M&F on ODEs PhL 12.17.09

I am pretty hot on ODE's right now, so maybe today is a good time to scan MF and see what little gems it might have to add. Chap 5 is on ODE's. I have already read much of Section 5.1 on separable coordinates, so let's start there to get up to speed, then get on to the ODE section 5.2,

Section 5.1 (tail end only, since this material is needed as background for Section 5.2)

M&F are not going to talk here about ODE's in general. They are interested in a *specific* ODE (Helmholtz which includes Laplace if $k_1 = 0$), and they are interested in this specific ODE in the various separable coordinate systems. So this is a very different slant than we had in W&W or Stakgold, and one that is close to what I am interested in right now with Smythe and the Q functions and completeness.

Some background is needed before we can get rolling here. I have not studied the Stackel determinant theory, but I know what it is about. In 3D space, if you write the Helmholtz equation out in some weird curvilinear coordinates, and assume $\psi = X_1X_2X_3$ as your separated form, you get equations of this type as the "separated equations"

$$\frac{1}{f_n}\frac{\partial}{\partial \xi_n}\left[f_n \frac{\partial X_n}{\partial \xi_n}\right] + [k_1^2\Phi_{n1} + k_2^2\Phi_{n2} + k_3^2\Phi_{n3}]X_n = 0 \qquad (5.1.28)$$

where the $f_n(\xi_n)$ are certain functions related to the coordinates chosen, where ξ_n are the three coordinates, and where the functions $\Phi_{nj}(\xi_n)$ are certain other functions that just arise from doing the separation. Elsewhere he shows what these f and Φ functions are for various coordinate systems. The Φ_{ij} functions are the elements of the "Stackel matrix" whose determinant S plays a key role in the theory of separation, but we don't need to learn that theory right now. The constant k_1^2 is the Helmholtz constant, while the other two k_i are the "separation constants". We also have certain M_i objects,

$$\begin{aligned} M_1 &= \partial S/\partial\Phi_{11} = \Phi_{22}\Phi_{33} - \Phi_{23}\Phi_{32} \\ M_2 &= \partial S/\partial\Phi_{21} = \Phi_{13}\Phi_{32} - \Phi_{12}\Phi_{33} \\ M_3 &= \partial S/\partial\Phi_{31} = \Phi_{12}\Phi_{23} - \Phi_{13}\Phi_{22} \end{aligned} \qquad (5.1.26)$$

and if we process our first equation above, we get ($S = \det(\Phi_{ij})$)

$$\sum_n \frac{M_n}{S f_n}\frac{\partial}{\partial \xi_n}\left[f_n \frac{\partial \psi}{\partial \xi_n}\right] + k_1^2\psi = 0; \quad \psi = X_1X_2X_3 \qquad (5.1.29)$$

and we *want* the thing on the left to be $\nabla^2\psi$ written out in the usual curvilinear way,

$$\nabla^2\psi = \sum_n \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial \xi_n}\left[\frac{h_1 h_2 h_3}{h_n^2}\frac{\partial \psi}{\partial \xi_n}\right] \qquad (5.1.30)$$

But this can only be realized in certain coordinate systems, and this is the discussion on pages 509-510 which I am skipping for now. It is the conditions that make the above identification true which causes only the famous 11 systems to be "separable". The h_i are of course the scale factor functions.

Now comes some important stuff: [p 515]

Confluence of Singularities. We have arranged the scale of all these coordinates so that the functions f_n and Φ_{nm} are algebraic functions of the coordinate ξ_n and so that, when the separated equation is put in the form

$$\frac{d^2X_n}{d\xi_n^2} + p_n(\xi_n)\frac{dX_n}{d\xi_n} + q_n(\xi_n)X_n = 0; \quad p_n = \frac{d}{d\xi_n}[\ln f_n(\xi_n)];$$

$$q_n = \sum_{m=1}^{3} k_m^2 \Phi_{nm}(\xi_n) \qquad (5.1.39)$$

the functions p and q have singularities at the concentration points for the corresponding coordinates. For instance, for the ellipsoidal coordinates,

He is now casting each separated ODE into the canonical form with the famous p and q functions! And we see exactly the general form these functions have for our specific ODE in some curvilinear coordinates! His phrase "concentration points" refers to places which are limits of the curvilinear coordinates of interest. Recall for ellipsoidals that $0 \le \xi_3 \le b \le \xi_2 \le a \le \xi_1 \le \infty$ so we expect that maybe the p and q functions will be singular at perhaps 0,b,a,∞. They continue,

p and q have poles at $\xi = \pm a, \pm b$, and at infinity (*i.e.*, if we change to the variable $u = 1/\xi$ the corresponding functions p and q have poles at $u = 0$, or $\xi = \infty$). We obtain the degenerate forms of the coordinate systems by merging two or more of these singularities. A point which is a singularity for p or q is called a *singular point* of the corresponding equation, and a merging of singular points is called a *confluence* of singular points.

So NOW I finally think I understand what is going on here. In W&W we saw that generalized Lamé equation with its 5 regular singular points, and we did various confluences to get other families of equations. That same concept is going to be operative here. Maybe the ellipsoidal most general coordinates will give that same generalized Lamé equation, or something like it! Continue:

In the case of prolate spheroidal coordinates, for instance, there is confluence between the singular points at a and b and at $-a$ and $-b$ which, together with a change of scale, puts the singular points for the ξ_1 equation at $\pm a$ and infinity, for the ξ_2 and ξ_3 equations at ± 1 and ∞. In the spherical coordinates we let a go to zero, so that the singular points for the ξ_1 equation are at 0 and ∞, and so on.

I think I get the point. I won't keep quoting text, but M&F are making a lot of sense, and they say that yes, the ellipsoidal equation does have those 5 singular points, but they are just commenting. [p 516]
They also comment that when a coordinate is a simple azimuth, two of the three related Φ_{ij} vanish,

[p 517] If we start out blindly and work toward separation, we get this situation with just slight processing of the Helmholtz,

$$\sum_{n=1}^{3} \frac{1}{h_n^2 f_n X_n} \frac{d}{d\xi_n}\left[f_n \frac{dX_n}{d\xi_n}\right] + k_1^2 = 0 \qquad (5.1.40)$$

They are pointing out that separability depends on the nature of the h_i functions -- what variables they are functions of. They now want to consider some general cases. Recall that you are trying to get an isolated term in the above sum that involves only one coordinate, then you can set that term to a separation constant.

In Case A, he assumes that two of the three coordinates fall out that way at once and their separation constants are then k_2^2 and k_3^2 and he assumes these are generic coordinates ξ_2 and ξ_3. He then states the remaining third equation, and how the separated functions depend on the constants:

constant k_3^2. Consequently the remaining equation for X_1 is

$$\frac{\mu}{h_1^2 f_1 X_1} \frac{d}{d\xi_1}\left(f_1 \frac{dX_1}{d\xi_1}\right) + k_2^2 + k_3^2 + \mu k_1^2 = 0$$

and the factored solution takes on the form

$$(A) \qquad \psi = X_1(\xi_1;k_1,k_2,k_3)X_2(\xi_2;k_2)X_3(\xi_3;k_3) \qquad (5.1.41)$$

The obvious nice feature here is that X_2 and X_3 each depend only on *one* constant. Obviously, this is the case you would like to have, and it is the simplest case. Sadly, only the Cartesian and cylindrical coordinate systems work this way. [for Cartesian, those constants are integers which set $\psi = 0$ on the box walls ? For cylindrical the z constant is such an integer, while azimuth has its usual m;] Now on to the next case:

B. Solution Partially Separable for Separation Constants. In this case only one term (the one for ξ_3, for example) can be separated the first time round; the remaining equation

$$\frac{\mu}{h_1^2 f_1 X_1}\frac{d}{d\xi_1}\left(f_1\frac{dX_1}{d\xi_1}\right)+\frac{\mu}{h^2 f_2 X_2}\frac{d}{d\xi_2}\left(f_2\frac{dX_2}{d\xi_2}\right)+\mu k_1^2+k_3^2=0$$

must be multiplied by still another factor, $\nu(\xi_1,\xi_2)$, in order that a second term can be split off. The factored solutions, therefore, have one of the following forms:

$$\begin{aligned}(B_1)\qquad &\psi = X_1(\xi_1;k_2,k_3)X_2(\xi_2;k_2k_3)X_3(\xi_3;k_1,k_3)\\ (B_2)\qquad &\psi = X_1(\xi_1;k_1,k_2,k_3)X_2(\xi_2;k_1,k_2,k_3)X_3(\xi_3;k_3)\end{aligned}\tag{5.1.42}$$

Reference to the table shows that parabolic cylinder coordinates correspond to (B_1) and elliptic cylinder, parabolic, oblate, and prolate spherical coordinates (all the rest of the cylindrical and rotational coordinates) correspond to (B_2). Here only the last row of the Stäckel determinant has two zeros.

The main point is to notice how the constants k_2 and k_3 are distributed into the three X_i functions. And he treats k_1 as being of interest at the same time, even though it is not a separation constant. They are smart I think to be handling Helmholtz throughout! Continuing:

C. Solution Nonseparable for Separation Constants. In this case no Stäckel element in the second or third column is equal to zero and the full machinery of the Stäckel determinant must be used to achieve separation. The possible forms are

$$\begin{aligned}(C_1)\qquad &\psi = X_1(\xi_1;k_1,k_2,k_3)X_2(\xi_2;k_2,k_3)X_3(\xi_3;k_2,k_3)\\ (C_2)\qquad &\psi = X_1(\xi_1;k_1,k_2,k_3)X_2(\xi_2;k_1,k_2,k_3)X_3(\xi_3;k_1,k_2,k_3)\end{aligned}\tag{5.1.43}$$

Only conical coordinates give form (C_1). Ellipsoidal and paraboloidal coordinates result in form (C_1), where no Stäckel element is zero.

It should be obvious that forms (A) are comparatively simple to apply to a given problem, forms (B) are more difficult, and forms (C) are very much more difficult to use.

So this is the ugliest case where all the k_i appear everywhere as in the C_2 case.

MF then ask if perhaps there are not more separable systems for Laplace than for Helmholtz? The answer is a modified yes, and this involves a certain "modification factor" and a variation in the Stackel theory. I skip this detail for now.

[p 520-523] My interest has been in the quadric surface type coordinate systems, but there are fancier systems known as confocal cyclides which are quartic (4th order) instead of quadric (2nd order). This cyclides discussion goes on through p 523, and I skip it for now. MF want to be encyclopedic in their work, and I praise them for that. Some day I will be back for these details and I can add them right here.

Section 5.2 general properties of ODEs and their series solutions"

Now to the main act. Here is our starting point:

$$\mathfrak{L}(\psi) = \frac{d^2\psi}{dz^2} + p(z)\frac{d\psi}{dz} + q(z)\psi = 0 = \frac{1}{f_n}\frac{d}{dz}\left(f_n\frac{d\psi}{dz}\right) + q\psi \quad (5.2.1)$$

$$\text{where} \qquad p = \frac{d}{dz}\ln f_n = \left(\frac{1}{f_n}\right)\left(\frac{df_n}{dz}\right); \quad q = \sum_m k_m^2\Phi_{nm}$$

They comment that the p and q functions have only "poles". They have not used the terms regular or irregular yet, but if the poles are "right", we know we can have some regular points here.

We have some text hinting at linear operators and dimensions of nullspace and such, all in very soft terms.

p 524: The Wronskian. This thing is defined it seems for the first time to the reader in this book. They show how, if you start with one solution, you can "build out" a second one where the Wronskian of the two does not vanish, and in this way they construct a "second solution", given a "first solution". This discussion of course relates to my general question of "how do you know there are n independent solutions to an nth order ODE?". So fine. By the way, in this discussion I think they have developed Abel's Theorem, all of which is more clearly presented in Stakgold. MF refer to the Wronskian as $\Delta(u,v)$, not W(u,v) which is the Stak notation.

p 525: Independent Solutions.

Independent Solutions. We have just seen that, if one solution, $y_1(z)$, of $\mathfrak{L}(y) = 0$ is known, a second solution can be obtained by integration,

$$y_2(z) = By_1(z)\int e^{-\int p\,dz}\frac{dz}{y_1^2(z)} \quad (5.2.6)$$

This is a famous result where you see the second solution as a multiple of the first, and it involves an integral of the first squared in the bottom, and the exponentiated but first integrated p(z) function in the top. The notation here is a little vague, but that is OK. Later they use this to get second kind solutions from first kind ones. Note that often the integral fzctor will cancel out the leading $y_1(z)$ factor, so the result is not really a "multiple" of $y_1(z)$. I don't think this little gem appears in Stakgold. I wonder if it has a name associated with it? They go on to show that y_1 and y_2 is all there is, there cannot be some third independent solution. They talk about these two independent functions as spanning a "plane", meaning a

2D space. For me, we have L_2 as an infinite dimensional Hilbert Space, and we have our ODE as Lu=0 and the dimension of the nullspace is 2. So only a very tiny set of functions solves the ODE compared to the size of L_2. They note that a n^{th} order linear differential operator L would have nullspace of dimension n.

p 526: Integration Factors and Adjoint Equations. Some excellent text here! The point is made that doing an integral (by looking up in tables) is earlier than solving an ODE, so maybe we can convert an ODE into "looking up an integral". Even if the integral cannot be done, it would be nice to have the solution of the ODE as some integrals which could at least be treated by expansion or numerically.

formed by expanding in series (or computing numerically). This is called "reducing to quadratures," a phrase by which the mathematician symbolically washes his hands of the remainder of the task of finding the solution.

Here then is the notion of an "integration factor", something I have not heard of in a long time:

For instance, the simplest linear differential equation

$$(dy/dz) + p(z)y = 0 \qquad (5.2.8)$$

can be reduced to quadratures by rearrangement of terms

$$\int (dy/y) = -\int p(z)\, dz: \quad \ln y = -\int p\, dz + C; \quad y = Ae^{-\int p\, dz} \qquad (5.2.9)$$

or by the use of an *integration factor*. We notice that, if we multiply Eq. (5.2.8) by the factor $e^{\int p\, dz}$, the result is a complete differential, which may be integrated immediately:

$$e^{\int p\, dz} y' + e^{\int p\, dz} py = (d/dz)[ye^{\int p\, dz}] = 0; \quad ye^{\int p\, dz} = A; \quad y = Ae^{-\int p\, dz}$$

An extensive literature has developed, reporting useful techniques for finding integration factors for more complicated first-order equations.

So here, using an integration factor, we have our solution written "as an integral". Notice how that p function likes to end up in the position shown, we saw it earlier in our "second solution" stuff. Maybe we can't do the integral in the exponent, but we have an answer, we are reduced to "quadratures", a word which means integrals. [OED quadrature: the expression of an area bounded by a curve, esp. a circle, by means of an equivalent square. More widely, the calculation of the area bounded by, or lying under, a curve.]

Now, before getting into the 2^{nd} order ODE version of "integration factors", let's digress to Stakgold page 40 in his PDE chapter 5 where we have

vLu - uL*v = div **J**(u,v)

and its integral form where **J** is seen to be a Green's boundary surface current. I invested a lot of effort into understanding this idea (see "parts and greens.doc") where I realized this is just an expression of generalized parts integration, and the **J** you get is not unique. Stak did not give **J** a name, I called it a surface current, and he does point out that it is "bilinear" in u and v. I show it is bilinear just from the parts integration basics. I don't know if Stak has this in Volume 1. He has something dimly like it on page 269 where he does not use the adjoint symbol on L, and where the J current seems to be the product of the p function and the Wronskian of u and v.

Now, what do we get if we apply this PDE result to a 1D ODE? We get this:

vLu - uL*v = d/dz **J**(u,v)

which MF represent in this way

$$v\mathfrak{L}(y) - y\tilde{\mathfrak{L}}(v) = (d/dz)P(v,y) \qquad (5.2.10)$$

$$\tilde{\mathfrak{L}}(v) = \frac{d^2v}{dz^2} - \frac{d}{dz}(pv) + qv = v'' - pv' + (q - p')v \qquad (5.2.11)$$

$$P = vy' - v'y + vpy = vy\left[\frac{y'}{y} - \frac{v'}{v} + p\right]$$

So their solutions are v and y, not u and v, and their "current" is P, not J. And MF now have a name for this thing: the "**bilinear concomitant**", which name I remember very dimly from ages past. Notice that I have quoted the form of L* above, the formal adjoint. L* does not look any simpler than L, really.

L(u) = u" + p u' + q u.

So how does this help us solve our ODE Lu=0 ? "Suppose", they say, that for some reason L*v=0 is easy to solve for v. Then 5.2.10 above says vLy = dP/dz. If you could solve P(v,y) = constant for y, given your known v, then that y would solve Ly = 0, the thing you really want to solve. Here then is how this works:

If, now, it is easy to solve the adjoint equation $\bar{\mathfrak{L}}(v) = 0$, then the solution of the original equation, $\mathfrak{L}(y) = 0$, is equivalent to solving the first-order equation

$$P = vy[y'/y) - (v'/v) + p] = \text{constant}$$

Since any solution will suffice, we choose the easiest one by setting the constant equal to zero, so that, if v is a solution of $\bar{\mathfrak{L}}(v) = 0$, then

$$\frac{dy}{y} = \frac{dv}{v} - p\,dz; \quad y_1 = ve^{-\int p\,dz} \tag{5.2.13}$$

(we pay no attention to arbitrary constants, since we can introduce them later). A second solution independent of y_1 can be found by using Eq. (5.2.6):

$$y_2 = ve^{-\int p\,dz} \int e^{\int p\,dz} \frac{dz}{v^2} \tag{5.2.14}$$

where we have to perform another integration. Then the general solution is $\psi = Ay_1 + By_2$.

So fine, IF L*v=0 can be solved for v, we have our two solutions to Ly = 0 in "reduced to quadratures" form. Now we yet again get that exp(±∫p(z)dz) structure appearing. The second equation above is just our usual (by now) "second solution" finder. I guess I would like to see an example where Ly=0 is hard to solve, but L*v = 0 is easy to solve. Well this is their next little effort:

A general type of second-order differential equation which may be solved in this manner is the case where $q = dp/dz$, for then the adjoint equation (5.2.11) for v takes on the simple form

$$v'' - pv' = 0: \quad v' = e^{\int p\,dz}; \quad v = \int e^{\int p\,dz}\,dz$$

and two independent solutions for y are

$$\begin{aligned} y_1 &= e^{-\int p\,dz} \int e^{\int p\,dz}\,dz = \frac{v}{v'} \\ y_2 &= -e^{-\int p\,dz} \left[\int e^{\int p\,dz}\,dz\right] \int \frac{e^{\int p\,dz}\,dz}{[\int e^{\int p\,dz}\,dz]^2} = \frac{1}{v'} \end{aligned} \tag{5.2.15}$$

for proper choice of the constants of integration.

On scratch I have verified all these results. The very last one involves $\int v'\, dz / v^2 = \int \partial_z(v^{-1}) = 1/v$. Of course I don't know how common it is to have q = p' to get this special case. Well, they go on to claim that this does in fact exactly happen "in certain coordinate" systems, I will skip the details.

[p 529] Solution of inhomogeneous equation. In this section, whose details I skip, MF show how you can solve the inhomo equation $L\psi = r$ (where r is the driving function). The solution is this

$$\psi = vy_1 = y_1\left[c_1 - \int \frac{ry_2\,dz}{\Delta(y_1,y_2)}\right] + y_2\left[c_2 + \int \frac{ry_1\,dz}{\Delta(y_1,y_2)}\right] \quad (5.2.19)$$

where y_1 and y_2 are our homo solutions and Δ is the Wronskian. There are four terms shown, and obviously two of them are the homo solution, so the two terms with integrals are the particular solution, and he writes this out in the following more compact form

$$\int^z r(w)\,dw\left[\frac{y_1(w)y_2(z) - y_2(w)y_1(z)}{y_1(w)y_2'(w) - y_2(w)y_1'(w)}\right]$$

This seems a bit new to me. In one sense, you would think that the square bracket thing was a Green's function and we are just saying that $\psi(z) = \int dw\, r(w)\, g(w|z)$, but it is not obvious to me that this fits with any of the "forms" I saw in Stak Chapter 4. Well go to Stak p 272 where you see W = C/p and where you also see g = - w z/C = -wz / [pW] where w and z are homo solutions for the appropriate "sides". I might then rename things and say $g = - y_1 y_2 / p\,\Delta$ and we are then *pretty close*, but where is his p ? Also, he has a Volterra type endpoint which I know changes things a bit, so $w \le z$ always in his case. He gets p to go away in his processing. I can only conclude that this is some kind of "Volterra Green's Function" approach. Just something to keep in mind, another way to solve the inhomo equation !

[p 530] Series solutions about ordinary points. Now we are where I want to be, reviewing familiar ground to see if "new gems" somehow appear in the MF presentation. Well, their approach is quite different, and I will try to just quote results.

> **Series Solutions about Ordinary Points.** As we pointed out several times before and shall prove shortly, the general solution of $\mathfrak{L}(y) = 0$ has its singularities at the points where the functions p and q have their poles. All other values of z, where p and q are analytic functions, are called *ordinary points for the equation.* The points where p or q (or both) have singularities are called *singular points for the equation.*

Notice the phrase "general solution" which I guess means a lincom of the two independent solutions. The first point here is that the solutions of Ly=0 will never have singularities at a point where p and q are

analytic, and that seems reasonable to me. If the solutions have singularities, they will be at places where p and q are singular. Note also that they talk about p and q having "poles". It is not clear to me whey p and q could not have other singularities. [They could, but this always leads to solutions with essential singularities which are pretty ugly, I think that is the point.]

Next, MF expand everything around some point z=a and get the recursion stuff, but they then write the solutions as follows,

$$
\begin{aligned}
y &= a_0 y_1 + a_1 y_2 \\
y_1 &= 1 - \tfrac{1}{2}q(a)(z-a)^2 + \tfrac{1}{6}[q(a)p(a) - q'(a)](z-a)^3 + \cdots \\
y_2 &= (z-a) - \tfrac{1}{2}p(a)(z-a)^2 \\
&\qquad + \tfrac{1}{6}[p^2(a) - p'(a) - q(a)](z-a)^3 + \cdots
\end{aligned}
\tag{5.2.20}
$$

[Note: when I first saw the above result, I was quite confused. But then I updated my ODE meta notes to include a solution at an ordinary point, and the above just falls out, the exponents are r = 0 and 1, etc. So this was a good benefit of reading MF, to get this little matter cleared up.]

To recap: we assume everything is analytic, so we are at an ordinary point. We expand everything. We stumble in this way into series for what appear to be the two independent solutions. These solutions have a name in fact, called **the basic set** of solutions at the ordinary point z = a. Notice these facts:

$y_1(a) = 1$ $\quad$ $y_1'(a) = 0$
$y_2(a) = 0$ $\quad$ $y_2'(a) = 1$

Then consider $y = Ay_1 + By_2$. We then have y(0) = A and y'(0) = B. So this set is very useful if you want to have "initial" BC's like this. Can write

$y(z) = y(a)\, y_1(z) + y'(a)\, y_2(z)$

as your solution at an ordinary point z = a with initial BC's there. This is new to me (and now in meta).

> fied by setting $y = Ay_1 + By_2$. This series solution will hold within a circle of convergence of radius equal to the distance between a and the nearest singular point of the differential equation. For instance, the basic set of solutions of the differential equation $y'' + y = 0$ at the ordinary point $z = 0$ is $y_1 = \cos z$, $y_2 = \sin z$. Since the nearest singular point is at infinity, the series expansions for cosine and sine are valid over the entire finite part of the z plane.

An impressive result. Now they start looking into pole situations for p and q. First an observation, where the word "pole" on the first line means a pole of any finite order,

It is of interest to note that, when p has a pole at $z = a$ but q is analytic at a, *one solution* of the equation is analytic, though the second solution has a singularity. Setting $p = F(z)/(z - a)^n$, where F is an

They go on to show that if p has a pole of order n, as shown above, then a_1 thru $a_n = 0$ so one of your solutions is killed off. You still have your a_0 solution which is y_1-like, but now the terms are all different because we have thrown in this pole thing so we now have F-dependent stuff. As an example, they set n = 2 and then write out the series for y_1 and it is just a series. In order to obtain another independent solution, we do the usual 5.2.6 "trick" (see above quoted), and he shows that this solution has an "essential singularity". This was defined back on page 380 (unread by me).

Digression: In that section, MF first talk about **Taylor series** with positive only powers. The **Laurent series** can be infinite in both positive and negative powers. If, when you expand a function near a singularity of interest, you find that the Laurent series has an infinite number of negative powers, then this singularity is "stronger than a pole of any order", you might say, and this is exactly what the **essential singularity** is. The simple example is that $e^{1/z}$ has such a problem at z = 0. We sort of know this blows up worse than any order pole. If f(z) has an ES at z=a, then 1/f also has an ES at z=a. They quote the famous fact that near an ES, f(z) takes all possible values.

By the way, earlier on page 481 MS state that poles (of any finite order) and essential singularities are **isolated singularities**, which means there is no branch cut to them and you can put a contour around the singularity (as they just did above). The Branch Point is NOT an isolated singularity and if you go around it, you don't get back where you started.

[p 532] So when we have an n=2 pole in p(z), we get a nice analytic $y_1(z)$ solution near z = a, but the other solution has an essential singularity at z = a. In this case z = a is called an **irregular singularity**. If we only have an n = 1 pole in p(z), be instead get a branch cut in the y_2 second solution, and this is going to be called an example of a being a regular singular point.

Singular points, Indicial Equation. Here is a nice summary to this point: (again, "pole" means a pole of some finite order)

Singular Points, Indicial Equation. We have just seen that, near those singular points where q is analytic but p has a pole, one solution is analytic and the second solution has a branch point if p has a simple pole or has an essential singularity if p has a pole of higher order. This distinction is of general utility; we can define a *regular singular point* as one where the general solution has a pole or branch point, an *irregular singular point* as one where the general solution has an essential singularity.

Then we go on (ignore the details of what J is). He is asking how "bad" p and q can be where you avoid the essential singularity problem so that z = a is a regular singular point. He arrives at the classical conclusion, but I must admit is new to me: if things are worse than the pole situation shown, that does mean P and Q are not analytic, and that in fact causes one of the two solutions to have an essential

singularity! I did not know that. The first solution remains analytic it seems in all cases ??? I don't think that is true, since both exponents could be fractional, so both have branch cuts. Hold on that for now:

In order that J be analytic, the pole for q must be second order or less ($n = 2$) and the pole for p first order or less ($m = 1$). Consequently *if p has the form $F(z)/(z - a)$* and *q the form $G(z)/(z - a)^2$*, where F and G are analytic, *then the point $z = a$ is a regular singular point* for the equation $\mathfrak{L}(y) = 0$ and *the general solution has a branch point there.* The equation

[p 533] MF then go on to the usual discussion of the indicial equation, the two exponents they call s_1 and s_2, and then they treat the usual special cases of equal or differ by integer.

This logarithmic term always appears in the second solution when $s_1 = s_2$ (see page 529) and nearly always appears when $(s_1 - s_2)$ is an integer. Therefore we can say that the general solution of Eq. (5.2.1)

Now here is more new information:

When q has a pole of higher order than second or when p has a pole of order higher than first, or both, one or both solutions must have an essential singularity and we have an irregular singular point. When q has a pole of order one higher than p plus a term in $(z - a)^{-2}$, only one of the solutions has an essential singularity.

They give a proof of this which I did not trace. They then claim that not all essential singularities are the same, some are "worse" than others. We finally see the word **species** appear :

sort of essential singularity the solution has. For example, the form

$$y = [c_0(z - a)^s + c_1(z - a)^{s+1} + \quad \cdot] \cdot \cdot \exp\,[A_0(z - a)^{-k} + A_1(z - a)^{-k+1} + \cdot \quad \cdot]$$

is a solution at some irregular singular points; the nature of the essential singularity of y at $z = a$ will be determined by the value of k required. One can order the singular points into different *species* according to the value of k required. When p or q has branch points or essential singularities, the singularities of y are even more "pathological." Luckily we need not penetrate this jungle of complications to solve the equations we are to work with.

So our example $e^{1/z}$ would have k = 1 and perhaps be "first species". I don't know if this species word is the same word which appears in the Lamé discussion. Allowing functions p or q to *themselves* have essential singularities (or even just branch points) does really sound horrible!

[p 534] Classification of Equations, Standard Forms. This is all just what I want. We open here with an *excellent* paragraph:

Classification of Equations, Standard Forms. It should be apparent by now that the first thing to do when attempting to solve a linear differential equation of unfamiliar form is to locate the position of all the singularities of the functions p and q. If all these singularities are poles, we can go on to the next step; if some of them are branch points or essential singularities, we try to change the independent variable z so they all become poles (if this cannot be done, we are reduced to numerical integration; this sometimes happens!). We then distinguish the regular singular points from the irregular ones, and by solving the indicial equations for all the regular points, we find the values of the *indices* s, which determine the nature of the branch points in the solutions at these places. If there are irregular singular points, we also determine the nature of the essential singularity there by the methods discussed above.

I like the part about changing variable to try to get things more regular. They call things indices and not exponents. They then deal with the **z = ∞ singularity situation**, where W&W just gave a result . Here is the MF analysis:

The nature of the point at infinity is determined by setting $z = 1/w$; the resulting equation is

$$\frac{d^2y}{dw^2} + P(w)\frac{dy}{dw} + Q(w)y = 0; \quad P(w) = \frac{2}{w} - \frac{1}{w^2}p\left(\frac{1}{w}\right); \quad Q = \frac{1}{w^4}q\left(\frac{1}{w}\right) \tag{5.2.23}$$

The nature of the poles of P and Q for $w = 0$ tells us the nature of the singular point at infinity. For instance, if a solution of the indicial equation for $w = 0$ is s_1, then the solution has the form $w^{s_1}F(w) = (1/z)^{s_1}F(1/z)$, where $F(w)$ is analytic at $w = 0$.

And here is a pretty important thing to note:

The singular points of all the equations arising from the separation of the equation $\nabla^2\psi + k^2\psi = 0$ are listed in the table at the end of this chapter.

Now the next piece of advice seems pretty sound:

It is usually best next to transform the equation into a standard form, with standard positions for the singular points and with indices s for the solutions which are as simple as can be. For instance, if there are but two singular points, it is usually best to place them at zero and infinity; if there are three, the usual location is zero, unity, and infinity. This is done by changing the independent variable. If the points are originally at a, b, and c, the transformation

$$z = (w - a)\gamma/(w - c); \quad w = (\gamma a - cz)/(\gamma - z); \quad \gamma = (b - c)/(b - a) \tag{5.2.24}$$

will change the position of the singular points but will not change their nature or the indicial equations at each point. The related equations are

$$\frac{d^2\psi}{dw^2} + P(w)\frac{d\psi}{dw} + Q(w)\psi = 0; \quad \frac{d^2\psi}{dz^2} + p(z)\frac{d\psi}{dz} + q(z)\psi = 0$$

$$p(z) = \frac{\gamma(a - c)}{(\gamma - z)^2} P\left(\frac{\gamma a - cz}{\gamma - z}\right) - \frac{2}{\gamma - z} \tag{5.2.25}$$

$$q(z) = \frac{\gamma^2(a - c)^2}{(\gamma - z)^4} Q\left(\frac{\gamma a - cz}{\gamma - z}\right)$$

Here, and for the rest of this chapter, we use ψ for the general solution, y for specific solutions.

So in other words, if you have three singular points at a,b,c in your z-world, shift to the w-world as shown. I imagine these are the usual Riemann P equation ideas. They go on:

It is impossible to devise a transformation which will change four arbitrary points to four standard positions and not change the indices, so there is no single standard form for equations with more than three singular points. We usually set the irregular singular point at infinity (if there is only one), for the factor v discussed above then has the comparatively simple form $\exp(a_k z^k + a_{k-1}z^{k-1} + \cdots + a_{k-s}z^{k-s})$.

This chapter is a gold mine of important comments!

If the singular point at zero is regular, it is usually advisable next to change the dependent variable, by setting $\psi = vu$ and using Eq. (5.2.21), in order to obtain an equation for u which is comparatively easy to solve. Usually this involves letting v have a branch point corresponding to the lowest root of the indicial equation for the point at zero, so that one solution of the equation for u is analytic at $z = 0$. But the detailed considerations guiding the choice of transformations for both independent and dependent variable are best shown in terms of examples.

Above they say "change variable", but I think they mean to change the "functional form". I have certainly seen many examples of such form changes.

They now propose to consider ODEs with 1,2,3,4.... singular points, building up from the simple to the more complicated.

N =1: I don't follow this little discussion, let's just move on to the next case:

Two Regular Singular Points. Here the general form of the equation is

$$\frac{d^2\psi}{dw^2} + \frac{2w + c(\lambda + \mu - 1) - a(\lambda + \mu + 1)}{(w - a)(w - c)} \frac{d\psi}{dw} + \frac{\lambda\mu(a - c)^2}{(w - a)^2(w - c)^2}\psi = 0 \quad (5.2.28)$$

OK, the points in question are a and c. Recall from now on, they said, we have P(w), Q(w), ψ(w) etc. I am not sure I would have done that notational change. The general form here is carefully tuned to make w = ∞ be an ordinary point! Let's read through the treatment of this example:

where the form of $P(w)$ is closely specified (particularly the $2w$ in the numerator) in order that the point at infinity be an ordinary point. The indicial equation (5.2.22) for $w = a$ is $s^2 - (\lambda + \mu)s + \lambda\mu = 0$, with roots λ and μ, so that the solutions are $y_1 = (w - a)^\lambda u_1^a$, $y_2 = (w - a)^\mu u_2^a$. The indicial equation for $w = c$ is $\alpha^2 + (\lambda + \mu)\alpha + \lambda\mu = 0$, with roots $-\lambda$ and $-\mu$, so that the solutions are $y_1 = (w - c)^{-\lambda} u_1^c$, $y_2 = (w - c)^{-\mu} u_2^c$. Direct substitution (or series solution) shows that $u_1^a = (w - c)^{-\lambda}$, $u_1^c = (w - a)^\lambda$, $u_2^a = (w - c)^{-\mu}$, $u_2^c = (w - a)^\mu$, so that the general solution is

$$\psi = A\left(\frac{w - a}{w - c}\right)^\lambda + B\left(\frac{w - a}{w - c}\right)^\mu \quad (5.2.29)$$

I have several comments. First, somehow the exponents are λ and μ for a, and -λ and -μ at c. I guess his form shown does this, but in general you would expect exponents α and β at c, and somehow they are able to move these to "standard positions", nothing said about that. Second. $u_1^a(w)$ is the power series part of the solution, and they separate off the possible branch cut factor. WW did not do this separation. It is OK, no problem. This problem is simple enough that one can just write down the solution as shown (I wonder why the photocopy loses the horizontal lines so completely in the two ratios shown?)

They then show how you could move the points a,c to 0,∞ and everything gets simpler. No quote.

And finally, if λ = μ, we get a log appearing, as expected. So much for two regular points. All solutions are "elementary functions".

[p 537:]

One Irregular Singular Point. The equation for a single irregular point at $w = a$ turns out to be

$$\frac{d^2\psi}{dw^2} + \frac{2}{w-a}\frac{d\psi}{dw} - \frac{k^2}{(w-a)^4}\psi = 0 \qquad (5.2.32)$$

The term in $(w-a)^{-4}$ is responsible for the irregular singular point, and the term $2/(w-a)$ prevents a singular point from occurring at $w = \infty$.

When they say "the equation", they must mean this is "the simplest form" and any other equation you might have with one irregular singular point can somehow be jockeyed into this standard simple form. The point is of course at w=a and is "induced" by too high a pole in the q function. I wonder why a third order pole would not have been adequate? Again, this thing is tuned so that w=∞ is ordinary (factors of 2 often are required to do this, I see). So [recall these expos are the essential singularity poster children]

The solution of this equation is

$$\psi = Ae^{k/(w-a)} + Be^{-k/(w-a)} \qquad (5.2.33)$$

as can be found by direct substitution or else by transformation to $z = 1/(w-a)$, for the standard form of this type of equation is obtained by setting the irregular singular point at infinity:

$$(d^2\psi/dz^2) - k^2\psi = 0 \qquad (5.2.34)$$

The solutions are then just $e^{\pm kz}$ which means z=∞ is now where the exponent blows up instead of w=a. Very good. We continue:

We should note that Eq. (5.2.34) is not the only equation having one irregular singular point at ∞, for there can be different *species* of irregular singular points. For instance, the equation coming from the wave equation in parabolic cylinder coordinates,

$$y'' + (a^2 + b^2z^2)y = 0$$

has one irregular singular point, at ∞, but its solutions are not the simple exponentials.

where the word species mentioned a bit above comes back. At this point they do a little confluence example letting the c→a and letting the exponents get infinite, a very strange limit. A nice addition, but I will ignore it.

Three Regular Singular Points. Equations with more complexity than the former have solutions which cannot be expressed in terms of elementary functions. Reference to the table at the end of this chapter indicates, however, that a majority of the separated equations listed there have either three regular singular points or one regular and one irregular singular point, so it behooves us to analyze these cases in detail.

[p 538] MF go on to explicitly (with full explanation and derivation) derive the most general form of the equation with three regular points at a,b,c. This is of course the same Papperitz or Riemann P equation quoted in WW.

$$\frac{d^2\psi}{dw^2} - \left[\frac{\lambda+\lambda'-1}{w-a} + \frac{\mu+\mu'-1}{w-b} + \frac{\nu+\nu'-1}{w-c}\right]\frac{d\psi}{dw} + \left[\frac{\lambda\lambda'(a-b)(a-c)}{(w-a)^2(w-b)(w-c)} + \frac{\mu\mu'(b-a)(b-c)}{(w-a)(w-b)^2(w-c)} + \frac{\nu\nu'(c-a)(c-b)}{(w-a)(w-b)(w-c)^2}\right]\psi = 0 \quad (5.2.36)$$

This is called the *equation of Papperitz*. As we have seen, this equation (and therefore its solutions) is completely determined by specifying the position of its three singular points and the values of the two indices at each of the three points [or, rather, five out of the six indices, the sixth being fixed by Eq. (5.2.35)]. In other words the symbolic tabular display

$$\psi = P\begin{Bmatrix} a & b & c & \\ \lambda & \mu & \nu & w \\ \lambda' & \mu' & \nu' & \end{Bmatrix} \quad (5.2.37)$$

is completely equivalent (if the sum of the second and third rows is unity) to a statement that ψ is a solution of Eq. (5.2.36). This symbol, due to Riemann, will occasionally be a space saver.

MF then go on to consider series expansions at each of the three singular points. At each point, you end up with a triple recursion relation, as I well know. It is not very pleasant. You can regard such a recursion relation as a second order difference equation, and so we have basically replaced our 2nd order ODE with continuous variable w with an infinite set of difference equations with integer index n, a kind of transform we have seen many times. How here are some ideas of what to do next:

As long as we are limiting ourselves to the use of power series for the solutions, it behooves us to transform the independent and dependent variables of the differential equation into the form which will give the simplest recursion formula possible. The three-term recursion formula of Eq. (5.2.41), for instance, corresponds to a second-order difference equation; perhaps a transformation of dependent variable will change this to a two-term recursion formula, which would correspond to a first-order difference equation and which would be very much simpler to solve.

[p 541] This leads directly to the hypergeometric equation. We have already assumed in the above that our singular points a,b,c and going to be at 0,1,∞ which is where they are in the HG equation. With lots of words, they do some transformations to clear out some of the exponents (they claim you can alter the

differences, but not the sums). They don't quote the full blown transformations that we have in Bateman or WW. When the dust settles, we this situation:

$$F = P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{matrix}\right\}$$

and the corresponding differential equation

$$z(z-1)F'' + [(a+b+1)z - c]F' + abF = 0 \qquad (5.2.42)$$

called the *hypergeometric equation*, is the standard equation for three regular singular points.

The analytic solution of this equation about $z = 0$ is called the *hypergeometric function*. To obtain its series expansion we set $F =$

Notice that we have managed to get one exponent at z = 0 and one at z = 1 to be 0, so one solution will always be analytic, no branch cut, no muss no fuss. That solution at z = 0 is "F". This situation gives a simple two-term recursion relation (I am not clear on what knocks out the third term, but I know you just do it and that is how it comes out). We can solve the recursion formula, and out pops our old friend,

The corresponding series

$$F(a,b|c|z) = 1 + \frac{ab}{c}z + \frac{a(a+1)b(b+1)}{2!c(c+1)}z^2 + \cdots \qquad (5.2.44)$$

is called the *hypergeometric series* (see page 388). It is the analytic

where now we have a "vertical bar" notation I have never seen used before. Now we know that the "other" solution at z=0 has exponent 1-c, so he "tries" a solution of the form $z^{1-c}F_2$ and then writes out the ODE for F_2 and low and behold, it has the same HG form, so we can read off its solution. So here then is the full solution expanded at z = 0:

$$AF(a,b|c|z) + Bz^{1-c}F(b-c+1,\, a-c+1|\, 2-c|\, z) \qquad (5.2.45)$$

They then shuffle things and end up with the general solution of the full Papperitz at w = a as 5.2.46 which is this:

and the general solution of Papperitz equation (5.2.36) at $w = a$ is

$$A\left(\frac{w-a}{w-c}\right)^{\lambda}\left(\frac{w-b}{w-c}\right)^{\mu} F\left(\lambda+\mu+\nu,\right.$$
$$\left.1-\nu-\lambda'-\mu'|\lambda-\lambda'+1|\frac{w-a}{w-c}\frac{b-c}{b-a}\right)$$
$$+B\left(\frac{w-a}{w-c}\right)^{\lambda'}\left(\frac{w-b}{w-c}\right)^{\mu} F\left(\lambda'+\mu+\nu,\right.$$
$$\left.1-\nu-\lambda-\mu'|\lambda'-\lambda+1|\frac{w-a}{w-c}\frac{b-c}{b-a}\right) \quad (5.2.46)$$

Solutions of Papperitz equation at $w = b$ or $w = c$ can be obtained by interchanging λ, λ' with μ, μ', etc., since the equation is symmetrical for interchange of singular points and corresponding indices (if we remember that $\nu' = 1 - \lambda - \lambda' - \mu - \mu' - \nu$).

Notice the sort of "characteristic form" of the solution above (expansion at w = a): out front we have certain $w\text{-}w_i$ *ratios* raised to powers which are some of the exponents, and then we have the analytic F functions/series of the same weird argument which is a scaled w-a. Recall from above that we might "think of" the above solution in this notation

$$\psi = P\begin{Bmatrix} a & b & c & \\ \lambda & \mu & \nu & w \\ \lambda' & \mu' & \nu' & \end{Bmatrix}$$

If I were fiddling with this, I would try for a more "cyclic looking" notation.

So, if someone hands you a Papperitz matrix, you can at once write down the general solution as above if you want an expansion about w=a, and similarly for w = b or w = c.

PL Question: What to you do if, say, c = ∞ in the above Papperitz solution? There is a rule here which I remember from somewhere but I cannot find it right now. The rule is this (which rule of course could be derived with some effort by me) If you have a linear factor like w-c and c = ∞, you just replace that factor by a 1. This rule makes both the Chubby and Legendre stuff come out right below. It does seem odd that the encyclopedic M&F don't mention this rule.

[p 543] He then points out that if |1-c| is an integer, one of the terms above diverges, and this is nothing more than the case of exponents different by an integer, which we already know requires special treatment. Picking such a singular point, one of the above terms is OK, then we have to get "the other solution" using our usual second kind "trick" (they use this a lot!). Using symbols that look very much like those of WW, the second solution then looks like this (c = positive integer, so second term *above* is divergent).

$$y_2 = F(a,b|c|z)g_{c-1} \ln z + (1/z^{c-1})(h_0 + h_1 z + \cdots + h_{c-2}z^{c-2} + h_c z^c + \cdots)$$

Recall those h_n from WW and that unusual g factor that might be zero in certain cases, killing off the log term.

[p 544] Functions Expressible as HG Series. They first show some very elementary functions that can be written as HG functions,

$$(1+z)^n = F(-n,b|b| - z); \quad \ln(1+z) = zF(1,1|2| - z)$$

and then they start into those separated curvilinear Helmholtz equations1 For certain cases I am not interested in right now, you get the fat functions (Chebychev written in Russian by them) which of course have the corresponding fat polynomials -- as I like to call them -- they're chubby. Then he comes to something more interesting to me. In sphericals and spheroidal-Laplace we get a Legendre equation which they write in this interesting way, (for Laplace, so $k_1 = 0$)

$$\frac{d^2X}{d\xi^2} + \left[\frac{1}{\xi-1} + \frac{1}{\xi+1}\right]\frac{dX}{d\xi} - \left[\frac{k_2^2}{\xi^2-1} + \frac{k_3^2/2}{(\xi^2-1)(\xi+1)} - \frac{k_3^2/2}{(\xi^2-1)(\xi-1)}\right]X = 0$$

which is called the *Legendre equation*. It corresponds to Eq. (5.2.36) if $a = +1$, $b = -1$, $c = \infty$, $\lambda = -\lambda' = m/2 = \mu = -\mu'$, $\nu = -n$, $\nu' = n+1$, where we have set $k_3 = m$ and $k_2 = n(n+1)$ in order to make the results easier to write out. Therefore the solution corresponds to the Riemann symbol

$$X = P\begin{Bmatrix} 1 & -1 & \infty & \\ m/2 & m/2 & -n & \xi \\ -m/2 & -m/2 & n+1 & \end{Bmatrix}$$

and the general solution about $\xi = 1$ is [see also Eq. (5.2.52)]

$$X = A(1-\xi^2)^{m/2} F\left(m-n,\, m+n+1 \middle|\, 1+m \middle|\, \frac{1-\xi}{2}\right) + B\left(\frac{1+\xi}{1-\xi}\right)^{m/2} F\left(-n,\, n+1 \middle|\, 1-m \middle|\, \frac{1-\xi}{2}\right) \quad (5.2.47)$$

The first solution is known as the *Legendre function of the first kind*, and the second solution is called the *Legendre function of the second kind.*

Verify the Rule for Legendre. Let me explicitly verify my little "rule" claimed above. First, here is the general Papperitz solution:

$$A\left(\frac{w-a}{w-c}\right)^{\lambda}\left(\frac{w-b}{w-c}\right)^{\mu} F\left(\lambda+\mu+\nu,\, 1-\nu-\lambda'-\mu' \middle|\, \lambda-\lambda'+1 \middle|\, \frac{w-a}{w-c}\frac{b-c}{b-a}\right) + B\left(\frac{w-a}{w-c}\right)^{\lambda'}\left(\frac{w-b}{w-c}\right)^{\mu} F\left(\lambda'+\mu+\nu,\, 1-\nu-\lambda-\mu' \middle|\, \lambda'-\lambda+1 \middle|\, \frac{w-a}{w-c}\frac{b-c}{b-a}\right) \quad (5.2.46)$$

and we want to set in the values (and w = ξ)

$$\begin{Bmatrix} a & b & c & \\ \lambda & \mu & \nu & w \\ \lambda' & \mu' & \nu' & \end{Bmatrix} = \begin{Bmatrix} 1 & -1 & \infty & \\ m/2 & m/2 & -n & \xi \\ -m/2 & -m/2 & n+1 & \end{Bmatrix}$$

We then have (setting c factors to 1)

$$A\,(w-1)^{m/2}\,(w+1)^{m/2}\; F(m/2 + m/2 - n,\, 1 - (-n) - (-m/2) - (-m/2) \mid m+1 \mid (w-1)/(-1-1))$$

$$+ B\,(w-1)^{-m/2}\,(w+1)^{m/2}\; F(-m/2 + m/2 - n,\, 1 - (-n) - m/2 + m/2 \mid -m+1 \mid ((w-1)/(-1-1))$$

$$= A\,(w^2-1)^{m/2}\; F(m-n,\, 1+n+m \mid m+1 \mid (1-w)/2)$$

$$+ B\,[(w+1)/(w-1)]^{m/2}\; F(-n,\, 1+n \mid 1-m \mid (1-w)/2)$$

and we can compare this to the MF result

$$X = A(1-\xi^2)^{m/2} F\left(m-n,\, m+n+1 \middle|\, 1+m \middle|\, \frac{1-\xi}{2}\right) + B\left(\frac{1+\xi}{1-\xi}\right)^{m/2} F\left(-n,\, n+1 \middle|\, 1-m \middle|\, \frac{1-\xi}{2}\right) \quad (5.2.47)$$

The only difference is a factor $(-1)^{m/2}$ phase in the first term. Since A and B are general constants, this is irrelevant. I have therefore shown that if you take the Papperitz general solution, you should set linear factors to 1 if the constant is infinite. As I said, I read this somewhere but cannot find it today.

Now one comment on the above Legendre form. These two series are expansions at the ξ=1 singular point, so you see $(1-\xi)^{m/2}$ out in front for the first solution, and $(1-\xi)^{-m/2}$ for the second, where these are the exponents in the first column of the matrix for this singular point.

Main Point: The Legendre equation is just a special case of the general Papperitz equation with certain values of the singular points (1,-1,∞) and certain values of the exponents.

MF's comments about first and second kind seem wrong. If you look at Bateman p 122 you see that in fact the second term above (the one with the B) is in fact proportional to $P_n^m(\xi)$ which is the first kind function. Moreover, the first term is not the corresponding Q function, but is some combination of P and Q. Here are their comments which I think are wrong:

> The first solution is known as the *Legendre function of the first kind*, and the second solution is called the *Legendre function of the second kind*. If m is a positive integer, this second solution, as written, will "blow up" and a second solution will have to be constructed by use of Eq. (5.2.6) (see page 532). These Legendre functions and the related Gegenbauer functions will be discussed again in a few pages and in very considerable detail in Secs. 5.3 and 10.3, for they are of very great importance in our future discussions.

Maybe this error snuck in because they are going to do most of their "work" with the more general Gebenbauers.

[p 545] Analytic Continuation of the HG Series. Their first exercise here is to write the z=0 HF function as a linear combination of the z=1 expansions which we already know. In other words, they first establish that the "general solution" at z = 1 (using the various exponents) is this:

$$AF(a,b|a+b-c+1|1-z) + B(1-z)^{c-a-b}F(c-b,\, c-a|\, c-a-b+1|\, 1-z)$$

Then if we set this equal to F(a,b,c,z), there must be some A and B that "work", since there are after all only two possible independent solutions. They call A and B by names α and β and then solve for α and β by looking close to the point z = 1. Their result is then this:

$$F(a,b|c|z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,\ b|\ a+b-c+1|\ 1-z)$$
$$+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} F(c-a,\ c-b|\ c-a-b+1|\ 1-z) \qquad (5.2.49)$$

and this is then one of those Kummer relations seen on pages 107,8 of Bateman. MF are calling this analytic continuation because the original series F(a,b,c,z) does not converge in a region near z = 1, but the form shown here does, so we have found a way to "continue" the meaning of the function F(a,b,c,z) beyond its normal range. In this general manner I know we can continue it everywhere. MF call this relation a "joining relation".

Next, they derive this interesting fact:

$$F(2\alpha,\ 2\beta|\ \alpha+\beta+\tfrac{1}{2}|\ z) = F(\alpha,\ \beta|\ \alpha+\beta+\tfrac{1}{2}|\ 4z-4z^2) \qquad (5.2.50)$$

This formula, relating an F for 2α, 2β, and z to an F for α, β, and z^2 might be called a *duplication formula* for the hypergeometric function.

[p 547] Gegenbauer Functions.

Gegenbauer Functions. We mentioned earlier that it is sometimes desirable to have the canonical form for three regular singular points so that two of them are at ± 1 rather than 0 and 1. This is particularly true when the indices at $+1$ are the same as the indices for -1. This suggests the Riemann symbol

$$\psi = P \begin{Bmatrix} -1 & 1 & \infty & \\ 0 & 0 & -\alpha & z \\ -\beta & -\beta & \alpha+2\beta+1 & \end{Bmatrix}$$

The corresponding equation

$$(z^2-1)\psi'' + 2(\beta+1)z\psi' - \alpha(\alpha+2\beta+1)\psi = 0 \qquad (5.2.51)$$

is called *Gegenbauer's equation* [compared with the Legendre equation on page 544]. It is a satisfactory form for the equations for ξ_2 for circular cylinder and spherical coordinates. The solutions can, of course, be expressed in terms of hypergeometric functions:

Before continuing, I would like to see what "**the world**" has to say about Gegenbauer stuff. I think the most basic idea is this:

$$(1 - 2hz + h^2)^{-\nu} = \Sigma_{n=0} C_n^{\nu}(z)\, h^n$$

$$(1 - 2hz + h^2)^{-1/2} = \Sigma_{n=0} P_n(z)\, h^n$$

This type of formula is called a "generating function" . As MM say p 132, the generating function is a function of two variables such that, if you expand in a power series in one variable, the coefficients are a set of polynomials of interest. [There are other math meanings of generating function.] So you see that the **Gegenbauer polynomial** is a generalization of the Legendre where the negative power is ν and not just 1/2. So obviously we have this connection

$$P_n(z) = C_n^{1/2}(z)$$

so in some sense we are generalized "by one parameter". You can of course then think of ν and n as general complex numbers along with z, and then you have Gegenbauer functions. I think this is now the standard notation for Gegenbauer. Then we go to GR p 1031 and we see that the ODE does NOT agree with that used by MF, which is annoying.

$(z^2-1)\psi'' + (2\nu+1)z\psi' - n(2\nu+n)\psi = 0$ solution $C_n^{\nu}(z)$

So we have to say $2\nu+1 = 2\beta + 2$ and $n(2\nu+n) = \alpha(\alpha+2\beta+1)$

$\nu = \beta+1/2$ and $\alpha = n$

The MF functions will be called $T_\alpha^\beta(z)$ [their "Gegenbauer function"] so we generally have

$$T_\alpha^\beta(z) = \text{constant}(\alpha,\beta) * C_\alpha^{\beta+1/2}(z)$$

which can be called a *Gegenbauer polynomial* (it differs, by a numerical factor, from the polynomial $C_n^{\beta+\frac{1}{2}}$ often called a Gegenbauer polynomial).

If we set $\beta = 0$, we get

$$T_\alpha^0(z) = \text{constant}(\alpha,0) * C_\alpha^{1/2}(z) = 1 * P_\alpha(z)$$

$$P_n(z) = T_n^0(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}(z^2 - 1)^n$$

This fact is also interesting

$$T_n^m(z) = \frac{d^m}{dz^m} P_{n+m}(z)$$

= **tesseral polynomials**

Notice that the "direction" of the Gegenbauer generalization is not that of the $P_n^m(z)$ associated Legendre functions. Yes, we have an extra parameter, but it is a different parameter.

The Gegenbauer polynomials are sometimes called **ultraspherical polynomials**.

If we consider $C_n^\nu(z)$ and set $\nu = 1$ ($\beta = 1/2$) , we are talking $C_n^1(z)$ and a peek at GR page 1033 shows that these functions must be the same as the "**second kind Chebyshev polynomials**" called $U_n(z)$.

$U_n(z) \sim C_n^1(z) \sim T_n^{1/2}(z)$ // see cases listed on GR p 1032.

Interestingly, this completely disagrees with MF who say on p 549,

$$T_n^{-\frac{1}{2}}(z) = \frac{1}{n}\sqrt{\frac{2}{\pi}}\cosh\left[n\cosh^{-1}(z)\right]$$
$$T_n^{\frac{1}{2}}(z) = \sqrt{\frac{2/\pi}{z^2-1}}\sinh\left[(n+1)\cosh^{-1}(z)\right] \qquad (5.2.54)$$

called *Tschebyscheff polynomials.*

At least they certainly don't LOOK the same. For example, GR say $U_0(z) = 1$. On the other hand, Maple tells us

```
sinh( (1)*arccosh(z) );
```

$$\sqrt{z-1}\,\sqrt{z+1}$$

so perhaps they are the same. GR shows some forms that are similar to the above, but with trig and not hyperbolic functions. Let's let this strange subject lie. MF are just trying to pack in stuff they will use way later in their book. [I don't see anyone else that does Gegenbauer functions the MF way.]

One Regular and One Irregular Singular Point. For this equation it is usual to put the regular singular point at $z = 0$ and the irregular one at $z = \infty$. Equations of this sort occur for the following separable coordinates:

MF list off 4 examples they yield this kind of equation. One is the Bessel function in several coordinates, and another is the SE with a Coulomb potential in spherical coordinates. So these all seem to have a certain form, we MF look into this form:

This suggests that we study the general equation

$$\frac{d^2\psi}{dz^2} + p(z)\frac{d\psi}{dz} + q(z)\psi = 0; \quad p = \frac{(1-\lambda-\lambda')}{z}; \qquad q = -k^2 + \left(\frac{2\alpha}{z}\right) + \left(\frac{\lambda\lambda'}{z^2}\right) \tag{5.2.55}$$

This is not the most general equation with a regular singular point at $z = 0$ and an irregular one at infinity. The function p could have a constant, and both p and q could have terms like az or bz, etc., added to the expressions shown in Eq. (5.2.55). But these additional terms would make the essential singularity in the solution at $z = \infty$ more "singular," and since none of the equations coming from the separation of the wave equation exhibit such additional terms, we shall not include them here. A few more complicated cases will be included in the problems, however!

The first let $\psi = z^{\lambda}f(z)$. Next they go to z = 1/w. Next, they go to $f = e^{-k/w}F$ and then we have

$$F'' + \left[\frac{2k}{w^2} + \frac{1-\lambda+\lambda'}{w}\right]F' - \left[\frac{k(1+\lambda-\lambda') - 2\alpha}{w^3}\right]F = 0$$

OK, this is just leading them to the **confluent hypergeometric function** stuff F(a|c|x). This seems a very strange and painful way to introduce the student to all these special functions! But the main point is that this CHG function falls into the class of one irregular (put at ∞) and one regular(put at 0) point. We are conflowing two of the HG's three regular points somehow, and this is creating an irregular point. I think I noted this earlier.

[p 554] Here they look into a 1/z series expansion for the CHG function

$$F_2 = Bz^{a-c}e^{z}\left[1 + \frac{(1-a)(c-a)}{1!}\frac{1}{z} + \frac{(1-a)(2-a)(c-a)(c-a+1)}{2!}\frac{1}{z^2} + \cdots\right]$$

The chief trouble with these two series solutions is that *they do not converge*, except at $z = \infty$. The divergence is of a peculiar type, however, for when z is large but finite, the series *first converges* and then, as we take more and more terms, *eventually diverges*. To be specific, we

so they are taking this CHG example to introduce the (famous) subject of "asymptotic series" which I have run into before.

Such series, which are divergent but which can be used to compute a value that never exactly equals the "correct" value but rapidly approaches this "correct" value as z is increased, are called *asymptotic series*. They were discussed in some detail in Sec. 4.6. In some respects they turn out to be more useful than convergent series as long as they are handled with tact and understanding. Some of the requisite understanding will be supplied in next section; the tact must be left to the user.

Two Regular, One Irregular Singular Point.

He claims this applies for ξ_1 and ξ_2 of both prolate and oblate spheroidal coordinates! In my notes, I thought that these things were Legendre P and Q function guys, one of imaginary coordinates. Maybe somehow the z = ∞ point is going to change from regular to irregular. But here MF claim we end up somehow with Mathieu's Functions! I am rapidly losing interest now because I feel we are just randomly doing things in this chapter. Mention of Floquet's Theorem of some sort, periodicity issues.

[p 558] Next subject is "continued fractions". I am not interested.

[p 560] The Hill Determinant. On and on, don't care.

[p 562] Mathieu Functions.

OK, so Section 5.2 sort of dies with a whimper. I have no interest in Section 5.3 on integral representations of things right now. I know that special functions have integral representations and these help you "continue" things and all that stuff.

The end of this huge Chapter 5 contains two huge summary sections:

1) data for all the separable coordinate systems

2) review of the various classes of ODE solutions in terms of their singular points! That really was the main thrust of Section 5.2.

More MF Notes on Gegenbauer.

They introduce this T function above in Section 5.2

$$T_\alpha^\beta(z) = \text{constant}(\alpha,\beta) * C_\alpha^{\beta+1/2}(z)$$

but they are going to say a lot more about it in Section 5.3, so let's right now wander into that section and see what they are saying about the T functions. We got to **page 600** in Section 5.3. We have

case, equals the mth derivative of the Legendre function $P_{m+\alpha}(z)$. When *both* α and β are integers,

$$T_n^m(z) = (1 - z^2)^{-\frac{1}{2}m} P_{n+m}^m(z) = \frac{d^m}{dz^m} T_{m+n}^0(z)$$
$$= \frac{(n + 2m)!}{2^{n+m} n!(n + m)!} \frac{1}{(z^2 - 1)^m} \frac{d^n}{dz^n} (z^2 - 1)^{n+m}$$
$$= \frac{1}{2^{n+m}(n + m)!} \frac{d^{n+2m}}{dz^{n+2m}} (z^2 - 1)^{n+m} \qquad (5.3.37)$$

as indicated in Eq. (5.2.53). These polynomials are called *associated Legendre polynomials.*

He calls these "associated Legendre" but I would call $P_n^m(z)$ associated Legendre, so MF are being confusing to the reader here. In any event, we see how T and P are related in this "both integers" case. If n = 0, the derivative is this [he is assuming m ≥ 0, there is some phase rule for negative m]

$\partial_z^{2m} (z^2\text{-}1)^m$

Now $(z^2\text{-}1)^m$ is a poly of order 2m, so this derivative will be a constant. So $T_o^m(z)$ *has no z dependence*. But the above says

$P_m^m(z) = (1\text{-}z^2)^{m/2} T_0^m(z) = = \sin^m\theta\ T_0^m(z)$ // one node at equator for z.

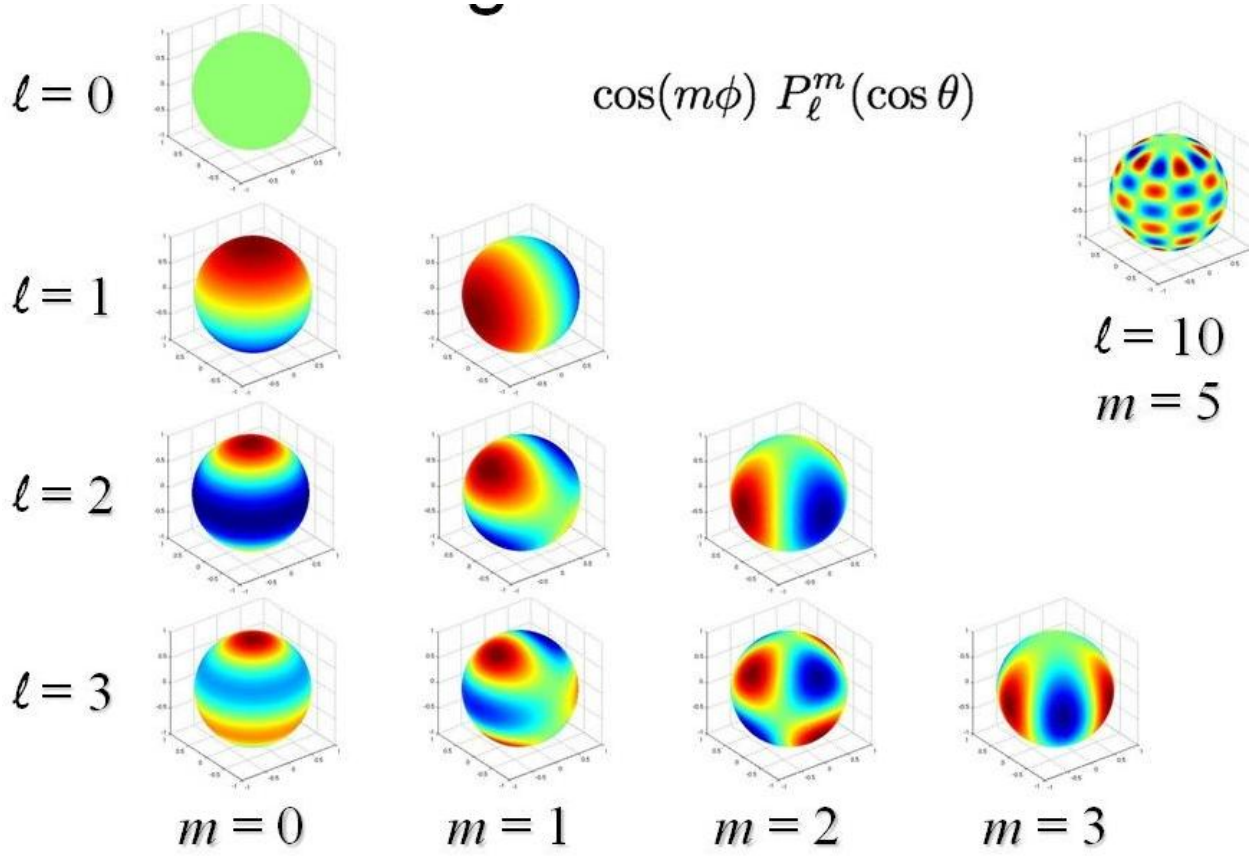
so P_m^m has only the z=dependence shown. For any m, is something that is 0 at the poles and peaks at the equator. So for any value of m, this looks more or less the same.

Viewing Spherical Harmonics.

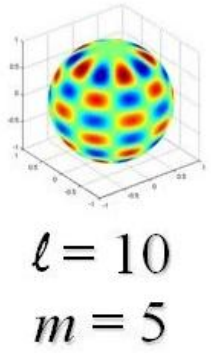
Here is a java deal that draws these things, I will clip some pix:

http://icgem.gfz-potsdam.de/ICGEM/potato/Tutorial.html

Well, all I really want is a few pictures, here we are:



So now we see clearly why the m = 0 are called **zonal**, and why the m = ℓ (also -ℓ) are called **sectorial**, and why everything else is **tesseral**, for example,



The zonal ones really are constant along a latitude line, whereas the sectorial are not really constant on a longitude line, but have the simple one node shape noted above.

So, perhaps this T function is better at showing this kind of stuff. Back to page 600 MF now.

$$T^{\beta}_{-\alpha-2\beta-1}(z) = \left[\frac{\sin \pi(\alpha + 2\beta)}{\sin(\pi\alpha)}\right] T^{\beta}_{\alpha}(z); \quad P^{\beta}_{\alpha+\beta}(z) = P^{\beta}_{-\alpha-\beta-1}(z)$$

This I think shows the normalization of his functions:

$$\frac{2^{\beta}\Gamma(\beta + \frac{1}{2})/\sqrt{\pi}}{(1 + h^2 - 2hz)^{\beta+\frac{1}{2}}} = \sum_{n=0}^{\infty} h^n T^{\beta}_{n}(z); \quad |h| < 1 \qquad (5.3.39)$$

I don't see much more crucial T stuff in this general book section.

The next Gegenbauer cache of data is on page 782, a sort of summary section. The "density function" is the weight function that will show up in orthogonality below.

I Range $-1 \leq z \leq +1$; Density Function $(1 - z^2)^{\beta}$: Gegenbauer polynomials: $T^{\beta}_{n}(z)$ (see page 748).

Generating function:

$$\frac{2^{\beta}}{(1 + t^2 - 2tz)^{\beta+\frac{1}{2}}} = \frac{\sqrt{\pi}}{\Gamma(\beta + \frac{1}{2})} \sum_{n=0}^{\infty} t^n T^{\beta}_{n}(z)$$

Special cases: $T^{0}_{a}(z) = P_n(z)$, Legendre polynomials (see Eq. 5.3.24)

$(1 - z^2)^{m/2} T^{m}_{n-m}(z) = P^{m}_{n}(z)$, associated Legendre functions [see Eq. (5.3.38)]

$nT^{-\frac{1}{2}}_{n}(z) = \sqrt{\frac{2}{\pi}} \cosh [n \cosh^{-1} z]$, Tschebyscheff polynomials [see Eq. (5.3.43)]

$\sqrt{z^2 - 1}\, T^{\frac{1}{2}}_{n-1}(z) = \sqrt{\frac{2}{\pi}} \sinh [n \cosh^{-1} z]$, Tschebyscheff polynomials [see Eq. (5.3.43)]

$$T^{\beta}_{0}(z) = \frac{2^{\beta}}{\sqrt{\pi}} \Gamma(\beta + \tfrac{1}{2}) = 1 \cdot 1 \cdot 3 \cdot 5 \cdot \cdot \quad (2\beta - 1); \text{ when } \beta = 0, 1, 2, \ldots$$

$$T^{\beta}_{1}(z) = \frac{2^{\beta+1}}{\sqrt{\pi}} \Gamma(\beta + \tfrac{3}{2}) z = 1 \cdot 3 \cdot 5 \cdot \quad \cdot (2\beta + 1) z; \text{ when } \beta = 0, 1, 2, \ldots$$

Recurrence formulas, relating these polynomials, obtained from the generating function

$$\frac{d}{dz}\,T_n^{\beta}(z) = T_{n-1}^{\beta+1}(z);$$

$$\frac{d}{dz}\,[(z^2-1)^{\beta}T_n^{\beta}(z)] = (n+1)(n+2\beta)(z^2-1)^{\beta-1}T_{n+1}^{\beta-1}(z)$$

$$(2\beta+2n+1)zT_n^{\beta}(z) = (n+1)T_{n+1}^{\beta}(z) + (2\beta+n)T_{n-1}^{\beta}(z)$$

$$(2\beta+2n+1)T_n^{\beta}(z) = \frac{d}{dz}\,[T_{n+1}^{\beta}(z) - T_{n-1}^{\beta}(z)] = T_n^{\beta+1}(z) - T_{n-2}^{\beta+1}(z)$$

$$(2\beta+1)T_n^{\beta}(z) + 2z\,\frac{d}{dz}\,T_n^{\beta}(z) = T_n^{\beta+1}(z) + T_{n-2}^{\beta+1}(z)$$

$$(n+2\beta+1)T_n^{\beta}(z) = T_n^{\beta+1}(z) - zT_{n-1}^{\beta+1}(z)$$

$$nT_n^{\beta}(z) = zT_{n-1}^{\beta+1}(z) - T_{n-2}^{\beta+1}(z)$$

$$(2\beta + 2n + 1)(z^2 - 1)T_{n-1}^{\beta+1}(z) = n(n+1)T_{n+1}^{\beta}(z) - (n + 2\beta)(n + 2\beta + 1)T_{n-1}^{\beta}(z)$$

$$T_n^{\beta}(z) = \frac{2^{\beta}\Gamma(\beta + \frac{1}{2})}{n!\sqrt{\pi}} \cdot \frac{(2\beta+1)(2\beta+3)\cdots(2\beta+2n-3)(2\beta+2n-1)}{(2\beta+n+1)(2\beta+n+2)\cdots(2\beta+2n-1)(2\beta+2n)} \cdot \frac{1}{(z^2-1)^{\beta}} \frac{d^n}{dz^n}(z^2-1)^{n+\beta}$$

$$(z^2 - 1)\frac{d^2}{dz^2}T_n^{\beta}(z) + 2(\beta+1)z\frac{d}{dz}T_n^{\beta}(z) - n(n+2\beta+1)T_n^{\beta}(z) = 0$$

which is related to the hypergeometric equation, having three regular singularities. $\beta = m$, an integer, results in *associated Legendre* polynomials,

$$T_n^m = \frac{P_{n+m}^m}{(1-z^2)^{\frac{1}{2}m}} = \frac{d^m}{dz^m}(P_{n+m})$$

$T_0^0 = 1$; $T_0^1 = 1$; $T_0^2 = 3$; $T_0^3 = 15$; . . .
$T_1^0 = z$; $T_1^1 = 3z$; $T_1^2 = 15z$; $T_1^3 = 105z$; . . .
$T_2^0 = \frac{1}{2}(3z^2 - 1)$; $T_2^1 = \frac{3}{2}(5z^2 - 1)$; $T_2^2 = \frac{15}{2}(7z^2 - 1)$; . .
$T_3^0 = \frac{1}{2}(5z^3 - 3z)$; $T_3^1 = \frac{5}{2}(7z^3 - 3z)$; $T_3^2 = \frac{105}{2}(3z^3 - z)$; . .
$T_4^0 = \frac{1}{8}(35z^4 - 30z^2 + 3)$; $T_4^1 = \frac{15}{8}(21z^4 - 14z^2 + 1)$;
. . . . . .

$$T_n^m = \frac{(2n+2m)!}{2^{n+m}n!(n+m)!}\left[z^n - \frac{n(n-1)}{2(2n+2m-1)}z^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n+2m-1)(2n+2m-3)}z^{n-4} \cdots\right]$$

see also Eqs. (6.3.37) and (6.3.40).

Normalization integral:

$$\int_{-1}^{1} (1 - z^2)^{\beta} T_m^{\beta}(z) T_n^{\beta}(z)\, dz = \delta_{mn} \frac{2\Gamma(n + 2\beta + 1)}{(2n + 2\beta + 1)\Gamma(n + 1)}$$

Special values:

$$T_n^{\beta}(z) = (-1)^n T_n^{\beta}(-z)$$

$$T_n^{\beta}(1) = \frac{\Gamma(n + 2\beta + 1)}{2^{\beta} n! \Gamma(\beta + 1)}; \quad T_n^{\beta}(0) = 0; \quad n = 1, 3, 5, \quad \ldots$$

$$T_n^{\beta}(0) = (-1)^{\frac{1}{2}n} \frac{2^{\beta}\Gamma(\beta + \frac{1}{2}n + \frac{1}{2})}{\sqrt{\pi}\ (\frac{1}{2}n)!}; \quad \text{when } n = 0, 2, 4, \ldots$$

Relation with hyygeomctric function:

$$T_n^{\beta}(z) = \frac{\Gamma(n + 2\beta + 1)}{2^{\beta} n! \Gamma(\beta + 1)} F\left(-n, n + 2\beta + 1 | 1 + \beta | \frac{1 - z}{2}\right)$$

$$= (-1)^n \frac{\Gamma(n + 2\beta + 1)}{2^{\beta} n! \Gamma(\beta + 1)} F\left(-n, n + 2\beta + 1 | 1 + \beta | \frac{1 + z}{2}\right)$$

Addition formula:

$$T_n^{\beta}(\cos\vartheta \cos\vartheta_0 + \sin\vartheta \sin\vartheta_0 \cos\varphi)$$

$$= \sqrt{2\pi} \sum_{m=0}^{n} \frac{(\beta + m)(n - m)!}{\Gamma(2\beta + n + m + 1)} [\sin\vartheta \sin\vartheta_0]^m \cdot$$

$$\cdot T_{n-m}^{\beta+m}(\cos\vartheta)\ T_{n-m}^{\beta+m}(\cos\vartheta_0) T_m^{\beta-\frac{1}{2}}(\cos\varphi)$$

Good heavens! I wonder if anyone else deals with these functions in this manner and notation. AS mention a T thing in Legendre stuff, but not the same thing. GR nothing. Bateman nothing. This is really an alternative to "associated Legendre" notation.

I have been hunting for some other source that uses this T notation, and I found (amazingly) what seems to be the article that proposed it in 1935, and I saved it. It is by Stratton, also at MIT. Here is a quip

$$T_l^a(z) = \frac{\Gamma(2a)}{2^{a-1}\Gamma(a)} C_l^{a+1/2}(z), \tag{2}$$

or as a hypergeometric function

$$T_l^a(z) = \frac{\Gamma(2a+l+1)}{2^a\Gamma(a+1)\Gamma(l+1)} F\left(2a+l+1, -l, a+1, \frac{1-z}{2}\right). \tag{3}$$

Then if $a = m$, an integer,

$$T_l^m(z) = \frac{d^m}{dz^m} P_{l+m}(z) = (1-z^2)^{-m/2} P_{l+m}^m(z). \tag{4}$$

One has furthermore

$$(2l+2a+1)zT_l^a = (l+2a)T_{l-1}^a + (l+1)T_{l+1}^a, \tag{5}$$

$$\int_{-1}^{1}(1-z^2)^a T_l^a(z)T_k^a(z)dz = \frac{2\Gamma(l+2a+1)}{(2l+2a+1)\Gamma(l+1)}\delta_{lk}. \tag{6}$$

[warning: ℓ in the above is assumed to be a positive integer ℓ = 1,2,3,.. When people talk about the C functions, they always have the lower index a positive integer or maybe 0. It is never a complex ν.]

which finally directly shows the connection to the other notation. But he uses ℓ and a instead of α and β, but OK. Stratton says he invented this notation because it makes a simple connection to the associated Legendres as shown above. By the way, the above article is from PNAS which is not a free journal, but if you poke around there are free PDF's. Stratton has another paper where he does a Q version of this T function, and he calls it T .

So OK, Morse was around MIT at this time and I think we can assume he has all the stuff right. It is a little local pool of usage, probably long forgotten now. We can just think of T as defined by (4) above.