Simple Examples of Non-Square R PhL 3.26.16

I should have done examples earlier, but better late than never.

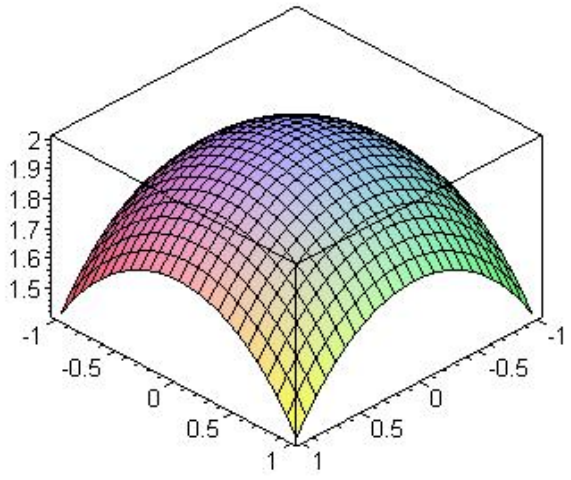
Example 1: Suppose in x-space we have a unit square centered at the origin as the domain. Take these equations:

$x'^1 = x^1$
$x'^2 = x^2$
$x'^3 = (R^2 - (x^1)^2 - (x^2)^2)^{1/2}$ = height of dome at (x^1, x^2) =

```
restart; R := 2;
```

$$R := 2$$

```
plot3d([x1,x2,sqrt(R^2 - x1^2-x2^2)],x1=-1..1,x2=-1..1,axes = boxed);
```



So this example maps the unit square in x-space to a spherical dome in x'-space. Now let's compute everything in sight! First

$x'^3 = (R^2 - (x^1)^2 - (x^2)^2)^{1/2}$

$\partial_j x'^3 = (1/2)\ (R^2 - (x^1)^2 - (x^2)^2)^{-1/2} * (-2x^j) = -x^j\ (R^2 - (x^1)^2 - (x^2)^2)^{-1/2} = -x^j/h$

$R^i{}_j \equiv (\partial x'^i/\partial x^j)$:

$R^1{}_1 = (\partial x'^1/\partial x^1) = 1$ $\qquad$ $R^1{}_2 = (\partial x'^1/\partial x^2) = 0$
$R^2{}_1 = (\partial x'^2/\partial x^1) = 0$ $\qquad$ $R^2{}_2 = (\partial x'^2/\partial x^2) = 1$
$R^3{}_1 = (\partial x'^3/\partial x^1) = -x^1\ (R^2 - (x^1)^2 - (x^2)^2)^{-1/2}$ $\qquad$ $R^3{}_1 = (\partial x'^3/\partial x^1) = -x^2\ (R^2 - (x^1)^2 - (x^2)^2)^{-1/2}$

So

$$\begin{matrix} 1 & 0 \end{matrix}$$

$$R = \begin{matrix} 0 & 1 \\ -x^1/h & -x^2/h \end{matrix}$$ = the "tall" R matrix

I have this all going in Maple now.

So what do I think is the inverse equation $\mathbf{x} = \mathbf{F}^{-1}(\mathbf{x}')$?

Well, try this

$x^1 = x'^1$
$x^2 = x'^2$

This seems to "exist". Then what is S?

$S^i{}_j \equiv (\partial x^i/\partial x'^j)$

$S^1{}_1 = (\partial x^1/\partial x'^1) = 1$ $\quad S^1{}_2 = (\partial x^1/\partial x'^2) = 0$ $\quad S^1{}_3 = (\partial x^1/\partial x'^2) = 0$
$S^2{}_1 = (\partial x^2/\partial x'^1) = 0$ $\quad S^2{}_2 = (\partial x^2/\partial x'^2) = 1$ $\quad S^2{}_3 = (\partial x^1/\partial x'^2) = 0$

So then

$$S = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}$$

There is my "non-existent" S matrix. I am able to do the derivative $(\partial x^i/\partial x'^j)$ in the j axis direction in x'-space even though this takes me off the surface of the sphere. This is probably because my initial transformation functions are "diffeomorphic".

Recall from "inverting non-square matrices":

Theorem: If R is a "tall" matrix of full rank, then $(R^{-1}{}_\ell)$ exists and $(R^{-1}{}_r)$ does not exist.

So here is what Maple has to say:

```
R := matrix(3,2,R_);
```

$$R := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{x_1}{\sqrt{4-x_1{}^2-x_2{}^2}} & -\frac{x_2}{\sqrt{4-x_1{}^2-x_2{}^2}} \end{bmatrix}$$

```
S := matrix(2,3,[1,0,0,0,1,0]);
```

$$S := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

```
evalm(R &* S);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{x_1}{\sqrt{4-x_1{}^2-x_2{}^2}} & -\frac{x_2}{\sqrt{4-x_1{}^2-x_2{}^2}} & 0 \end{bmatrix}$$

```
evalm(S &* R);
```

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Here you see that my tall R matrix has SR = 1 so (R^{-1}_{ℓ}) exists and in fact $(R^{-1}_{\ell}) = S$.

On the other hand, you see that with this same S and R, RS = a 3x3 matrix which is not 1. This shows only that $(R^{-1}_{r}) \neq S$, but from the simple theory I know that there exists no S shaped matrix B so RB = 1.

Conclusions:

(1) My Section 10.6 v6 doc which claims that "S does not exist" and "anything depending on S does not exist" is completely wrong! Here we have a simple counterexample which shows that S does exist for a tall R matrix. You write out the inverse transformation, and you compute the S derivatives. Yes, the inverse transformation $\mathbf{x} = \mathbf{F}^{-1}(\mathbf{x}')$ is defined only for $\mathbf{x}'$ on the sphere manifold. But we write out the detailed equations for $\mathbf{x} = \mathbf{F}^{-1}(\mathbf{x}')$ with $\mathbf{x}'$ on the manifold and we find that we can differentiate those equations to get S just fine, even though derivatives take us in directions off the manifold.

(2) We compute R and compute S and we find that SR = 1 while RS ≠ 1 and this would be true for any size tall R matrix. Specifically I know that for a tall R matrix, you can find at least one left inverse if R is full rank, but you cannot find any right inverse. That is consistent with SR = 1 and RS ≠ 1/

(3) What does the chain rule have to say here?

$$R^{i}{}_{j} S^{j}{}_{k} = (\partial x'^{i}/\partial x^{j})(\partial x^{j}/\partial x'^{k}) = \frac{\partial x'^{i}}{\partial x^{j}} \frac{\partial x^{j}}{\partial x'^{k}} \neq \delta^{i}{}_{k} \qquad RS \neq 1$$

So how do I explain that "the chain rule fails in this direction? Totally new to me. How do you prove the chain rule in the first place? Bucks do a very long proof starting on page 250 for the case that you have the same number of variables on each side. It is a non-trivial proof.

Wiki suggests that the chain rule should be valid in the non-square situation as well.

But this is not really what the chain rule says! The "chain rule" is a statement about the derivative of a concatenation of two functions for example

$$\mathbf{f}(\mathbf{x}') \qquad \mathbf{x}' = \mathbf{F}(\mathbf{x}) \qquad \mathbf{f}(\mathbf{F}(\mathbf{x})) = (\mathbf{f} \circ \mathbf{F})(\mathbf{x})$$

$$\partial f^{k}(\mathbf{x}')/\partial x^{i} = \Sigma_{j} \frac{\partial f^{k}(\mathbf{x}')}{\partial x'^{j}} \frac{\partial x'^{j}}{\partial x^{i}}$$

Now suppose you take this particular function

$$\mathbf{f}(\mathbf{x}') = \mathbf{x}' \qquad f^{k}(\mathbf{x}') = x'^{k}$$

Then the chain rule above says

$$\partial x'^{k}/\partial x^{i} = \Sigma_{j} \frac{\partial x'^{k}}{\partial x'^{j}} \frac{\partial x'^{j}}{\partial x^{i}} = \frac{\partial x'^{k}}{\partial x^{i}} = \text{a triviality, no } \delta \text{ is involved here}$$

On the other hand consider

$$\mathbf{f}(\mathbf{x}) \qquad \mathbf{x} = \mathbf{F}^{-1}(\mathbf{x}') \qquad \mathbf{f}(\mathbf{F}^{-1}(\mathbf{x}')) = (\mathbf{f} \circ \mathbf{F}^{-1})(\mathbf{x}')$$

$$\partial f^{k}(\mathbf{x})/\partial x'^{i} = \Sigma_{j} \frac{\partial f^{k}(\mathbf{x})}{\partial x^{j}} \frac{\partial x^{j}}{\partial x'^{i}}$$

Now suppose you take this particular function

$$\mathbf{f}(\mathbf{x}) = \mathbf{x} \qquad f^{k}(\mathbf{x}) = x^{k}$$

Then the chain rule above says

$$\partial x^{k}/\partial x'^{i} = \Sigma_{j} \frac{\partial x^{k}}{\partial x^{j}} \frac{\partial x^{j}}{\partial x'^{i}}$$

or

$$\partial x^{k}/\partial x'^{i} = \frac{\partial x^{k}}{\partial x'^{i}} = \text{a triviality, no } \delta \text{ is involved here.}$$

So where am I getting the idea that you should have

$$\frac{\partial x'^{i}}{\partial x^{j}} \frac{\partial x^{j}}{\partial x'^{k}} = \delta^{i}{}_{k}$$

This fact does not "follow from the chain rule" in the non-square world!

Discussion of coordinate lines for this example

The inverse transformation (for points on M) was given above as

$$x^{1} = x'^{1}$$
$$x^{2} = x'^{2}$$

Let's now look at the "coordinate lines" in x-space generated by reverse-mapping lines of constant coordinate in x'-space. For example, if we fix x'^{2} and x'^{3} and vary only x'^{1}, we get a horizontal line in x-space which lies at vertical position $x^{2} = x'^{2}$. Conversely, if we fix x'^{1} and x'^{3} and vary only x'^{2}, we get a vertical line in x-space located at position x'_{1}.

But if we fix x'^{1} and x'^{2} and vary only x'^{3}, the coordinate line in x-space is just a point!

So the coordinate lines exist, but they might be points instead of lines.

What are the three $\mathbf{e}_{n}$ vectors in x-space? From my tensor doc definition,

$$\mathbf{e}_{n} \equiv S\mathbf{e}'_{n}.$$

I did these on scratch paper using the S matrix shown above and I find that

$\mathbf{e}_1 = (1,0)$	tangent to the x'_1 varying coordinate line discussed above
$\mathbf{e}_2 = (0,1)$	tangent to the x'_2 varying coordinate line discussed above
$\mathbf{e}_3 = (0,0)$	the x'_3 coordinate line is just a point, what is the tangent to a point?

Notice that the S matrix does have full rank 2. These $\mathbf{e}_n$ are the columns of matrix S just as it says in tensor doc.

So this is what happens in my simple example. What happens in a more general case?

Do some coordinate lines always become points? I need to do some more simple examples!!

Tilted plane example.

Here are my equations (the Maple plot shows a nice flat tilted plane at a weird angle).

$$x'_1 = x_1+2x_2 \qquad \mathbf{x}' = R\mathbf{x}$$
$$x'_2 = 2x_1+x_2$$
$$x'_3 = x_1 + 3x_2$$

To invert this, we have to solve 3 equations in 2 unknowns. In theory solution is not unique. So let's try a few methods:

Solution 1. From first two have

$$2x'_1 = 2x_1+4x_2$$
$$x'_2 = 2x_1+x_2 \qquad \Rightarrow \qquad 2x'_1 - x'_2 = 3x_2 \Rightarrow x_2 = (2x'_1 - x'_2)/3 = 2/3\ x'_1 - 1/3\ x'_2$$

Then first one says

$$x'_1 = x_1+2x_2 = x_1 + 2(2x'_1 - x'_2)/3 = x_1 + 4/3\ x'_1 - 2/3\ x'_2$$

$$\Rightarrow x'_1 - 4/3\ x'_1 + 2/3\ x'_2 = x_1 = -1/3\ x'_1 + 2/3\ x'_2$$

So my solution then is

$$x_1 = -1/3\ x'_1 + 2/3\ x'_2$$
$$x_2 = 2/3\ x'_1 - 1/3\ x'_2$$

This should satisfy the first two equations. The third equation then says

$$x'_3 = x_1 + 3x_2 = -1/3\ x'_1 + 2/3\ x'_2 + 2\ x'_1 - x'_2 = 5/3\ x'_1 - 1/3\ x'_2$$

and this then must be the equation of the tilted plane. Write as

$5/3\ x'_1 - 1/3\ x'_2 - x'_3 = 0$

Solution 2. From the last two we have

$x'_2 = 2x_1+x_2$

$2x'_3 = 2x_1 + 6x_2 \quad \Rightarrow \quad x'_2 - 2x'_3 = -5\ x_2 \quad \Rightarrow \quad x_2 = -1/5\ x'_2 + 2/5\ x'_3$

Now pick either of our last two equations of the triplet, such as

$x'_3 = x_1 + 3x_2 \Rightarrow \quad x_1 = x'_3 - 3x_2 = x'_3 - 3[-1/5\ x'_2 + 2/5\ x'_3] = x'_3 + 3/5\ x'_2 - 6/5\ x'_3$

$= -1/5\ x'_3 + 3/5\ x'_2$

So my solution so far is

$x_1 = 3/5\ x'_2 - 1/5\ x'_3$

$x_2 = -1/5\ x'_2 + 2/5\ x'_3$

Then first equation $x'_2 = 2x_1+x_2$ says

$x'_1 = x_1+2x_2 = 3/5\ x'_2 - 1/5\ x'_3 - 2/5\ x'_2 + 4/5\ x'_3$

$= 1/5\ x'_2 + 3/5\ x'_3$

which is the plane equation, write as

$x'_1 - 1/5\ x'_2 - 3/5\ x'_3 = 0$

Multiply through by 5/3 to get

$5/3\ x'_1 - 1/3\ x'_2 - x'_3 = 0$

and this agrees with the plane from Solution 1.

If I picked the first and third equations, I would get a third solution. But forget that, here are the two solutions I found :

<u>Solution 1</u>

$x_1 = -1/3\ x'_1 + 2/3\ x'_2$

$x_2 = 2/3\ x'_1 - 1/3\ x'_2$

$S_{11} = -1/3 \qquad S_{12} = 2/3 \qquad S_{13} = 0$

$S_{21} = 2/3 \qquad S_{22} = -1/3 \qquad S_{23} = 0$

I have Maple "examples non-squre v2" and I enter things.

```
R := matrix(3,2,R_);
```

$$R := \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 3 \end{bmatrix}$$

```
S := matrix(2,3,[-1/3,2/3,0,2/3,-1/3,0]);
```

$$S := \begin{bmatrix} \frac{-1}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & \frac{-1}{3} & 0 \end{bmatrix}$$

```
evalm(R &* S);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{5}{3} & \frac{-1}{3} & 0 \end{bmatrix}$$

```
evalm(S &* R);
```

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

As expected, I get RS ≠ 1 and SR = 1. Now consider the second solution

<u>Solution 2</u>

$x_1 = 3/5\ x'_2 - 1/5\ x'_3$

$x_2 = -1/5\ x'_2 + 2/5\ x'_3$

$S_{11} = 0$	$S_{12} = 3/5$	$S_{13} = -1/5$
$S_{21} = 0$	$S_{22} = -1/5$	$S_{23} = 2/5$

I enter this different S matrix and get

```
R := matrix(3,2,R_);
```

$$R := \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 3 \end{bmatrix}$$

```
S := matrix(2,3,[0,3/5,-1/5,0,-1/5,2/5]);
```

$$S := \begin{bmatrix} 0 & \frac{3}{5} & \frac{-1}{5} \\ 0 & \frac{-1}{5} & \frac{2}{5} \end{bmatrix}$$

```
evalm(R &* S);
```

$$\begin{bmatrix} 0 & \frac{1}{5} & \frac{3}{5} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
evalm(S &* R);
```

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Again we get RS ≠ 1 and SR = 1, and the RS matrix is not the same as with the solution 1 S.

Comments: The new feature of this Example is there are at least two solutions for matrix S, and for the two that I found, both have RS ≠ 1 and SR = 1. The previous Example had only one solution matrix S. This problem has a third solution matrix S and it would do the same thing. We know from SR = 1 that R must have a left inverse called S, and we know that S = Dx/Dx', and any S we find will be a left inverse.

Discussion of coordinate lines for this example

$x'_1 = x_1+2x_2$ $\quad\quad$ $\mathbf{x}' = R\mathbf{x}$

$x'_2 = 2x_1+x_2$

$x'_3 = x_1 + 3x_2$

$x_1 = -1/3\ x'_1 + 2/3\ x'_2$ $\quad$ **Solution 1** from first two of triplet

$x_2 = \ \ 2/3\ x'_1 - 1/3\ x'_2$

Let's now look at the "coordinate lines" in x-space generated by reverse-mapping lines of constant coordinate in x'-space. For example,

If we fix x'_2 and x'_3 and vary only x'_1, then both x_1 and x_2 vary according to the doublet. They must vary according to T2 $x'_2 = 2x_1+x_2$ which gives the coordinate line as a certain line in x-space. For different fixed values of x'_2 we get a family of parallel lines in x-space.

If we fix x'_1 and x'_3 and vary only x'_2, then both x_1 and x_2 vary according to the doublet. They must vary according to T1 $x'_1 = x_1+2x_2$ which gives the coordinate line as a certain line in x-space. For different fixed values of x'_1 we get a family of parallel lines in x-space. These lines have a different slope from the previous set of lines.

If we fix x'_1 and x'_2 and vary only x'_3, then x_1 and x_2 of the doublet do not vary at all, so the coordinate line is a point.

What about the e_n ? They are the columns of the solution 1 matrix S, so

$$S := \begin{bmatrix} \frac{-1}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & \frac{-1}{3} & 0 \end{bmatrix}$$

you read off the first two, and $\mathbf{e}_3 = (0,0)$. As in our first example, when a coordinate line comes out being a point, the tangent base vector is 0. !!

What happens with +? Same triplet, but doublet is

$x_1 = 3/5\ x'_2 - 1/5\ x'_3$
$x_2 = -1/5\ x'_2 + 2/5\ x'_3$

In this case we get lines if we vary either x'_2 or x'_3, holding others constant. But if we vary x'_1 we get a point for the coordinate. Line. The S matrix is then

$$S := \begin{bmatrix} 0 & \frac{3}{5} & \frac{-1}{5} \\ 0 & \frac{-1}{5} & \frac{2}{5} \end{bmatrix}$$

In this case we get $\mathbf{e}_1 = 0$ which matches the fact that varying x'_1 must has a point for the coordinate line.

So look at this general idea

$\mathbf{e}_n = \partial \mathbf{x} / \partial x_n'$

It seems that you are going have three tangent base vectors, but one of the coordinate lines degenerates to a point and its corresponding tangent base vector is null. So the good news is that you end up with two reasonable tangent base vectors in x-space.

I am not sure what the general case would be. My second example really covers all the linear cases.