General comments about real-orthogonal matrices PhL 4.10.12

See Theorem at the end.

For any real orthogonal matrix we have:

$RR^T = 1 \quad \Rightarrow \det(R) = \pm 1$

The Case N = 2.

In Section 9 of my old matrix notes, I start with

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and note that $M^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} / \det M$

For a real orthogonal matrix we must then have $M^T = M^{-1}$ so

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} / (ad-bc)$$

or

$$(ad-bc)\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

or

$(ad-bc)a = d$ 1
$(ad-bc)c = -b$ 2
$(ad-bc)b = -c$ 3
$(ad-bc)d = a$ 4

How would you solve this set of 4 equations for 4 unknowns?

- Divide 2/3 to find that $c/b = b/c \Rightarrow c^2 = b^2$.
- Divide 1/4 to find that $a/d = d/a \Rightarrow a^2 = d^2$
- Divide 3/4 to find that $b/d = -c/a \Rightarrow ab = -cd$
- Divide 2/4 to find that $c/d = -b/a \Rightarrow ac = -bd$
- Divide 1/2 to find that $a/c = -d/b \Rightarrow ab = -cd$ but we already know this from 3/4
- Divide 1/3 to find that $a/b = -d/c \Rightarrow ac = -bd$ but we already know this from 2/4

So we now have four simpler equations in four unknowns

$c^2 = b^2$

$a^2 = d^2$
$ab = -cd$
$ac = -bd$

Well write

$c = \sigma_1 b$ $\quad\quad$ $\sigma_1 = \pm 1$
$a = \sigma_2 d$ $\quad\quad$ $\sigma_2 = \pm 1$

Then the last two equations say

$\sigma_2 db = -\sigma_1 bd$ $\quad\Rightarrow\quad$ $\sigma_2 = -\sigma_1$
$\sigma_2 d\, \sigma_1 b = -bd$ $\quad\Rightarrow\quad$ $\sigma_1\sigma_2 = -1$, same as above

So eliminate σ_2 so we have

$c = \sigma_1 b$
$a = -\sigma_1 d$

This is all that is left. You can pick b and d freely. The final form then is

$$M = \begin{pmatrix} -\sigma_1 d & b \\ \sigma_1 b & d \end{pmatrix}$$

So here are the two possible solutions

$$M = \begin{pmatrix} -d & b \\ b & d \end{pmatrix} \qquad \det M = -(d^2+b^2)$$

$$M = \begin{pmatrix} d & b \\ -b & d \end{pmatrix} \qquad \det M = d^2+b^2$$

Now if we want $\det M = \pm 1$ we can set $d = \cos\theta$ and $b = \sin\theta$ to get

$$M_2 = \begin{pmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \det M = -1$$

$$M_1 = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad \det M = +1$$

So these ARE the solutions to this problem. Now easy to show that

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Now $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ causes a reflection in the y-axis causing y→y but x→-x. Call this r_y. Then

$M_1 = r_y M_2$ and $M_2 = r_y M_1$ since $r_y^2 = 1$

So our second solution is to first reflect in the y axis, THEN to rotate. Note that this is not a parity transformation! So consider:

$$r_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad r_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -r_y$$

So consider again:

det = 1 solution: $M_1 = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ only 1 free parameter

det = -1 solution:

$$M_2 = r_x M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

$$M_3 = r_y M_1 = -\, r_x M_1 = -\, M_2$$

$$M_4 = M_1 r_x = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix}$$

$$M_5 = M_1 r_y = -M_4$$

Now the M_4 case arises from θ→-θ on the M_2 case, so if we think of θ in (-π,π), then M_4 is then not a new solution. Similarly, if we take M_2 and let θ → π-θ, we change the cosine sign only, but then we could do the other thing and change the sin only. So in effect there is only one det = -1 solution.

Conclusion: for N = 2, one solution is the usual rotation matrix, while the other solution is that rotation combined with either r_x or r_y.

The Case N = 3. It seems clear that one solution here is the usual rotation matrix, while all the other solutions with det = -1 come by combining this with combinations of r_x, r_y, r_z, or P. Note that $r_x r_y = r_z P$ So these are "combinations of reflections in axes". For N = odd, P is included in the set.

Theorem:

If we are told that N is a real orthogonal matrix, we know right away that detN = ± 1. When det = +1, our matrix N is a rotation matrix possibly combined with an even number of axis reflections. When det = -1, the matrix is a rotation matrix combined with an odd number of axis reflections. When N = odd, this includes the parity transformation where we reflect all axes.

The reflections affect the determinant in the obvious manner, but don't affect the fact that $MM^T = 1$. Here is why (based on the fact that $r_i r_i^T = 1$ since a simple diagonal matrix of ±1's)

$M' \equiv r_1 r_2 .. M\, r_3 r_4 ....$

$M'^{T} \equiv \; .....r_4^{T} r_3^{T} \; M^{T} \; ....r_2^{T} r_1^{T}$

$MM'^{T} \equiv r_1 r_2 .. M\, r_3 r_4 .... \; .....r_4^{T} r_3^{T} \; M^{T} \; ....r_2^{T} r_1^{T}$

$= r_1 r_2 .. M \; M^{T} \; ....r_2^{T} r_1^{T}$

$= r_1 r_2 .. 1 ....r_2^{T} r_1^{T}$

$= 1$