Inverting non-square matrices PhL 3.3.16

[See v2 of this doc for a more sophisticated analysis.]

So now I am off to Linear Algebra for Non Square Matrices world. I have some notes somewhere on this. I can find no PL notes, but I do find this in **wiki** on invertible matrix

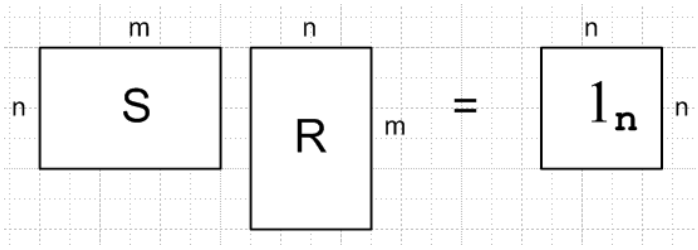
"Non-square matrices (*m*-by-*n* matrices for which $m \neq n$) do not have an inverse. However, in some cases such a matrix may have a left inverse or right inverse. If **A** is *m*-by-*n* and the rank of **A** is equal to *n*, then **A** has a left inverse: an *n*-by-*m* matrix **B** such that **BA** = **I**. If **A** has rank *m*, then it has a right inverse: an *n*-by-*m* matrix **B** such that **AB** = **I**."

So assuming this is correct, I will translate what it says my the current wedge doc Chapter 10 context. In this context, R is a matrix with m rows and n columns with m > n, so R is m x n. So

R has m rows and n columns and so is an m x n matrix. The max rank that this matrix could have I know is the smaller of the two, which for me means rank ≤ n. If R has this full rank n, then (R^{-1}_{ℓ}) exists and then of course we have $(R^{-1}_{\ell})\ R = 1_n$. Since m > n matrix R can never have rank m, so the second claim made above does not apply, and I guess I conclude that (R^{-1}_{r}) does not exist.

Theorem: If R is a "tall" matrix of full rank, then (R^{-1}_{ℓ}) exists and (R^{-1}_{r}) does not exist.

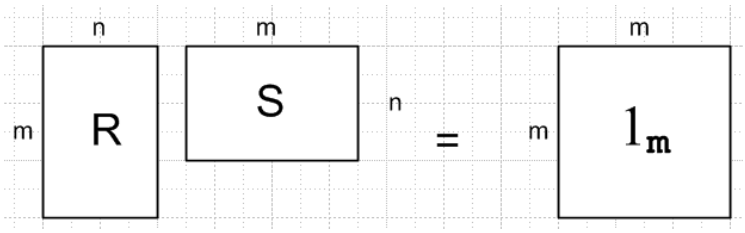
Consider again the graphic above where S is on the right,



Question: Does the matrix $S = (R^{-1}_{\ell})$ exist? Answer: yes, as shown below

If you write out all the equations, you get n^2 equations (one for each entry on the right). If you regard the elements of S as unknowns, there are then n*m unknowns. If you want to solve for the elements of S you have n^2 linear equations in n*m unknowns. Since n < m, there are then fewer equations than there are unknowns, so there are then many solutions and then you can say that you can always a candidate for $S = (R^{-1}_{\ell})$, in agreement with the claim made above. If you instead regard the elements of S as unknowns, you still have n^2 equations in n*m unknowns, so you always have a candidate for $(S_r{}^{-1})$.

On the other hand, consider

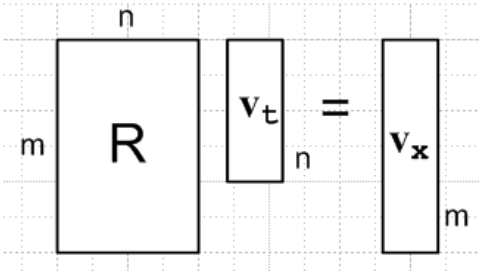


Now there are m^2 equations in n*m unknowns (one for each entry on the right), so in general there are too many constraints on those unknowns, so there will be no solution. Then $(S^{-1}\ell)$ and $(R^{-1}{}_r)$ do not exist.

So what does rank have to do with it?

I need to add this section later. I think a matrix has to be full rank for the inverse to really exist, but maybe not. ******

Related topic. Consider this equation $R\mathbf{v_t} = \mathbf{v_x}$ where as drawn, m > n.



Can we solve this for $\mathbf{v_t}$ given $\mathbf{v_x}$? Two methods to answer this question:

(1) There are clearly m equations. If $\mathbf{v_x}$ is known and we seek $\mathbf{v_t}$ which has n unknowns, then we have m equations in n unknowns. Problem is overspecified so there is no R^{-1} .

(2) We showed above that $(R^{-1}\ell)$ exists. So we apply that to both sides of $R\mathbf{v_t} = \mathbf{v_x}$ to get

$R\mathbf{v_t} = \mathbf{v_x}$

$(R^{-1}\ell)\, R\mathbf{v_t} = (R^{-1}\ell)\, \mathbf{v_x}$

$\mathbf{v_t} = (R^{-1}\ell)\, \mathbf{v_x}$

So yes, you can invert the equation $R\mathbf{v_t} = \mathbf{v_x}$

Question: Why do my two methods give conflicting results? Here is the answer. In the first method, we treat the vector $\mathbf{v_x}$ as if it were completely arbitrary. But in fact, $\mathbf{v_x}$ must be a vector that is obtained by application of R to some vector $\mathbf{v_t}$. So it is NOT completely arbitrary. Consider R: $R^n \rightarrow R^m$. The vectors $\mathbf{v_x}$ must be in the *range* of operator R and this range is some dimension-n subspace of R^m. If we consider

arbitrary vectors $\mathbf{v_x}$ in R^m , you will NOT be able to solve for $\mathbf{v_t}$ just as (1) says. But for vectors $\mathbf{v_x}$ in the range, you CAN find a solution, that given by (2).