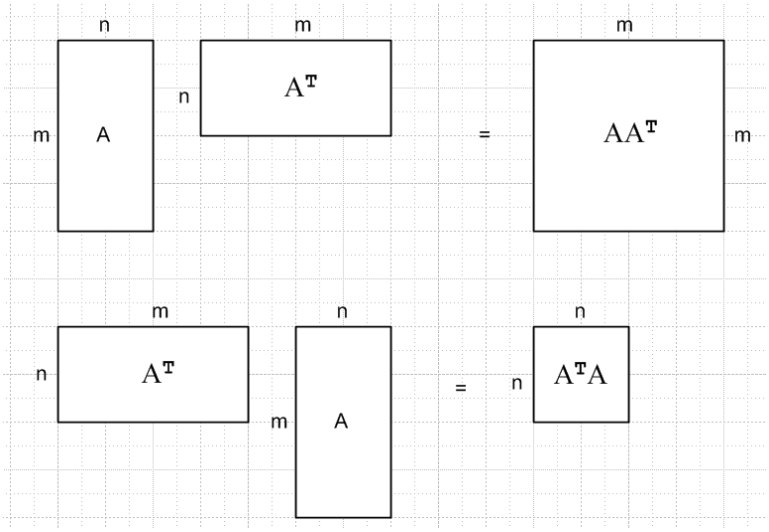
Determinant Theorems Associated with non-square matrices PhL 3.22.16

Suppose A is non-square. We know that both AA^T and A^TA are square, though of different sizes.

Suppose A is m x n (m rows, n columns, tall, m > n). Then we know

AA^T = m x m matrix = large matrix

A^TA = n x n matrix = small matrix



Theorem 1: If C = AB, then $C^T = B^TA^T$ even if A and B are non-square, as long as they conform.

Proof: $C_{ik} = A_{ij}B_{jk} = B_{jk}A_{ij} = (B^T)_{kj}(A^T)_{ji} = (B^TA^T)_{ki} = [(B^TA^T)^T]_{ik}$
Therefore $C = (B^TA^T)^T$ and $C^T = B^TA^T$. QED

Theorem 2: (rank theorems) A and B are conforming matrices where the common dimension is n

(a) $\text{rank}(AB) \le \min\{\text{rank}(A),\text{rank}(B)\}$ (4.10) "theorems on rank.pdf"
(b) $\text{rank}(A^\dagger A) = \text{rank}(AA^\dagger) = \text{rank}(A)$ for any complex matrix A (4.12) "same"
(c) $\text{rank}(A^TA) = \text{rank}(AA^T) = \text{rank}(A)$ for any real matrix A (special case of above)
(d) $\text{rank}(A) + \text{rank}(B) \le \text{rank}(AB) + n$ (Sylvester's Inequality, see web)
(e) $\text{rank}(AB) + \text{rank}(BC) \le \text{rank}(ABC) + \text{rank}(B)$ (Frobenius Inequality, see web)

Proofs: The first three have simple proofs in a pdf I saved. The last two have proofs are on the web. As far as I know, the first three theorems do not have names.

Theorem 3: Assume that A is an m x n matrix where $m \ne n$. Then:

(a) If m > n, then $\det(AA^T) = 0$ and thus AA^T, although square, is not invertible.
(b) If m < n, then $\det(A^TA) = 0$ and thus A^TA, although square, is not invertible.
(c) In general, $\det(AA^T) \ne \det(A^TA)$ in the above two cases.
(d) If m = n, then $\det(AA^T) = \det(A^TA) = \det^2(A)$ and the two agree.

Proof: (a) By Theorem 2 (c) we know that rank(AA^T) = rank(A) ≤ n since n is the smaller dimension of A. But for m > n the matrix AA^T is an m x m matrix shown in the graphic above. If it has rank ≤ n, then it does not have full rank, and thus it must have det(AA^T) = 0.

(b) replace A→A^T and (b) becomes (a).

(c) counterexample given below

In case (a) if instead we have m < n, then rank(AA^T) = rank(A) ≤ m but now AA^T is mxm so all we know in this case is that rank(AA^T) ≤ m so it could be full rank, or it could be less than full rank. In general then you will get det(AA^T) ≠ 0 in this case. So in our theorem, the claim that det(..) = 0 only applies to the case of m and n shown, and the opposite case will not in general have det(..) = 0.

Comment: I could not prove this theorem just writing out the determinant expansion for det(AA^T). It did not seem obvious that det(AA^T) = 0 for m > n. For example in the special case that m = 3 and n = 2 we have AA^T being a 3x3 matrix, so

$$\det(AA^T) = \varepsilon_{abc}(AA^T)_{1a}(AA^T)_{2b}(AA^T)_{3c} = \varepsilon_{abc}\ A_{1j}A_{aj}A_{2k}A_{bk}\ A_{3r}A_{cr}$$

$$= A_{1j}A_{2k}A_{3r}\ \varepsilon_{abc}A_{aj}A_{bk}A_{cr} = \text{etc} = \text{???}$$

Example: Here is a general matrix with m = 3 rows and n = 2 columns so m > n :

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A := matrix(3,2,[a,b,c,d,e,f]);
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$$A := \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

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AT := transpose(A);
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$$AT := \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$$

We compute AA^T and det(AA^T):

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X := evalm(A &* AT);
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$$X := \begin{bmatrix} a^2+b^2 & a\,c+b\,d & a\,e+b\,f \\ a\,c+b\,d & c^2+d^2 & c\,e+d\,f \\ a\,e+b\,f & c\,e+d\,f & e^2+f^2 \end{bmatrix}$$

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det(X);
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$$0$$

which agrees with Theorem 3 (a). Now compute A^TA and det(A^TA) and get some general non-zero result for the determinant. In this case

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B := evalm(AT &* A);
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$$B := \begin{bmatrix} a^2+c^2+e^2 & ab+cd+ef \\ ab+cd+ef & b^2+d^2+f^2 \end{bmatrix}$$

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det(B);
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$$a^2d^2+a^2f^2+c^2b^2+c^2f^2+e^2b^2+d^2e^2-2abcd-2abef-2cedf$$

Implication: Suppose the R matrix has m rows and n columns and is tall with m > n. Then

det(RR^T) = 0 and thus RR^T, although square, is not invertible

Paradox?: In wedge doc I say that $RR^T = 1$ for any R, where this is all say down-tilt. If we ignore the italic T, this seems to contradict the fact that det(RR^T) = 0 for m > n. Please explain.

Resolution: R^T simply is not the same as R^T for downtilt or uptilt, so the above discussion simply does not apply. $RR^T = 1$ is the orthog relation which remains true from the chain rule.

Theorem 4: det(CAA^T) = det(AA^TC) where C is a conforming square matrix

Proof: Obvious since one can write

det(CAA^T) = det(C[AA^T]) = det(C)det(AA^T) =det([AA^T]C) = det(AA^TC) QED

Theorem 5: det(CAA^T) = det(AA^TC^T) where C is a conforming square matrix

Proof : Let D = CAA^T . Then

det(CAA^T) = det(D) = det(D^T) = det(AA^TC^T) QED

Theorem 6: det(CAA^T) = det(A^TCA) ?? cyclic forward one position // not true

Proof: If m > n and C is m x m, then CA has the same shape as A, so at least the matrices conform on both sides of the above. But the theorem is not valid. Here is a simple counterexample: if C = 1, we know in general that det(AA^T) ≠ det(A^TA). When m ≠ n in fact one of these determinants is 0 and the other generally is not, as shown in Theorem 3.

So expunge the notion that you are allowed to cyclic permute things in a determinant with non-square matrices and maintain the determinant value.