aji & Ajj & Ajk \\ Aki & Akj & Akk \end{vmatrix} + \ldots + \det(A)$$

If n=4, then all terms are shown and the last term is just $A_{\mathtt{11}}A_{\mathtt{22}}A_{\mathtt{33}}A_{\mathtt{44}}$. The first term is 1, second term is just tr(A), the third term is the sum of all 2x2 matrices of the form shown, etc. When two or more indices are equal, the determinant vanishes so those terms in the sums give nothing. The factorials handle overcounting. In the n=4 case, there will be 6 distinct determinants in the second sum

ij = (1,2) + (1,3) + (1,4) + (2,3) + (2,4) + (3,4)

and 4 distinct determinants in the third sum

ijk = (1,2,3) + (1,2,4) + (1,3,4) + (2,3,4)

• If the matrix A is small, this is like a Taylor expansion of det(1+A).

• If $A_{ij} = f_i g_j$ for some reason, then the complete answer is det(1+A) = 1 + tr(A).

• If A is a continuous matrix like $A_{xy} = A(x,y)$, the theorem is still true providing things converge, and is called "the Fredholm expansion". The sums are replaced with integrals. For example, $\Sigma_i \rightarrow \int dx$ and $A_{ij} \rightarrow A(x,y)$. This situation arises when you work with integral equations where -A(x,y) = K(x,y) is the kernel of the integration (known as **Fredholm theory**). For a separable kernel with K(x,y) = f(x)g(y), the result is det(1-K) = 1 - tr(K) = 1 - $\int dx\ K(x,x)$. The full expansion det(1-K) is called "the Fredholm determinant".

3. Determinant Theorems:

$$\det(A) = \varepsilon_{i_1 i_2 i_3 \ldots i_n} A_{1 i_1} A_{2 i_2} A_{3 i_3} \ldots A_{n i_n}$$

det(AB) = det(A)det(B)

The first "theorem" is really a definition of the determinant and one can correlate this with the usual method of cofactors. The second theorem is proven here.

4. Another determinant theorem:

$\det(e^A) = e^{tr(A)}$ or $\det(M) = e^{tr(\ln M)}$

The second is just a corollary of the first. I crossed out my proof, but then realized it is correct. The proof is rather subtle and makes use of this fact:

$\det(\Pi_i A_i) = \Pi_i \det(A_i)$ which is the det(AB) = det(A)det(B) idea.

For a special group SX(n..) where we know that det(R) = 1 for all "rotations" R, this theorem tells us that the generators must be traceless. That is to say: $1 = \det(R) = \det(e^{-i\theta G}) = e^{-tr(i\theta G)}$ so tr(G) = 0.

5. Another theorem about det(1+A).

$$\det(1+A) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left[\sum_{m=1}^{\infty} \frac{(-1)^m}{m} tr(A^m) \right]^n$$

The proof says to write $\det(1+A) = e^{tr(\ln(1+A))}$ from 4 above, which gives the n-sum, and then we expand the ln(1+x) to get the m-sum. Not sure this is very useful, but it must have come up somewhere!

6. Why is a matrix inverse related to the cofactor matrix?

I give a proof of this fact and ways to write the cofactor.

7. The inverse of the matrix (1-A).

$$(1\text{-}A)^{-1} = \sum_{n=0}^{\infty} A^n \qquad \text{// just as for a scalar A = x, assuming RHS converges}$$

The object $(1\text{-}A)^{-1}$ is the inverse of the matrix 1-A. another way to note this is

$$(1\text{-}A)(1\text{-}A)^{-1} = (1\text{-}A)\,(\,1 + A + A^2 + ...) \qquad \text{// all terms cancel but the first.}$$

8. Fundamental Matrix Theorems, a "reminder".

(a) The eigenvectors of a Hermitian (real symmetric) matrix A can be constructed via Gram-Schmid so as to be orthonormal with respect to the appropriate inner product.

(b) It is for this reason that a Hermitian (real symmetric) matrix A can always be diagonalized by a unitary (real orthogonal) transformation. In fact, a transformation matrix S that does the job has as its columns the eigenvectors of A. The diagonal elements are of course the eigenvalues of A.

(c) This transformation can always be written $\Lambda = S^{-1}A\,S$ where S is either unitary or real orthogonal in the two cases. In either case, we have a similarity transformation and these rules for a similarity apply:

similarity transformation preserves: eigenvalues, trace and determinant

I give quick proofs of this last set of claims.

9. The World of 2x2 matrices, comments on SU(2).

Just a set of facts. Start with $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ all numbers complex. It follows that $M^{-1} = \frac{1}{\det(M)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. If detM=1 (unimodular, "special") and if M is unitary, $M^{-1} = M^{\dagger}$, then we have

$$M = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \qquad \text{but detM=1 means } |a|^2 + |b|^2 = 1, \text{ so there are 3 free parameters}$$

This is the most general form of an SU(2) matrix. Both M and -M yield this corresponding SO(3) matrix

$$R_{ij} = \frac{1}{2}\,\mathrm{tr}(\sigma_i M \sigma_j M^{\dagger})$$

and

$$M\,(\boldsymbol{\sigma}\bullet\mathbf{x})\,M^{\dagger} = (\boldsymbol{\sigma}\bullet\mathbf{x}') \qquad \text{where } \mathbf{x}' = R\mathbf{x} \text{ and } (\boldsymbol{\sigma}\bullet\mathbf{x}) = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \quad \text{// = Hermitian}$$

10. On which index do you sum?

This page compares E^3 and the |a> Hilbert Spaces and shows that, in either case,

(1) when you rotate a vector, $x'_i = (Rx)_i = R_{ik}x_k$, the summation is on the *second* index k, and this index labels the *components* of the vector you are linearly combining in the rotation.

(2) when you rotate a basis vector $\hat{\mathbf{e}}^{(i)}$ [for example, $\hat{\mathbf{x}} = \hat{\mathbf{e}}^{(1)}$ = column (1,0,0)] , the summation is on the *first* index k, and this index represents the *labels* of the basis vectors whose linear combination describes the rotated unit vector.

$$R\,\hat{\mathbf{e}}^{(i)} = R_{ki}\,\hat{\mathbf{e}}^{(k)} \qquad \Leftrightarrow \qquad R|i\rangle = R_{ki}\,|k\rangle$$

11. Theorem about a unitary diagonal matrix.

If diagonal matrix $\mathbf{\Lambda}$ is unitary, its diagonal elements can be written as phases: $\Lambda_{ii} = e^{\delta_i}$.

Application: If S is a unitary scattering matrix, $SS^\dagger = 1$, then $\langle \ell m|S|\ell' m'\rangle$ is unitary, and the diagonal elements are equal to phases $\langle \ell m|S|\ell m\rangle = e^{2i\delta_\ell}$.

12. About the matrix $H = AA^\dagger$.

Regardless of A, the matrix $H = AA^\dagger$ is Hermitian, and of course all Hermitian matrices have real eigenvalues. In this case, the eigenvalues are both real and positive. Proof given. Also true of course for $H' = AA^\dagger$. Matrices like H and H' are called positive definite.

Meanwhile, we know that the tr(H) = sum of eigenvalues of a Hermitian matrix, so in this case we know that tr(H) > 0.

13. Different notations for matrix types:

Hermitian adjoint $A^\dagger$ is called the "associate" matrix in M&M.

The $(\text{cof}A)^{\mathbf{T}}$ matrix was historically called the "adjoint" and was written $\hat{A}$, so $A^{-1} = \hat{A}/\det A$.

14. Dingbat matrix theorems.

(a) If [A,B] = 0 for all matrices A, then B = k 1. The condition can also be written $A = B^{-1}AB$.

(b) $(A^\dagger)^{-1} = (A^{-1})^\dagger$ and $(A^T)^{-1} = (A^{-1})^T$ and $(A^*)^{-1} = (A^{-1})^*$

(c) Suppose $A = BA^*B^{-1}$. If A is unitary, so is B.

15. The Baker-Campbell-Hausdorff [BCH] Formula. (aka The Hadamard Lemma)

Define these commutators:

$C_0 = B$
$C_1 = [B,A]$
$C_2 = [C_1, A] = [[B,A], A]$
$C_3 = [C_2, A] = [[[B,A], A], A]$
...
$C_n = [C_{n-1}, A]$

Then $e^{-A} B e^{A} = C_0 + C_1/1! + C_2/2! + C_3/3! + ... = \sum_{n=0}^{\infty} C_n/n!$

I give a 2-page induction proof of this theorem which finds wide use in "sandwich" formulas.

16. About direct sum and direct product matrices.

(a) two pages summarize properties of the $\otimes$ $\oplus$ and * operators with comments about direct product representations of groups. Statement of theorem $e^{A*B} = e^{A} \otimes e^{B}$ as the direct product representation

(b) details of the direct product

(c) details of the direct sum

(d) theorem: $(A \oplus B) \otimes C = (A \otimes C) \oplus (B \otimes C)$

(e) details of the * operator: claim that Z = X*Y satisfies same Lie Algebra as X and Y do.

(f) direct product of group representations

(g) old historical note on direct product. Other names "cross product" and "Kronecker product".

17. Notes from book "Introduction to Matrix Analysis" by Bellman. (5p)

Nothing very useful.

18. Notes from book "Elementary Matrices" by Frazer, Duncan & Collar. (4p)

Last page quotes Sylvester's Law of Nullity (or Degeneracy).

19. Sylvester's Laws of Nullity and Inertia

Not sure of my source, but the presentation looks pretty good.

20. Matrix Representation Theorems

[1] Lemma is a repeat of item 12 above about $H = AA^{\dagger}$.

[2] Lemma shows that for an nxn Hermitian positive definite matrix like $H = AA^{\dagger}$, there are exactly 2^n square root matrices. For each diagonal element in the diagonalized H, select ± that element, hence 2^n.

[4] Corollary: Only one of these square roots is a positive definite matrix, the one with all + signs.

[3] Lemma. Since $\det\Lambda = \Pi$ eigenvalues, and since Hermitian H can be diagonalized to some Λ, if $\det H \neq 0$, there can be no zero eigenvalues for a Hermitian matrix.

[5] Let $B = AA^{\dagger}$. If $\det B \neq 0$, then all square roots of B are Hermitian and invertible. If $\det B = 0$, then none of the square roots of B are invertible.

[6] If $\det G \neq 0$, you can write **G = HU** = a Hermitian times a unitary matrix. Proof given, it's easy and is based on [5[above. Specifically, we have H = any $\sqrt{GG^{\dagger}}$ and $U = H^{-1}G$ = unitary. Could select H as the positive definite square root. This is a "matrix representation theorem".

[7] Could also use H = any $\sqrt{G^{\dagger}G}$ in the above, then get $U = GH^{-1}$.

[8] Application: In SL(2,C) can write any G = HU as above. But H = boost, U = rotation, and so in sense of SO(3,1), can write any group element as boost times rotation!

21. The Inverse Matrix Theorem.

Suppose matrix B = B(j), j = some continuous parameter (like angular momentum). Suppose detB(j) has a pole at $j = j_1$. Define $A(j) = \det B(j)\, B^{-1}(j) = [\mathrm{cof}(B(j))]^{T}$. The theorem claims that, at the pole $j = j_1$, all matrix elements of matrix A(j) "factorize" as follows:

$$A_{kn}(j_1) = \sqrt{A_{kk}(j_1)}\ \sqrt{A_{nn}(j_1)}$$

Application: This fact is used to show that certain regge poles have factorizing residues. I don't give a full proof here, but hints that could yield one.

22. The Matrix Expansion Theorem for $(1\text{-}K)^{-1}$.

In my Fredholm green binder section, you can represent an integral equation by $\phi = f + K\phi$ where we imagine the kernel K is a matrix, f is a driving term, and ϕ is the solution we seek. This is a typical situation arising in scattering theory.

Formally, the solution is given $\phi = (1\text{-}K)^{-1}f$, so this motivates our interest in this quantity $(1\text{-}K)^{-1}$. From item (7) above, we can make this formal solution: $(1\text{-}K)^{-1} = \Sigma_{n=0}^{\infty} K^n$, so then $\phi = \Sigma_{n=0}^{\infty} K^n f$. If we can compute all the K^n and they all converge, we have solved our problem.

One often writes $(1\text{-}K)^{-1} = 1 + \Sigma_{n=1}^{\infty} K^n = 1 + \Gamma$ where Γ is called **the resolvent**. We have just separated out the unity term in the sum. One can consider solving the integral equation by computing the resolvent as follows:

$\Gamma = -1 + (1\text{-}K)^{-1} = -1 + [\text{cof}((1\text{-}K)^{-1})]^T / \det(1\text{-}K). = -1 + N/D$

If K is in some sense "small", we might produce expansions for N and D and keep only the first terms. That is the subject matter of the theorem described in this section.

We already have an expansion for D = det(1-K) as described in item 2 above. If K is either small or factorizable we can write D = 1 - tr(K) + order(K^2), where order(K^2) = 0 if factorizable. So this gives our expansion for D. The expansion for N is not so easy and I think I have got something wrong in my writeup here. All equations on both sides of this page are suspect for this reason, but at least we know where the interest is coming from.

I think it is true that if K factorizes, $K^2 = K\text{tr}(K)$ and $K^n = K\ \text{tr}(K)^{n-1}$ for n=1,2.. In this case, our solution would be given by $(1\text{-}K)^{-1} = \Sigma_{n=0}^{\infty} K^n = 1 + K \Sigma_{n=1}^{\infty} \text{tr}(K)^n$ where now you only have to compute the scaling factor tr(K) and do the sum.

23. Completeness for non-orthogonal basis vectors.

The result quoted here was wrong, and I have written the corrected result. Just of academic interest to ask what this completeness looks like, can't imagine I would ever use it.

24. Special matrices

The inverse of an upper triangular matrix has a *very* simple form which I write out here. And $\det A^{-1} = 1/\det A$ as usual.

25. The non-invertibility of a "conserved" matrix (2 separate pages)

This section has applications to QED and gauge theory in general, but it involves some simple "matrix facts" which is why this section exists in our matrix binder section. In E&M we say a current is "conserved" if we have $\partial_\mu J^\mu = 0$. If we somehow transform to k-space, this says $k_\mu J^\mu = k \bullet J = 0$. More generally, we can talk about a "conserved matrix" meaning $\partial_\mu \Pi^{\mu\nu} = 0$ and hence $k_\mu \Pi^{\mu\nu} = 0$.

The matrix claims of this section are:

(1) if $k_\mu \Pi^{\mu\nu} = 0$, then $\det \Pi = 0$ and Π^{-1} does not exist. Example is self energy bubble propagator.

(2) in general, projection operators are not invertible. An example is $(\delta_{\mu\nu} - k_\mu k_\nu / k^2)$.

(3) in this last case and perhaps generally, the two ideas are related. Note that k_μ (..) = 0. The relation is discussed in length on page 2 which right now I cannot follow because I have forgotten QED.

26. Big Theorem on Hermitian matrices. (2p) Pretty much I give a proof of this on the first page.

If $H = H^\dagger$, then there exists a unitary U such that $U^{-1} H U = \Lambda$ (diagonal matrix of eigenvalues).

The columns of matrix U are the eigenvectors of H which have been made GS orthonormal.

It is possible to choose U such that det(U) = 1, meaning U is unimodular.

The second page reminds us that an nxn Hermitian matrix is guaranteed to have n eigenvectors with n real eigenvalues, and this information is required to prove diagonalizability. You cannot show that an arbitrary nxn matrix is diagonalizable, and in fact in general it is not so. However, as other matrix notes show, any "normal" matrix can be diagonalized, and this includes the Hermitian matrices.

27. Coleman's Integrals

I don't know where this arose in my past, I guess this refers to Sidney Coleman. Assume that matrix H is real symmetric nxn (hence Hermitian) and is positive definite. Then here are two theorems proved:

$$\int d^n x \;\; \exp[\, -(1/2)\, \mathbf{x} \bullet H\mathbf{x}\,] = (2\pi)^{n/2} / \sqrt{\det H})$$

$$\int d^n x \;\; \exp[\, -(1/2)\, \mathbf{x} \bullet H\mathbf{x} + \mathbf{b} \bullet \mathbf{x} + c\,] = \quad \exp[\, -(1/2)\, \mathbf{b} \bullet H^{-1}\mathbf{b} - c\,] \quad (2\pi)^{n/2} / \sqrt{\det H})$$

Recall that a positive-definite real-symmetric matrix H can be diagonalized with all positive elements, so detH > 0 and H^{-1}exists so the above formulas at least make sense. Was it Sidney Coleman who taught me Goldstein? I always thought it was Shelley Glashow, maybe I have mixed them up!

28. Test for linear independence of n-vectors in n-space

Just imagine the vectors as columns of a matrix. We have these two facts shown: (the same fact really)

(1) vectors are linearly dependent $\Rightarrow$ det(M) = 0
(2) det(M) $\neq$ 0 $\Rightarrow$ vectors are linearly independent

Nothing is said here about the $\Leftarrow$ direction, see elsewhere.

29. Notes from book "Vectors and Matrices" by MacDuffie (1943). (7p)

Topics include matrix polynomials, rings, remainder theorem, characteristic function, the Hamilton-Cayley Theorem (which says every matrix solves its own characteristic equation), the minimum function, index, derogatory, irreducible factors, companion matrix, norm of a matrix, nullspaces, invariant subspaces. This stuff relates to Galois Fields. Notes made in 1991 when I was writing my book on this subject. The last chapter of my book talks about all this stuff (matrix form of Galois fields).

30. Notes from book "Matrix Methods" by Bronson,AP (1969). (6p)

More on characteristic polynomials and eigenvalues, again notes in 1991 and related to my Galois work. Even has passing mention of Jordan Canonical Form, but mostly all scribble notes.