Some ODE Notes PhL 1.21.99

Differential and Difference Equations with constant coefficients.

Page 48 of MM is a good place to start. The ODE is written in 2.12 using the usual D operator. You can think of this as:

$Ay = 0$ $\quad A = D^n + a_1 D^{n-1} + ... + a_n\, 1$ $\quad$ = an operator

This operator A can be factored just as if it were a polynomial and D were x:

$$A = (D - r_1)(D - r_2)..... (D - r_n)$$

You can see for example that $- a_1$ = the sum of the r_i , and a_n is the product of the r_i. Consider now the corresponding x polynomial and its fully factored form:

$$p(x) = x^n + a_1 x^{n-1} + ... + a_n = (x - r_1)(x - r_2)..... (x - r_n)$$

This is called the **characteristic polynomial** for the ODE and it of course has roots that we can find by the usual methods of factorization.

Now back to the ODE. Notice that the factors $(D - r_i)$ can be put in any order, the factors commute because the r_i are constants! So we can pick any r_i and put its factor on the right, and we can then consider this equation:

$$(D - r_i)y = 0$$

A solution of this is then a solution of the full ODE. But we know the solution to this is $y(x) = \exp(r_i x)$. Therefore, we know this is true for any root and we have a linear equation, so we now have a fairly large class of solutions,

$$y(x) = \sum_{i=1}^{n} c_i \exp(r_i x)$$

The claim is now that this solution has n independent constants, and that therefore this is the most general solution of the equation. How do we know that?

I had to go off and write a whole separate document to see why this is so. The answer is that the n^{th} order ODE has a unique solution for the n usual boundary conditions on y, y'..... $y^{(n-1)}$ at some point, given certain conditions including Lipschitz. These n constants map into the n c_i constants above. There cannot be more or less than n free constants c_i to meet the boundary conditions. Thus, the dimensionality of the basis is n.

Now, if the roots are all different, we are done. If 4 roots are the same, then we replace those four indistinct expos with these: e^{rx} , $x\, e^{rx}$, $x^2 e^{rx}$, $x^3 e^{rx}$
This idea works for whatever root degeneracy you have.

Difference equations

Now suppose we have this, again with constant coefficients

$A = \Delta^n + a_1\Delta^{n-1} + ... + a_n\, 1$ = an operator $Ay_k = 0$

We can factor this thing as before

$A = (\Delta - s_1)(\Delta - s_2)..... (\Delta - s_n)$

Then we can look at this equation

$(\Delta - s_1)\, y_k = 0$

But I think instead they want you to think in terms of the raising operator $E = 1+\Delta$ where we have $Ey_k = y_{k+1}$. So let's reorder our difference equation like this:

$A = b_n E^n + b_{n-1}E^{n-1} + ... + b_0\, 1$ $Ay_k = 0$

which says

$b_n\, y_{k+n} + b_{n-1}\, y_{k+n-1} + ... + b_0\, y_k = 0$ // order n

then factor this thing

$A = (E - r_1)(E - r_2)..... (E - r_n)$

Then consider

$(E - r_1)y_k = 0$ => $y_{k+1} = r_1\, y_k$ => $y_k = (r_1)^k y_0$

so *now you see* how the little powers arise! As you iterate, each time you build up another factor. so these little root powers are the analogs of the exp functions in the continuous case. We can then write our general solution as:

$y_k = a_1\ (r_1)^k + a_2\ (r_2)^k + ...\ \ a_n\ (r_n)^k$

and you determine the a_i using initial conditions on say $y_0, y_1 .... y_{n-1}$.