**Laplace's Method for solving ODEs** PhL 12.5.08

The subject is a method for finding solutions of ODE's which are integral representations with somewhat adjustable integration paths. The method works when the ODE coefficients are at worst linear functions of the variable, which is a quite common situation. The method is called Laplace's Method. Laplace did so much stuff that it is hard to extract this specific subject on the web, which is different from Laplace Transforms and other methods he had for doing things. This subject might be a standard one in ODE books of which I have none.

These are some Google book pages taken from

http://books.google.com/books?id=FViAkJi9c7EC

Integral Transforms and their Applications by **Brian Davies** (**Hardcover** - Jan 2, **2002**) $56 Amazon

Here is the full link that got to these pages directly:

http://books.google.com/books?id=FViAkJi9c7EC&pg=PA303&lpg=PA303&dq=%22laplace's+method%22+%22integral+representation%22&source=web&ots=IXwWCKdvej&sig=PAk8wHI7WTpt8tdkXExvBkewnJY&hl=en&sa=X&oi=book_result&resnum=4&ct=result#PPA324,M1

I think this little snippet teaches several things of interest to me. One is how it is just an extension of the Laplace Transform idea to a different set of contours. Another is that it sheds light on the generating function idea, ad least for Hermites as in his example. The next page after my last was not on line. This subject relates directly to the $W_1$ and $W_2$ functions which are confluent hypergeometric functions involved in Coulomb scattering, see for example Messiah Vol 1 page 481.

# 18

# Laplace's Method for Ordinary Differential Equations

Transform methods are useful in finding solutions of ordinary differential equations far more complicated than those considered in Chapter 4. In fact, we have already seen in Section 4.4 that an explicit formula for the Bessel function $J_0(x)$, defined as the solution of an ordinary differential equation with variable coefficients, may be found with the Laplace transform. One advantage of the technique developed in this chapter, over the simpler method for solution in terms of a power series expansion, is that the transform method gives the solution required directly as an integral representation. In this compact form various properties of, and relations between, different solutions to an equation become quite clear, convenient asymptotic expansions can be obtained directly, and numerical computation may be facilitated. For applications the analytic properties, asymptotic expansions, and ease of computation of a function are of primary interest.

## 18.1 Laplace's Method

We consider the differential equation

$$\sum_{k=0}^{n}(a_k + b_k x)y^{(k)}(x) = 0, \tag{18.1}$$

the coefficients of which are linear polynomials. Laplace's method consists of assuming that the solutions have the integral representation

$$y(x) = \int_C S(p)\, e^{px}\, dp, \tag{18.2}$$

where the contour is to be chosen as part of the process of solution. Equation (18.2) therefore represents a straightforward extension of the Laplace transform method, for which the contour is a vertical line in the $p$-plane. If we insert (18.2) into (18.1), and integrate by parts to remove the terms in $x$, we get

$$\begin{aligned} 0 &= \int_C \{F(p) + xG(p)\} S(p)\, e^{px}\, dp \\ &= \int_C \left\{ F(p)S(p) - \frac{d}{dp} G(p)S(p) \right\} e^{px}\, dp + \left[ G(p)S(p)\, e^{px} \right]_1^2, \end{aligned} \tag{18.3}$$

where

$$F(p) = \sum_{k=0}^{n} a_k p^k,$$

$$G(p) = \sum_{k=0}^{n} b_k p^k.$$

The second term in (18.3) is evaluated at the end points of the contour, which are often at infinity. If we choose the contour so as to make this contribution vanish, then (18.2) will represent a solution to (18.1) if the function $S(p)$ satisfies the differential equation

$$F(p)S(p) - \frac{d}{dp} G(p)S(p) = 0. \tag{18.4}$$

There is an immediate formal solution, namely,

$$S(p) = \frac{A}{G(p)} \exp\left( \int^p \frac{F(q)}{G(q)}\, dq \right), \tag{18.5}$$

where $A$ is an arbitrary constant. Using (18.5), it is now possible to determine suitable contours to complete the solution.

The simplification that allows us to write the explicit formula (18.5) arises from the fact that (18.4) is a first-order differential equation. This, in turn, is due to our requirement that the coefficients of the original equation be only linear polynomials.[1] If they were quadratic, then the process of integrating by parts in (18.3) would lead to a second-order equation for $S(p)$. There is no general method for solving in this case; however, if a solution can be found, the method will still succeed.

---

[1] More complicated equations can sometimes be reduced to this form by suitable transformation.

## 18.2 Hermite Polynomials

As an illustration of Laplace's method, we consider Hermite's differential equation,

$$w''(x) - 2xw'(x) + 2\nu w(x) = 0, \tag{18.6}$$

where $\nu$ is an arbitrary constant. In this case (18.5) gives

$$\begin{aligned} S(p) &= -\frac{1}{2p}\exp\left(-\int^p \frac{q^2+2\nu}{2q}\,dq\right) \\ &= -\frac{1}{2p}\exp\left(-\frac{p^2}{4} - \nu\ln p\right) \\ &= -\frac{\mathrm{e}^{-p^2/4}}{2p^{\nu+1}}. \end{aligned}$$

This may be substituted into (18.2) to obtain an integral representation.

In order to conform with established convention, we first make the variable change $p \to 2p$, so that

$$w(x) = A\int_C \frac{\mathrm{e}^{-p^2+2px}}{p^{\nu+1}}\,dp, \tag{18.7}$$

where $A$ is an arbitrary constant, and the path of integration must be chosen so that

$$\left[\frac{\mathrm{e}^{-p^2+2px}}{p^{\nu+1}}\right]_1^2 = 0, \tag{18.8}$$

where $[\cdots]_1^2$ refers to the increment as we traverse the contour.

### *Choice of Contour*

In general, a variety of paths will satisfy the condition that the integrated term vanishes; each such path leads to a different solution. The particular path of interest is usually dictated by the physical or boundary conditions we impose. Possible paths that satisfy (18.8) are

(i) $-\infty < p < +\infty$, avoiding $p = 0$.

(ii) $0 < p < \infty$, with $\mathrm{Re}(\nu) < 0$.

(iii) If $\nu$ is an integer, a contour that encircles the origin.

(iv) The path shown in Figure 18.1.

We will prove in Section 18.3 that Hermite functions have the behavior $w(x) \sim A\exp(x^2)$ either for $x \to \infty$ or $x \to -\infty$, or both, except when

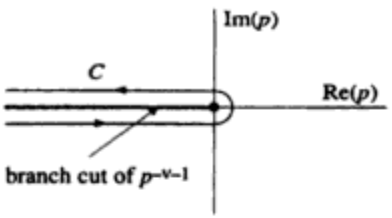



FIGURE 18.1. Contour for (18.11).

$\nu = 0, 1, 2, \ldots$, for which there are some bounded solutions.[2] Turning first to such bounded functions, for which $\nu$ must be an integer, we take $C$ to be a circle enclosing the origin and write

$$w_n(x) = A \int_C \frac{e^{-p^2+2px}}{p^{n+1}}\, dp, \qquad n = 0, 1, 2, \ldots .$$

These solutions, which turn out to be polynomials, are usually normalized by the convention that the coefficient of $x^n$ is $2^n$; this fixes the constant $A$ and gives

$$H_n(x) = \frac{n!}{2\pi i} \int_C \frac{e^{-p^2+2px}}{p^{n+1}}\, dp, \tag{18.9}$$

where we use the symbol $H_n$ as the (standard) notation for a Hermite polynomial.

## *Derivative Form*

A formula for $H_n(x)$ follows if we use the change of variables $u = p - x$ in (18.9), and apply the standard formula (1.12) for evaluating the residue at a pole or order $(n + 1)$. This gives

$$\begin{aligned} H_n(x) &= \frac{n!\, e^{x^2}}{2\pi i} \int_C \frac{e^{-u^2}}{(u+x)^{n+1}}\, du \\ &= (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} . \end{aligned}$$

## *Generating Function*

From (18.9), we have

$$H_n(x) = n! \times \left(\text{coefficient of } p^n \text{ in } e^{-p^2+2px}\right), \tag{18.10}$$

[2]This conclusion may also be reached using Watson's lemma for the paths (i), (ii), and (iv).