Laplace Transforms to solve simple ODEs PhL 1.17.11

This is not something I ever did much of, but I have an example on the whiteboard today and will write it down here (this became the first section of this document which then grew a bit)

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Laplace Transform method of solving a 2nd order ODE with constant coefficients.

Consider this ODE where we set the ∂_t^2 coefficient to 1 without loss of generality. We are only talking constant coefficients here.

$$[\,\partial_t^2 + B\partial_t + C\,]\, f(t) = g(t)$$

We know the following Laplace facts, which you prove with simple parts integration,

$$\int_0^\infty ds\, e^{-st} f(t) = F(s) \qquad \text{// Schaum page 162}$$

$$\int_0^\infty ds\, e^{-st} \partial_t f(t) = sF(s) - f(0)$$

$$\int_0^\infty ds\, e^{-st} \partial_t^2 f(t) = s^2F(s) - sf(0) - f'(0)$$

Thus, if we apply $\int_0^\infty ds\, e^{-st}$ to both sides of our ODE we get

$$[s^2F(s) - sf(0) - f'(0)] + B\,[sF(s) - f(0)] + C\,F(s) = G(s)$$

We want to solve this thing for F(s), so:

$$(s^2 + Bs + C)F(s) = G(s) + sf(0) + [\, f'(0) + Bf(0)\,]$$

The solution is then

$$F(s) = G(s)/(s^2 + Bs + C) + \{\, sf(0) + [\, f'(0) + Bf(0)\,]\}/(s^2 + Bs + C) \qquad (1)$$

One can factor $(s^2 + Bs + C) = (s - s_1)(s - s_2)$ and then write $s_1 + s_2 = -B$

$$1/(s^2 + Bs + C) = 1/[(s - s_1)(s - s_2)] = 1/(s_1 - s_2) * [\, 1/(s - s_1) - 1/(s - s_2)\,]$$

which is the sum of two poles. In this way we write F(s) as a sum of terms, and we look each term up in a Laplace transform table to get f(t).

A standard problem might have a driving function g(t) = cos(ωt) so G(s) = s/(s²+ω²).

If g(t)= 0, there is still a non-zero answer which describes a motion of the system caused by the initial conditions. In a damped system, these always die out in time, so these terms are called the transient response, and then the term G(s)/ (s² + Bs + C) is the significant response.

More details added 4.6.11.

The pole positions for s_1 and s_2 are these

$$s_{1,2} = -B/2 \pm\sqrt{B^2-4C}\,/2 \;=\; -B/2 \pm \sqrt{(B/2)^2-C}$$

If B is positive and $B^2 > 4C$, both poles are negative real. In a particle motion problem in which the two terms $\partial_t^2 + B\partial_t$ appear on the same side of the ODE, we are in effect saying F = ma ~ $-B\partial_t$. As shown in the Appendix below, we must have B>0 to have such a friction/damping problem. Therefore, both or our poles $s_{1,2}$ will be in the left half s-plane. And then each pole causes a term e^{+st} in the solution of our problem, which is an expo damping factor. So go back to our general solution to the driven equation shown above

$$F(s) = G(s)/ (s^2 + Bs + C) \;+\; \{ sf(0) + [f'(0) +Bf(0)]\}/ (s^2 + Bs + C) \qquad (1)$$

$$= G(s)\; 1/(s_1- s_2) * [\; 1/(s-s_1) – 1/(s-s_2) \;]$$

$$+ \{ sf(0) + [f'(0) +Bf(0)]\}\; 1/(s_1- s_2) * [\; 1/(s-s_1) – 1/(s-s_2) \;]$$

If G(s) = 0, which means there is no driving term g(t), our solution is just the second line. If you think about the inverse Laplace transform, you see that (I do this since not in any table)

$$\mathcal{L}^{-1}[\; s/(s-s_1) \;] = \mathcal{L}^{-1}[\; s_1/(s-s_1) \;] \;=\; s_1\, \mathcal{L}^{-1}[\; 1/(s-s_1) \;] \;= s_1\, e^{s_1 t}$$

so basically the entire second line gives these time domain results

$$\{ s_1 f(0) + [f'(0) +Bf(0)]\}\; 1/(s_1- s_2) * e^{s_1 t}$$
$$- \{ s_2 f(0) + [f'(0) +Bf(0)]\}\; 1/(s_1- s_2) * e^{s_2 t}$$

But $B = -(s_1+s_2)$ so rewrite as

$$= \{ s_1 f(0) + [f'(0) -(s_1+s_2)f(0)]\}\; 1/(s_1- s_2) * e^{s_1 t}$$
$$- \{ s_2 f(0) + [f'(0) -(s_1+s_2)f(0)]\}\; 1/(s_1- s_2) * e^{s_2 t}$$

$$= \{\; [f'(0) -s_2 f(0)]\}\; 1/(s_1- s_2) * e^{s_1 t}$$

$$- \{ \ [f'(0) - s_1 f(0)] \} \ 1/(s_1 - s_2) * e^{s_2 t}$$

so you see there are two different possible characteristic decay times implicit in the problem. These damped solutions are solutions to the *homogeneous problem* $[\partial_t^2 + B\partial_t + C] f(t) = 0$ so we claim that the nullspace of this operator has two dimensions assuming $s_1 \neq s_2$.

Let's try to duplicate the above results in a simpler manner. We can write our homo ODE as

$$(\partial_t - s_1)(\partial_t - s_2) f(t) = 0$$

so at once we see what the two solutions are and our general solution is then

$$f(t) = \alpha e^{s_1 t} + \beta e^{s_2 t} \qquad f(0) = \alpha + \beta$$

$$f'(t) = s_1 \alpha e^{s_1 t} + s_2 \beta e^{s_2 t} \qquad f'(0) = s_1 \alpha + s_2 \beta$$

So we have a little Cramer's problem here

$$s_1 f(0) = s_1 \alpha + s_1 \beta$$
$$f'(0) = s_1 \alpha + s_2 \beta$$

$$s_1 f(0) - f'(0) = (s_1 - s_2)\beta \qquad \beta = [s_1 f(0) - f'(0)]/(s_1 - s_2)$$

Then of course

$$\alpha = f(0) - \beta = f(0) - [s_1 f(0) - f'(0)]/(s_1 - s_2)$$

and then our homo problem solution is:

$$f(t) = \alpha e^{s_1 t} + \beta e^{s_2 t}$$

$$= \{ f(0) - [s_1 f(0) - f'(0)]/(s_1 - s_2) \} e^{s_1 t}$$
$$+ \{ [s_1 f(0) - f'(0)]/(s_1 - s_2) \} e^{s_2 t}$$

$$= \{ f(0)(s_1 - s_2) - [s_1 f(0) - f'(0)] \} \ 1/(s_1 - s_2) * e^{s_1 t}$$
$$+ \ [s_1 f(0) - f'(0)] \ 1/(s_1 - s_2) * e^{s_2 t}$$

$$= [-s_2 f(0) + f'(0)] \ 1/(s_1 - s_2) * e^{s_1 t}$$
$$- \ [-s_1 f(0) + f'(0)] \ 1/(s_1 - s_2) * e^{s_2 t}$$

which agrees with the Laplace method result.

Now earlier we assumed that $B > 0$ and $B^2 > 4C$. If $C<0$ we are fine, but if $C > 0$, then we have a different type of solution if $B^2 < 4C$. In this case we get

$$s_{1,2} = -B/2 \pm i\sqrt{4C-B^2}/2 \qquad \text{now } s_1 = s_2^* \quad s_1-s_2 = s_1-s_1^* = 2\mathrm{Im}(s_1)$$

The poles still have negative real part so our transient terms still damp out exponentially, but they oscillate as they do so at frequency $\omega=\sqrt{4C-B^2}/2$. We can express our solution in this case as

$$s_{1,2} = -B/2 \pm i\omega$$

$$\begin{aligned}
f(t) &= [\, -s_2 f(0)+f'(0)]\, 1/(s_1- s_2)\, *e^{s_1 t} \\
&\quad -\, [-s_1 f(0) + f'(0)]\, 1/(s_1- s_2)\, *e^{s_2 t} \\
&= [\, -s_1^* f(0) + f'(0)]\, 1/2\mathrm{Im}(s_1)\, *\exp(s_1 t) \\
&\quad -\, [-s_1\, f(0) + f'(0)]\, 1/2\mathrm{Im}(s_1)\, *\exp(s_1^* t) \\
&= [\, -s_1^* f(0) + f'(0)]\, 1/2\mathrm{Im}(s_1)\, *\exp(s_1 t) \;-\; \text{c.c} \\
&= 2\,\mathrm{Re}\,\{ [\, -s_1^* f(0) + f'(0)]\, 1/2\mathrm{Im}(s_1)\, *\exp(s_1 t)\} \\
&= (1/\mathrm{Im}(s_1))\,\mathrm{Re}\,\{ [\, -s_1^* f(0) + f'(0)]\, \exp(s_1 t)\} \\
&= (1/\mathrm{Im}(s_1))\,\mathrm{Re}\,\{ [-(B/2- i\omega)\, f(0) + f'(0)]\, \exp(-Bt/2)\, e^{i\omega t}\} \\
&= (1/\omega)\, \{-(B/2)\, f(0) + f'(0)\}\, \exp(-Bt/2)\, \}\, \cos(\omega t) \\
&+ \; (1/\omega)\, \{-\omega f(0)\, \}\, \exp(-Bt/2)\, \sin(\omega t) \\
&= (1/\omega)\, \{-(B/2)\, f(0) + f'(0)\}\, \exp(-Bt/2)\, \cos(\omega t) \\
&+ \qquad \{- f(0)\, \} \qquad\quad \exp(-Bt/2)\, \sin(\omega t)
\end{aligned}$$

OK, I may have made some algebra errors, but the point is clear. I have *assumed* that f(t) is a real solution and what we get is two terms each having the same damping expo envelope but different phase. We could combine them into a single term of the form $\cos(\omega t+\varphi)$.

So the point is that in this case the homo equation has two possible damped solutions, but they damp at the same rate, but one is $\sin(\omega t)$ and the other $\cos(\omega t)$ as they die out. Again the nullspace has dimension 2 since these two solutions are linearly independent.

Main Point of this Entire Discussion

We consider the 2nd order ODE

$$[\, \partial_t^2 + B\partial_t + C\,]\, f(t) = g(t)$$

where B > 0. We find that there is a particular solution + a homogeneous solution. The particular solution will be the inverse Laplace of

$$F(s) = G(s)/ (s^2 + Bs + C) = G(s)\ 1/(s_1 - s_2) * [\ 1/(s-s_1) - 1/(s-s_2)\ ]$$

If $g(t) = \cos(\omega_1 t)$ then $G(s) = s/(s^2+\omega_1^2)$, so this F(s) will have four poles in the s plane. The two on the imaginary axis at $\pm\ \omega_1$ will cause an endless sinusoidal solution of f(t), while the other poles which are in the left half plane will give rise to some transients which don't last long. So the particular solution has a long term sine result in this case, plus two transient pieces. But in addition to all this, the problem also has a homogeneous solution which also has two transient pieces with the same decay rates. This homo solution is the Laplace inverse transform of

$$\{\ sf(0) + [\ f'(0) + Bf(0)\ ]\}\ 1/(s_1 - s_2) * [\ 1/(s-s_1) - 1/(s-s_2)\ ]$$

So to get the total transient pieces of the solution, you have to add those of the particular solution to those of the homo solution. The upshot is that the amplitudes of the transient solutions will then be a function of the initial conditions f(0) and f'(0), *and also* of the nature of the driving function g(t).

The First Order Analogous Situation

If instead we have the equation

$$(\partial_t + C)\ f(t) = g(t)$$

our Fourier analysis will go like this:

$$[sF(s)-f(0)] + CF(s) = G(s)$$

$$(s + C)F(s) = G(s) + f(0)$$

$$F(s) = G(s)/(s+C) + f(0)/(s+C)$$

If C > 0, the pole at s = -C is in the left half plane and the homo solution is just a decaying transient of the form e^{-Ct} . The particular solution might also have a long term piece and its own contribution to the decaying transient.

Now, Stakgold page 242 vol II (Chap 7) presents a different equation (a 1D heat conduction PDE equation) where the second term of page 242 C is a *continuum* of decaying transients, something that does not happen in an ODE with constant coefficients as we have seen in this little section.

Appendix on Damping Force Physics. In a physics problem F = ma we might have (B>0)

$$F = ma = m\partial_t^2 x(t) = -B\partial_t x(t)$$ // the only force is that of friction or damping F=-bv

We can integrate once from t=0 to t to get

$$\int_0^t dt\, m\partial_t^2 x(t) = \int_0^t dt\, [-B\partial_t x(t)]$$

$$\int_0^t dt\, \partial_t[m\partial_t x(t)] = \int_0^t dt\, \partial_t\, [-Bx(t)]$$

$$[m\partial_t x(t)]|^t_0 = [-Bx(t)]|^t_0$$

$$mv(t) - mv(0) = -Bx(t) + Bx(0) \qquad c = B/m$$

$$v(t) - v(0) = -cx(t) + cx(0)$$

$$\partial_t x(t) + cx(t) = cx(0) + v(0) \qquad \text{LHS} = \text{RHS}$$

As a candidate solution we try

$$x(t) = Ae^{-ct} + D$$

$$\partial_t x(t) = -cAe^{-ct}$$

$$\text{LHS} = -cAe^{-ct} + c\,[Ae^{-ct} + D] = cD$$
$$\text{RHS} = cx(0) + v(0)$$

$$cD = cx(0) + v(0) \qquad D = x(0) + v(0)/c$$

Our solution is then

$$x(t) - x(0) = Ae^{-ct} + v(0)/c$$

If we start our particle with 0 velocity, if does not move. But if it has some v(0), then this tells how far it moves in time t. When $t = \infty$, it has moved a total distance of $v(0)/c = v(0)m/B$. More mass and it moves farther. More friction and it moves less far.

$$m\partial_t x(t) = -Bx(t) + Bv(0)$$

The solution to this first order ODE is given by

$$x(t) = e^{-(B/m)t} + (B/m)\, v(0)t + x(0) \qquad \Rightarrow$$

$m\partial_t x(t) = m\,(-B/m)\,e^{-(B/m)t} + m(B/m)\,v(0) = -B\,e^{-(B/m)t} + Bv(0) = -B$

$\Rightarrow\ m\,\partial_t x = -Bx + mv(0)$

$\Rightarrow\quad x(t) = e^{-Bt} + v(0)t + x(0)$

$\partial_t x(t) = -B\,e^{-Bt} + v(0)$

$\partial_t^2 x(t) = B^2\,e^{-Bt} + v(0)$