The Matrix Rollup Problem PhL 11.18.10

Suppose you have a system of equations like this

$a_{00}x_0 + a_{01}x_1 = 0$
$a_{10}x_0 + a_{11}x_1 + a_{12}x_2 = 0$
$a_{20}x_0 + a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0$
$a_{30}x_0 + a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = 0$
.....

Define these matrices and notice how each gives an equation above

$$A_2 = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \qquad \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = (-1/a_{12})\begin{pmatrix} 0 \\ x_2 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} a_{00} & a_{01} & 0 \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \qquad \begin{pmatrix} a_{00} & a_{01} & 0 \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = (-1/a_{23})\begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} a_{00} & a_{01} & 0 & 0 \\ a_{10} & a_{11} & a_{12} & 0 \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{33} & a_{34} \end{pmatrix} \quad \text{etc}$$

Theorem: The solution to this infinite system of equations can be written this way

$$x_n = (-1)^n x_0 \, \det A_n / (a_{01}\, a_{12}\, a_{23} \ldots\ldots a_{n-1,n})$$

$$= (-1)^n x_0 \, \det A_n / \Pi_{i=0}^{n-1}(a_{i,i+1})$$

So this is just a fancy way to "do the rollup". But in my Maple world, it would be more work to compute all these determinants that to just do the rollup in code like this

for n = 1 to 10 do
$x_n = (-1/a_{n-1,n})\,[a_{n-1,0}x_0 + a_{n-1,1}x_1 + \ldots. a_{n-1,n-1}x_{n-1}]$

for n = 1 to 10 do
$x[n] = (-1/a_{n-1,n})\,[\,\Sigma_{k=0}^{n-1} a_{n-1,k}\; x[k]\,]$

I fiddle with this theorem below but did not do the required induction proof.

One idea: Notice that

$\det A_3 = a_{22}\ \det A_2 - a_{12} \det A_2(1x \to 2x)$

$\det A_4 = a_{34} \det A_3 - a_{23} \det A_3(2x \to 3x)$

where $1x \to 2x$ means replace A_{1x} by A_{2x} for all x.

Meanwhile,

$\det A_3 = \varepsilon_{ijk} a_{0i} a_{1j} a_{2k}$

$\det A_3(2x \to 3x) = \varepsilon_{ijk} a_{0i} a_{1j} a_{3k} = a_{3k}$

So

$\det A_4 = a_{34} \varepsilon_{ijk} a_{0i} a_{1j} a_{2k} - a_{23} \varepsilon_{ijk} a_{0i} a_{1j} a_{3k}$

The matrices are triangular plus one upper off diagonal.

We know we can "roll things up" in this way

$x_1 = (-1/a_{01})\, a_{00}\, x_0$

$x_2 = (-1/a_{12}) [a_{10} x_0 + a_{11} x_1] = (-1/a_{12}) [a_{10} x_0 + a_{11}\{(-1/a_{01})\, a_{00}\, x_0\}]$

$= (-1/a_{12})(-1/a_{01})[\, a_{10}(-a_{01}) + a_{11}\, a_{00}]\, x_0 = (-1/a_{12})(-1/a_{01}) \det A_2 x_0$

$x_3 = (-1/a_{23}) [a_{20} x_0 + a_{21} x_1 + a_{22} x_2]$

$= (-1/a_{23}) [a_{20} x_0 + a_{21}\{ (-1/a_{01})\, a_{00}\, x_0\} + a_{22}\{(-1/a_{12})(-1/a_{01}) \det A_2\} x_0]$

$= (-1/a_{23})(-1/a_{12})(-1/a_{01}) [(-a_{12})(-a_{01}) a_{20} x_0 + (-a_{12}) a_{21}\{ a_{00}\, x_0\} + a_{22}\{ \det A_2\} x_0]$

$= (-1/a_{23})(-1/a_{12})(-1/a_{01}) [(-a_{12})(-a_{01}) a_{20} + (-a_{12}) a_{21}\{ a_{00}\} + a_{22}\{ \det A_2\}]\, x_0$

$= (-1/a_{23})(-1/a_{12})(-1/a_{01}) [(-a_{12})[\, a_{00} a_{21} - a_{01} a_{20}] + a_{22}\{ \det A_2\}]\, x_0$

$= (-1/a_{23})(-1/a_{12})(-1/a_{01}) \det A_3\, x_0$

We can see the pattern and we could do an induction proof, but here is the pattern

$x_1 = (-1/a_{01}) \det A_1 x_0$
$x_2 = (-1/a_{01})(-1/a_{12}) \det A_2 x_0$
$x_3 = (-1/a_{01})(-1/a_{12})(-1/a_{23}) \det A_3\, x_0$
$x_4 = (-1/a_{01})(-1/a_{12})(-1/a_{23})(-1/a_{34}) \det A_4\, x_0$
....

$x_n = (-1)^n \det A_n / (a_{01}\, a_{12}\, a_{23} \ldots\ldots a_{n-1,n}) x_0$

Rewrite as

$x_1/x_0 = (-1)^1 \, detA_1 / (a_{01})$
$x_2/x_0 = (-1)^2 \, detA_2 / (a_{01} \, a_{12})$
$x_3/x_0 = (-1)^3 \, detA_3 / (a_{01} \, a_{12} \, a_{23})$
$x_4/x_0 = (-1)^4 \, detA_4 / (a_{01} \, a_{12} \, a_{23} \, a_{34})$
. . .
$x_n/x_0 = (-1)^n \, detA_n / (a_{01} \, a_{12} \, a_{23} \, ... \, a_{n-1,n})$

The start of the induction would be

$x_k/x_0 = (-1)^k \, detA_k / (a_{01} \, a_{12} \, a_{23} \, ... \, a_{k-1,k})$

The equation of interest is

$a_{k0}x_0 + a_{k1}x_1 + a_{k2}x_2 + a_{k3}x_3 + a_{k4}x_4 + ... + a_{k,k+1} \, x_{k+1} = 0$

$x_{k+1}/x_0 = (-1/a_{k,k+1}) \, [a_{k0} + a_{k1}x_1/x_0 + a_{k2}x_2/x_0 + a_{k3}x_3/x_0 + a_{k4}x_4/x_0 + a_{k-1,k} \, x_k/x_0]$

$= (-1) \, [a_{k0} + a_{k1}x_1/x_0 + a_{k2}x_2/x_0 + a_{k3}x_3/x_0 + a_{k4}x_4/x_0 + a_{k-1,k} \, x_k/x_0]/a_{k,k+1}$

$= (-1) \, [\, a_{k0} + a_{k1} \, (-1)^1 \, detA_1 / (a_{01}) + a_{k2}(-1)^2 \, detA_2 / (a_{01} \, a_{12})$
$+ a_{k3} \, (-1)^3 \, detA_3 / (a_{01} \, a_{12} \, a_{23}) +$
$.. + a_{k-1,k}(-1)^k \, detA_k / (a_{01} \, a_{12} \, a_{23} \, ... \, a_{k-1,k}) \,] \, /a_{k,k+1}$

we need some supporting theorem which says

$(-1)^{k+1} \, detA_{k+1} / (a_{01} \, a_{12} \, a_{23} \, ... \, a_{k,k+1})$

$= (-1) \, [\, a_{k0} + a_{k1} \, (-1)^1 \, detA_1 / (a_{01}) + a_{k2}(-1)^2 \, detA_2 / (a_{01} \, a_{12})$
$+ a_{k3} \, (-1)^3 \, detA_3 / (a_{01} \, a_{12} \, a_{23}) +$
$.. + a_{k-1,k}(-1)^k \, detA_k / (a_{01} \, a_{12} \, a_{23} \, ... \, a_{k-1,k}) \,] \, /a_{k,k+1}$

or

$(-1)^k \, detA_{k+1} / (a_{01} \, a_{12} \, a_{23} \, ... \, a_{k,k+1})$

$= [\, a_{k0} + a_{k1} \, (-1)^1 \, detA_1 / (a_{01}) + a_{k2}(-1)^2 \, detA_2 / (a_{01} \, a_{12})$
$+ a_{k3} \, (-1)^3 \, detA_3 / (a_{01} \, a_{12} \, a_{23}) +$
$.. + a_{k-1,k}(-1)^k \, detA_k / (a_{01} \, a_{12} \, a_{23} \, ... \, a_{k-1,k}) \,] \, /a_{k,k+1}$

or

$detA_{k+1}$

$$
\begin{aligned}
= [\quad & a_{k0} && (a_{01}\, a_{12}\, a_{23} \ldots\ldots a_{k,k+1}) \\
& + a_{k1}\,(-1)^{k-1}\, \det A_1 && (\, a_{12}\, a_{23} \ldots\ldots a_{k,k+1}) \\
& + a_{k2}(-1)^{k-2}\, \det A_2 && (\, a_{23} \ldots\ldots a_{k,k+1}) \\
& + a_{k3}\,(-1)^{k-3}\, \det A_3 && (\, a_{34} \,.\, a_{k,k+1}) \\
& + \ldots. \\
& + a_{k-2,k-1}\,(-1)^{1}\, \det A_{k-1} && (a_{k,k+1}) \\
& + \; a_{k-1,k} \det A_k
\end{aligned}
$$

So the secret to this induction proof is showing this decomposition of the determinant for a matrix of this nearly triangular form.

Ah yes. I think you use those row operations which preserve the determinant!

Theorem 2D: Add a multiple of one row to another, det does not change.

So start with $\det A_4$ and our picture above. Multiply third row by $-a_{34}/a_{23}$ and add to 4th row to get

$$
\begin{pmatrix}
a_{00} & a_{01} & 0 & 0 \\
a_{10} & a_{11} & a_{12} & 0 \\
a_{20} - a_{30}a_{34}/a_{23} & * & * & 0 \\
a_{30} & a_{31} & a_{33} & a_{34}
\end{pmatrix}
$$

You just keep doing this.

OK, enough fiddling on this. I could do it if I had to. I wonder if this little theorem has a name or some kind of handle people use to talk about it. Here is the theorem