Frobenius Method Applied to the Associated Legendre Equation PhL 7.26.09

I have run into a lot of confusion and contradiction here, so we now have a separate document to do battle with the subject.

First, here is the ODE of interest:

$(1-z^2)\,u'' - 2zu' + [\,\lambda - m^2/(1-z^2)\,]\,u = 0$ // this form appears Schaum p 149

Expand around the point z = 1

To this end, I want to cast this equation into "standard Frobenius form" with respect to z = 1. This will take a few processing steps:

$(1-z^2)\,u'' - 2zu' + [\,\lambda - m^2/(1-z^2)\,]\,u = 0$ // as above

$(1+z)\,(1-z)\,u'' - 2zu' + [\,\lambda - m^2/\{(1+z)\,(1-z)\}\,]\,u = 0$ // trivial

$(1-z)\,u'' - 2z/(1+z)u' + [\,\lambda/(1+z) - m^2/\{(1+z)^2\,(1-z)\}\,]\,u = 0$ // divide through by (1+z)

$(1-z)^2\,u'' - 2z(1-z)/(1+z)u' + [\,\lambda(1-z)/(1+z) - m^2/\{(1+z)^2\}\,]\,u = 0$ // mult thru by (1-z)

$(z-1)^2\,u'' + 2z(z-1)/(1+z)u' + [\,-\lambda(z-1)/(1+z) - m^2/\{(1+z)^2\}\,]\,u = 0$ // change some signs

$(z-1)^2\,u'' + (z-1)\,\{2z/(1+z)\}u' + \{\,-\lambda(z-1)/(1+z) - m^2/(1+z)^2\,\}\,u = 0$ // std form

We can now read off our standard form which is this for general point z_0

$(z-z_0)^2 u''(z) + (z-z_0)\,P(z)u'(z) + Q(z)u(z)$

We therefore find that

$P(z) = \{2z/(1+z)\}$ $Q(z) = \{\,-\lambda(z-1)/(1+z) - m^2/(1+z)^2\,\}$

$P(1) = 1$ $Q(1) = -m^2/4$

The indicial equation with reference to z = 1 expansion has this form

$r(r-1) + rP(1) + Q(1) = 0$

$r(r-1) + r - m^2/4 = 0$

$r^2 = m^2/4$

$r = \pm\, m/2$

Thus, we can take

$r_1 = |m|/2 \qquad r_2 = -|m|/2$

Case 3: $r_1 = r_2 + s \qquad s = \text{integer} > 0$

$$w_1(z) = (z-1)^{|m|/2}\,[\,1 + \Sigma_{n=1}^{\infty} A_n^{(1)}\,(z-1)^n]$$
$$w_2(z) = (z-1)^{-|m|/2}\,[\,-1/s + \Sigma_{n=1}^{\infty} h_n\,(z-1)^n\,] + g_s\,w_1(z)\ln(z-1)$$
$$w_2(z) = (z-1)^{-|m|/2}\,[\,1 + \Sigma_{n=1}^{\infty} (-sh_n)\,(z-1)^n\,] + (-sg_s)\,w_1(z)\ln(z-1)$$

taken from my "regular singular points" doc. Now for m = positive integer, we know we can write solutions in this manner: (AS p 334)

$P_\nu^m(z) = (z-1)^{m/2}\,\{(z+1)^{m/2}\,\partial_z^m\,P_\nu(z)\}$ where $\nu(\nu+1) = \lambda$ = arbitrary value

This is consistent with the $w_1(z)$ solution shown above, that is my main point. The series in w_1 must be an infinite series in (z-1) which adds up to $(z+1)^{m/2}\,\partial_z^m\,P_\nu(z)$.

More detail added 10.8.10. For example, consider $f(z) = (z+1)^a$ where we have in mind a = m/2. Expand this in a Taylor series $\Sigma_{n=0}^{\infty} b_n\,(z-1)^n$ about the point z = 1. We have

$f(z) = (z+1)^a = 2^a$
$f'(z) = a(z+1)^{a-1} = a\,2^{a-1}$
$f^{(2)}(z) = a(a-1)(z+1)^{a-2} = a(a-1)\,2^{a-2}$ etc

$$(z+1)^a = 2^a + (z-1)\,a\,2^{a-1} + (z-1)^2/2!\,*\,a(a-1)\,2^{a-2} + \ldots$$

If you have doubts about this series, try it for a = 2 so $f = (z+1)^2$. This is recovered by the series for which only the first three terms are non-vanishing. So, my point is that, in this same manner, you could expand the quantity $\{(z+1)^{m/2}\,\partial_z^m\,P_\nu(z)\}$ in a power series about (z-1) by doing a similar Taylor series and relying on the fact that the P functions derivatives are continuous at z = 1. So when you see the total form of the result $P_\nu^m(z) = (z-1)^{m/2}\,\{(z+1)^{m/2}\,\partial_z^m\,P_\nu(z)\}$, if you are thinking about the regular point z = 1 and m>0, it really is showing here as the form $w_1(z) = (z-1)^{|m|/2}\,[\,\Sigma_{n=0}^{\infty} A_n^{(1)}\,(z-1)^n]$. The factor $(z+1)^{m/2}$ is analytic at z=1 so there is no problem doing a power series about z = 1, as shown above.

we know that we can expand:

$(z+1)^{m/2} =$

Expand around the point z = -1

Start in the same place:

$(1-z^2)\,u'' - 2zu' + [\,\lambda - m^2/(1-z^2)\,]\,u = 0$ // as above

$(1+z)(1-z)\,u'' - 2zu' + [\,\lambda - m^2/\{(1+z)(1-z)\}\,]\,u = 0$ // trivial

$(1+z)\,u'' - 2z/(1-z)u' + [\,\lambda/(1-z) - m^2/\{(1-z)^2(1+z)\}\,]\,u = 0$ // divide through by (1-z)

$(1+z)^2\,u'' - 2z(1+z)/(1-z)u' + [\,\lambda(1+z)/(1-z) - m^2/\{(1-z)^2\}\,]\,u = 0$ // mult thru by (1+z)

$(z+1)^2\,u'' - 2z(z+1)/(1-z)u' + [\,\lambda(z+1)/(1-z) - m^2/\{(1-z)^2\}\,]\,u = 0$ // replace 1+z = z+1

$(z+1)^2\,u'' + (z+1)\{-2z/(1-z)\}u' + \{\,\lambda(z+1)/(1-z) - m^2/\{(1-z)^2\}\,\}\,u = 0$ // standard form

$P(z) = \{-2z/(1-z)\}$ $Q(z) = \{\,\lambda(z+1)/(1-z) - m^2/\{(1-z)^2\}$

$P(-1) = 1$ $Q(-1) = -m^2/4$

The indicial equation with reference to z = -1 expansion has this form

$r(r-1) + rP(-1) + Q(-1) = 0$

$r(r-1) + r - m^2/4 = 0$

$r^2 = m^2/4$

$r = \pm\, m/2$

So we have the same two exponents that we had at the z = 1 point.

Case 3: $r_1 = r_2 + s$ s = integer > 0

$w_1(z) = (z+1)^{|m|/2}\,[\,1 + \Sigma_{n=1}^{\infty} A_n^{(1)}\,(z+1)^n]$
$w_2(z) = (z+1)^{-|m|/2}\,[\,-1/s + \Sigma_{n=1}^{\infty} h_n\,(z+1)^n\,] + g_s\,w_1(z)\,\ln(z+1)$
$w_2(z) = (z+1)^{-|m|/2}\,[\,1 + \Sigma_{n=1}^{\infty} (-sh_n)\,(z+1)^n\,] + (-sg_s)\,w_1(z)\,\ln(z+1)$

Now for m = positive integer, we know we can write solutions in this manner: (AS p 334)

$P_\nu^m(z) = (z+1)^{m/2}\,\{(z-1)^{m/2}\,\partial_z^m\,P_\nu(z)\,\}$

This is consistent with the $w_1(z)$ solution shown above, that is my main point. The series in w_1 must be an infinite series which adds up to $(z-1)^{m/2}\,\partial_z^m\,P_\nu(z)$.

Conclusion: For positive integer m and general ν (λ), our known form for the associated Legendres

$$P_\nu^m(z) = (z^2 - 1)^{m/2}\, \partial_z^m\, P_\nu(z)$$

is consistent with our Frobenius analysis at both z = 1 and z = -1. By the way, if z = ±1, then if m is any positive integer, we get $P_\nu^m(1) = 0$ (if ν = n). Then Jackson p 65 3.51 makes this true also for m being any negative integer. If m = 0, we get $P_\nu^0(1) = P_\nu(1) = 1$.