Differential Equations PhL 12.12.08
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These notes are extremely disorganized, I am just accumulating and then refining information. We are talking about second order ODE's with general function coefficients. The issue of the differential operator L being self-adjoint never comes up in these notes which are strongly oriented to the Frobenius style series solutions.

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My Problem with Stakgold: Stakgold's Chapter 4 is about "differential equations", but only in a general sense where L is a general second-order differential operator and we want to learn about the *spectrum* of this operator under specific circumstances which include a certain range on the real axis, and a specified norm, usually the L_2 norm of square-integrable functions. His second volume then discusses the main *partial* differential equations of physics. His book just does not really address the ordinary differential equations of physics (1 variable) which lead to the so-called Special Functions. You don't really learn about singular and regular points, Frobenius is not even mentioned, etc etc.

When I am wondering about ODE's, Stakgold is the only book I own that really has differential equations as its topic, but it never has the information I want, so I should stop opening it! It is all about Hilbert Space and formal theorems, although they are at least readable, to his credit.

The Bateman books have more information on ODE's for special functions, but they are not teaching you the basics, such as what is the Frobenius method, or what are branch cuts, etc. My "library" needs a good book on this stuff, but I don't know what the title would be. "Modern Analysis" ? " Differential Equations for Dummies" ? Maybe Whittaker and Watson is what I am missing, some book like that. Dover has lots of no-name books but not this one. // Just ordered Whittaker and Watson at $22.50!

Question: What is a regular point or a singular point of an ODE?

Wiki:

"In mathematics, in the theory of **ordinary differential equations** in the complex plane C, the points of C are classified into **ordinary points**, at which the equation's coefficients are analytic functions, and **singular points**, at which **some coefficient** has a singularity. Then amongst singular points, an important distinction is made between a **regular singular point**, where the growth of solutions is bounded (in any small sector) by an algebraic function, and an **irregular singular point**, where the full solution set requires functions with higher growth rates. This distinction occurs, for example, between the hypergeometric equation, with three regular singular points, and the Bessel equation which is in a sense a limiting case, but where the analytic properties are substantially different."

So, one of these "points" relates to the coefficient functions in your ODE, not to some abstract general thing about the equation overall, though that is likely also true.

Formal definitions [edit]

More precisely, consider an ordinary linear differential equation of *n*-th order

$$\sum_{i=0}^{n} p_i(z) f^{(i)}(z) = 0$$

with $p_i(z)$ meromorphic functions. One can assume that

$$p_n(z) = 1.$$

If this is not the case the equation above has to be divided by $p_n(x)$. This may introduce singular points to consider.

The equation should be studied on the Riemann sphere to include the point at infinity is a possible singular point. A Möbius transformation may be applied to move ∞ into the finite part of the complex plane if required, see example on Bessel differential equation below.

Then the Frobenius method based on the indicial equation may be applied to find possible solutions that are power series times complex powers $(z - a)^r$ near any given *a* in the complex plane where *r* need not be an integer; this function may exist, therefore, only thanks to a branch cut extending out from *a*, or on a Riemann surface of some punctured disc around *a*. This presents no difficulty for *a* an ordinary point (Lazarus Fuchs 1866). When *a* is a **regular singular point**, which by definition means that

$$p_{n-i}(z)$$

has a pole of order at most *i* at *a*, the Frobenius method also can be made to work and provide *n* independent solutions near *a*.

Otherwise the point *a* is an **irregular singularity**. In that case the monodromy group relating solutions by analytic continuation has less to say in general, and the solutions are harder to study, except in terms of their asymptotic expansions.

The regularity condition is a kind of Newton polygon condition, in the sense that the allowed poles are in a region, when plotted against *i*, bounded by a line at 45° to the axes.

An ordinary differential equation whose only singular points, including the point at infinity, are regular singular points is called a **Fuchsian** ordinary differential equation.

There are lots of words here I am not familiar with. I will dispense with them in Appendix A, so we can not loose track of the main thread here.

So, for an n^{th} order ODE with various weird coefficient functions, if you can do Frobenius, you will get an n^{th} degree polynomial in r for your indicial polynomial and that means in general that there will be n distinct solutions. For our normal ODE with n=2, there are normally 2 solutions for this reason. An example I think would be spherical Bessels where you get j and n functions with different "r".

Examples: Bessel, Legendre, Hermite, Hypergeometric

Now consider second order only:

Examples for second order differential equations [edit]

In this case the equation above is reduced to:

$$f''(x) + p_1(x)f'(x) + p_0(x)f(x) = 0.$$

One distinguishes the following cases:

- Point a is **ordinary point** when functions $p_1(x)$ and $p_0(x)$ are analytic at $x = a$.
- Point a is **regular singular point** if $p_1(x)$ has a pole of order 1 at $x = a$ or p_0 has a pole of order up to 2 at $x = a$.
- Otherwise point a is **irregular singular point**.

Listed below are several examples from ordinary differential equations from mathematical physics that have singular points and known solutions.

Remember that you have to first divide through by anything that might multiply f"(x). Then if the two functions are analytic at a point x, point x is an "**ordinary point**" for this ODE. If there are poles at some point x that are *not too bad* (it says first order max in p_1 or second order max in p_2) then the point x is singular, but it is **regular singular**. If the poles are higher order than this, or if there is a branch cut at x or what have you, then point x is **irregular singular**. I presume you only care about points in general that are in the range over which you work with your ODE (but maybe you do still care).

Wiki then provides some down to earth examples:

Bessel differential equation [edit]

This is an ordinary differential equation of second order. It is found in the solution to Laplace's equation in cylindrical coordinates:

$$x^2\frac{d^2f}{dx^2} + x\frac{df}{dx} + (x^2 - \alpha^2)f = 0$$

for an arbitrary real or complex number α (the *order* of the Bessel function). The most common and important special case is where α is an integer *n*.

Dividing this equation by x^2 gives:

$$\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx} + \left(1 - \frac{\alpha^2}{x^2}\right)f = 0$$

In this case $p_1(x) = 1 / x$ has a pole of first order at $x = 0$. When $\alpha \neq 0$ $p_0(x) = (1 - \alpha^2 / x^2)$ has a pole of second order at $x = 0$. Thus this equation has a regular singularity at 0.

To see what happens when $x \to \infty$ one has to use Möbius transformation, for example $x = 1 / (w - b)$. After performing the algebra:

$$\frac{d^2f}{dw^2} + \frac{1}{w-b}\frac{df}{dw} + \left[\frac{1}{(w-b)^4} - \frac{\alpha^2}{(w-b)^2}\right]f = 0$$

Now $p_1(w) = 1 / (w - b)$ has a pole of first order at $w = b$. And $p_0(w)$ has a pole of fourth order at $w = b$. Thus this equation has an irregular singularity $w = b$ corresponding to x at ∞.

The solutions of this differential equation are Bessel functions.

Yes, you see that x=0 is just "regular singular". It is not obvious to me that x=∞ has the fourth order pole when you Mobius it, but once you know it has that, you know that x = ∞ is "irregular singular"

Legendre differential equation

This is an ordinary differential equation of second order. It is found in the solution of Laplace's equation in spherical coordinates:

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}f\right]+n(n+1)f=0.$$

Opening the square bracket gives:

$$(1-x^2)\frac{d^2f}{dx^2}-2x\frac{df}{dx}+n(n+1)y=0.$$

And dividing by (1 - x^2):

$$\frac{d^2f}{dx^2}-\frac{2x}{(1-x^2)}\frac{df}{dx}+\frac{n(n+1)}{(1-x^2)}f=0.$$

This differential equation has regular singular points at -1, +1, and ∞.

Here you see there are only single poles in both functions, so x= 1 and x = -1 are regular singular points. Not obvious to me that x=∞ is also regular. [For x= ∞, see the WW test in meta notes.]

Hermite differential equation

One encounters this ordinary second order differential equation in solution of one dimensional time independent Schrodinger equation

$$E\psi=-\frac{\hbar^2}{2m}\frac{d^2\psi}{d^2x}+V(x)\psi$$

for harmonic oscillator. In this case the potential energy $V(x)$ is:

$$V(x)=\frac{1}{2}m\omega^2x^2.$$

This leads to the following ordinary second order differential equation is:

$$\frac{d^2f}{dx^2}-2x\frac{df}{dx}+\lambda f=0.$$

This differential equation has an irregular singularity at ∞. Its solutions are Hermite polynomials.

For the HO as shown, x=0 is just an ordinary point. We have p(x) = -2x which is analytic at z = 0, and we have q(x) = λ which is analytic as well. They claim irregular at x=∞ . [See WW test.]

As outlined below, at an ordinary point I expect the indicial exponents to be r = 0 and r = 1. Is this correct? Well here is what http://mathworld.wolfram.com/HermiteDifferentialEquation.html has to say:

Hermite Differential Equation

The second-order ordinary differential equation

$$\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+\lambda y=0.$$

\

The linearly independent solutions are then

$$y_1 = a_0 \left[1 - \frac{\lambda}{2!}x^2 - \frac{(4-\lambda)\lambda}{4!}x^4 - \frac{(8-\lambda)(4-\lambda)\lambda}{6!}x^6 - \ldots\right] \quad (7)$$

$$y_2 = a_1 \left[x + \frac{(2-\lambda)}{3!}x^3 + \frac{(6-\lambda)(2-\lambda)}{5!}x^5 + \ldots\right]. \quad (8)$$

So there you are! The y_1 solution is r = 0 (and odd powers are missing), the y_1 solution has r = 1 since you could factor out an x. You can see that if λ = even integer, both types of solutions will truncate. In this case, y_1 and y_2 are providing the even and odd Hermite Polynomials.

Hypergeometric equation

Is defined as

$$z(1-z)\frac{d^2f}{dz^2} + [c-(a+b+1)z]\frac{df}{dz} - abf = 0.$$

Dividing both sides by $z\,(1 - z)$ gives:

$$\frac{d^2f}{dz^2} + \frac{c-(a+b+1)z}{z(1-z)}\frac{df}{dz} - \frac{ab}{z(1-z)}f = 0.$$

This differential equation has regular singular points at 0, 1 and ∞. The solutions is the hypergeometric series.

You see that z=0 and z=1 have single poles so we are regular singular at both these points. I guess if you try the Frobenius method here, your indicial equation will give r=0 as one solution, which would then apply for the usual HG series.

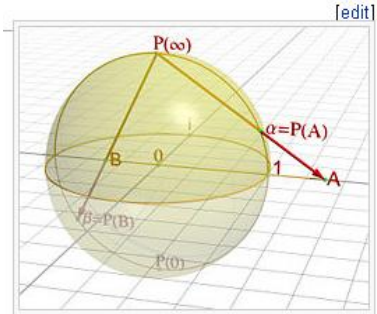
Appendix A.

1. The Riemann sphere is a way of extending the plane of complex numbers with one additional point at infinity, in a way that makes expressions such as

$1 / 0 = \infty$

well-behaved and useful, at least in certain contexts. It is named after 19th century mathematician Bernhard Riemann. In the picture below, the x-y plane is the complex plane C, and point A on it maps to a point α on the sphere. The entire boundary of C at ∞ them maps to the north pole.

[edit]



Stereographic projection of a complex number A onto a point α of the Riemann sphere.

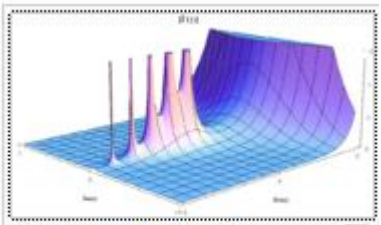
2. A holomorphic function is one (in a region D of the complex plane) which is fully "complex-differentiable" at all points in the region, meaning you can differentiate as many times as you want, which in turn implies that you can make a Taylor series at each point.

"Holomorphic functions are sometimes called **regular functions**.[1] A function that is holomorphic on the whole complex plane is called an **entire function**. The phrase **"holomorphic at a point z"** means not just differentiable at z, but differentiable everywhere within some open disk centered at z in the complex plane."

"The term **analytic function** is often used interchangeably with holomorphic function, although the term analytic is *also* used in a broader sense of any function (real, complex, or of more general type) that is equal to its Taylor series in a neighborhood of each point in its domain. The fact that the class of analytic functions coincides with the class of holomorphic functions is a major theorem in complex analysis."

I have always used the term analytic. Their point is that sometimes this just means Taylor-series-able at every point, while holomorphic seems to be a slightly stronger term meaning infinitely differentiable and I think then that holomorphic would imply analytic when this sense of "analytic" is used. So we might say that holomorphic => analytic. With the other meaning of analytic, you have holomorphic = analytic.

3. A meromorphic function is one that is holomorphic except at points where it has simple poles. An example is the gamma function



The Gamma function is meromorphic in the whole complex plane

Thus, a function that has branch points is not meromorphic in the entire complex plane.

4. Mobius transformation. See example below, just a transform from z to w such that $z=\infty$ is mapped into some w = b where you can think about it. For example, $z = 1/(w-b)$.

5. Frobenius Method. [see clarification below] This is the name of the method that Schiff used many times in his book. You try to find a power series solution of the form $f(x) = x^r \Sigma_{0,1,2} a_n x^n$. When you plug this into an ODE, you get an **indicial equation** for r which you solve for the values of r that make such a solution work. Here is what the math looks like in a nutshell:

The Frobenius method tells us that we can seek a power series solution of the form

$$u(z) = \sum_{k=0}^{\infty} A_k z^{k+r}, (A_0 \neq 0)$$

Differentiating:

$$u'(z) = \sum_{k=0}^{\infty} (k+r) A_k z^{k+r-1}$$

$$u''(z) = \sum_{k=0}^{\infty} (k+r-1)(k+r) A_k z^{k+r-2}$$

Substituting:

$$z^2 \sum_{k=0}^{\infty} (k+r-1)(k+r) A_k z^{k+r-2} + z p(z) \sum_{k=0}^{\infty} (k+r) A_k z^{k+r-1} + q(z) \sum_{k=0}^{\infty} A_k z^{k+r}$$

$$= \sum_{k=0}^{\infty} (k+r-1)(k+r) A_k z^{k+r} + p(z) \sum_{k=0}^{\infty} (k+r) A_k z^{k+r} + q(z) \sum_{k=0}^{\infty} A_k z^{k+r}$$

$$= \sum_{k=0}^{\infty} (k+r-1)(k+r) A_k z^{k+r} + p(z)(k+r) A_k z^{k+r} + q(z) A_k z^{k+r}$$

$$= \sum_{k=0}^{\infty} ((k+r-1)(k+r) + p(z)(k+r) + q(z)) A_k z^{k+r}$$

$$= (r(r-1)+p(z)r+q(z)) A_0 z^r + \sum_{k=1}^{\infty} ((k+r-1)(k+r)+p(z)(k+r)+q(z)) A_k z^{k+r}$$

The expression $r(r-1) + p(0) r + q(0) = I(r)$ is known as the *indicial polynomial*, which is quadratic in *r*.

where he has separated out the A_0 term as shown and we know $A_0 \neq 0$. Since we are near $z \approx 0$, we set z=0 in the coefficient functions like p(z). If there are two different r values, there are normally two different solutions to the ODE, which, by the way, is this for the above work

In mathematics, the **Frobenius method** describes a way to find an infinite series solution for a second-order ordinary differential equation of the form

$$z^2 u'' + p(z) z u' + q(z) u = 0$$

We can divide by z^2 to obtain a differential equation of the form

$$u'' + \frac{p(z)}{z} u' + \frac{q(z)}{z^2} u = 0$$

which will not be solvable with regular power series methods if either $p(z)/z$ or $q(z)/z^2$ are not analytic at $z = 0$. The Frobenius method enables us to create a power series solution to such a differential equation, provided that $p(z)$ and $q(z)$ are themselves analytic at 0 or, being analytic elsewhere, both their limits at 0 exist (and are noninfinite).

You can see there are conditions of p(z) and q(z) which make this possible or impossible.

Clarification of the Frobenius Method from WW. (4/18/09).

(1) The assumed starting form for the ODE is this:

$$[z^2D^2 + z\,P(z)\,D + Q(z)\,]\,u(z) = 0 \qquad\qquad u(z) = \Sigma_{k=0}^{\infty}\ A_k\, z^{r+k}$$

First of all, this is a homo equation Lu=0. Second, different people use different starting forms for different purposes, and the above form is right for the Frobenius purpose. Our wiki author includes a coefficient $A_0 \neq 0$ in the series, but W&W just set it to $A_0 = 1$ since we can always scale the solution in that manner. I will retain A_0 since it makes things look more regular below.

(2) Whittaker and Watson say the right words on page 191-192 that the above author does not provide. First of all, in the above we need to assume that z=0 is a "clean point" so that P(z) and Q(z) are analytic at z = 0 and may therefore be expanded in power series at z = 0. We then install the assumed form of the solution as shown above. However, we then need to expand both Q(z) and P(z) as power series with coefficients Q_i and P_i. When this is done, if you start with the lowest power and work upward, you get a list of equations. Here, according to W&W page 192, are the first several, where I translate into the notation above: ($\alpha = r$, $c=0$, $a_i = A_i$) [I write A_0 but will treat it as $A_o = 1$ as do W&W]

0 z^r: $A_0[\, r^2 + (P_0-1)r + Q_0\,] = 0$ // or $r(r-1) + P_0 r + Q_0 = 0$

1 z^{r+1}: $A_1[\, (r+1)^2 + (P_0-1)(r+1) + Q_0] + A_0\ [\, rP_1 + Q_1] = 0$

2 z^{r+2}: $A_2[\, (r+2)^2 + (P_0-1)(r+2) + Q_0] + A_1\,[\, (r+1)P_1 + Q_1]\ + A_0\ [\, rP_2 + Q_2] = 0$

....

Each of the above expressions is the coefficient of a power of z, and I have labeled each equation by a number which is (the exponent of z) less r. The coefficient in equation 0 is the indicial equation, and this equation MUST be satisfied or we are dead meat with our solution, so this determines two possible values for r. The coefficient of equation 1 involves only A_1 so we can solve that equation for A_1. Then knowing A_1 we can solve equation 2 for A_2 , and on and on. The n^{th} equation will involve A_n, A_{n-1}....A_2, A_1

(3) The conclusion is this: if we ignore issues like series convergence, we have in general two complete formal solutions to our ODE of the quite general form shown above. There are always two solutions, each corresponding to one of the values of r, which we can call r_1 and r_2 . Obviously the A_i are different for the two different solutions, call them eg $A_i^{(1)}$. We have assumed that P(z) and Q(z) are analytic at z = 0. As we shall see later, saying that P and Q are analytic at z=0 is saying that either z = 0 is an **ordinary** point (P has a zero and Q has double zero at z=0) **or** z= 0 is a **regular singular** point (P and Q analytic).

(4) The entire analysis above could be repeated replacing $z \rightarrow (z-c)$ as done by W&W. In that case, we would be assuming that P and Q were analytic at z = c and we would expand them around that point, and our solution would be around that point.

(5) What happens if $r_1 = r_2 \equiv r$? The two series solutions are then identical, so there must be some other independent solution. This is given in equation p 195 A where notice the h sum starts at n=1. In this case we have

$$w_1(z) = z^r [\, 1 + \Sigma_{n=1}^{\infty} A_n z^n]$$

$$w_2(z) = z^r \; [\Sigma_{n=1}^{\infty} h_n z^n \,] \; + w_1(z) \ln(z)$$

where we are not given a rule for computing the h_n, we just know they exist. You must include the series part of the second solution for w_2 to actually solve the ODE; the term $w_1(z) \ln(z)$ alone, although independent from $w_1(z)$, would not be a solution of the ODE.

(6) What happens if $r_1 = r_2 + s$ with s = positive integer ? In this case the solutions are p 195 B:

$$w_1(z) = z^{r1} [\, 1 + \Sigma_{n=1}^{\infty} A_n z^n]$$

$$w_2(z) = z^{r2} \; [- 1/s + \Sigma_{n=1}^{\infty} h_n z^n \,] \; + g_s \, w_1(z) \ln(z)$$

where r_2 is the "smaller" exponent. The constant g_s depends in a complicated way on both P and Q and so is really a function of the ODE itself. It is the coefficient of the power z^s in a function g(z) shown on page 194 which you see is a messy combination of an exponentiated integral of P(z) and $w_1(z)$.

The main point about g_s is that for some ODE's it *might vanish*. If that happens, there is no ln(z) term, and the second solution is just the power series part. I would guess that in this case, the h_n would be obtained from the ladder of equations just as the A_n are, as shown above, just a different value of r. Probably in the $g_s \neq 0$ case, the ladder causes some $A_n^{(2)} = \infty$ and that is why we have the ln(z) thing.

Here is an "alternate form" for the w_2 solution where we just multiply through by constant -s,

$$w_2(z) = z^{r2} \; [\, 1 + \Sigma_{n=1}^{\infty} (-s h_n) z^n \,] \; + (-s g_s) \, w_1(z) \ln(z)$$

The nice feature here is that the n=0 term in the series part is 1, just as in the $w_1(z)$ series. And of course the issue of $g_s = 0$ is unaltered.

Notes and clips from Special Functions By Z. X. Wang, D. R. Guo

1. What does Fuchsian Mean?

The following is from google book Special Functions By Z. X. Wang, D. R. Guo

2.7. Equations of Fuchsian Type

An equation with all its singular points being regular is called an equation of Fuchsian type. The most important special case of this type of equations is that with three regular singularities, of which the prototype is the hypergeometric equation [see Sec. 2.9 and Chap. 4]; the commonly occurred Legendre equation belongs to this category.

The first two chapters of this book are pedagogical , remaining chapters are specific to special functions. It is these "teaching chapters" that are missing from the Bateman and A&S books!

2. Index for first 2 chapters of Wang Guo

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3. What is the *significance* of a point being ordinary, regular singular, or irregular singular ?

In the wiki notes above, we saw the definition of the three kinds of points, and examples, but nothing was said about why we should care. So I have scanned to find some of the basic facts:

Meromorphic function is a single-valued function which has, in a finite domain, no singularities other than poles.

2.2. Solution of the Equation in the Vicinity of an Ordinary Point

Theorem: If $p(z)$ and $q(z)$ are single-valued and analytic in the circular region $|z - z_0| < R$, the equation

$$w'' + pw' + qw = 0 \qquad (1)$$

has a unique solution inside this circle satisfying the initial conditions

$$w(z_0) = c_0, \quad w'(z_0) = c_1 , \qquad (2)$$

where c_0 and c_1 are arbitrary constants, and $w(z)$ is single-valued and analytic in this circle.

[The above is proved at the start of W&W Chapter 10]

So this is my idea that away from any singular points of any kind, a solution will be smooth. If you apply some BC's as shown, this solution is unique.

2.3. Solutions of the Equation in the Vicinity of a Singular Point

According to the theorem in the preceding section, an ordinary point of the equation must also be an ordinary point of the solution, i.e., at such point and its vicinity, the solution is single-valued and analytic. But a singular point [Sec. 2.1] of the equation *may* yet be a singular point of the solution. Thus, if we still attempt to find the solution in the form of a power series, we have to employ the Laurent expansion. Before doing so, it is necessary to know whether this singular point is a branch point of the solution. We treat this problem by the method of analytic continuation.

The above results can be summarized in a theorem:

If z_0 is a singular point of equation $w''+pw'+qw=0$, then the two linearly independent solutions of the equation in the neighborhood $0<|z-z_0|<R$ of z_0 (where R is sufficiently small so that there is no singular point in the annular region) are

$$w_1(z)=(z-z_0)^{\rho_1}\sum_{n=-\infty}^{\infty}c_n(z-z_0)^n\ , \tag{15}$$

$$w_2(z)=(z-z_0)^{\rho_2}\sum_{n=-\infty}^{\infty}d_n(z-z_0)^n\ , \tag{16}$$

$$(\rho_1-\rho_2\neq \text{integer})$$

or,

$$w_2(z)=gw_1(z)\ln(z-z_0)+(z-z_0)^{\rho_1}\sum_{n=-\infty}^{\infty}d_n(z-z_0)^n$$

$$(\rho_1-\rho_2=\ \text{integer})\ . \tag{17}$$

Comment: (1) Notice that all sums are -∞ to +∞ in the above results. WW above gave us more specific results. (2) For infinite sums, it does not matter which exponent you put as the power in (17), since they differ by a shift of the series. (3) The special case that g = 0 is not mentioned, but g is sitting there.

Comment on Significance of ordinary, regular singular, and irregular singular.

The classification into these three bins depends on the nature of the P(z) and Q(z) functions. The indicial polynomial says $[\ r^2 + (P_0\text{-}1)r + Q_0] = 0$. Whether the two roots are the same or not depends on the constant value in the series for P and Q. If z = 0 is an "ordinary point", we know that P has a zero and Q has a double zero at z = 0. This means that $P_0 = 0$ and $Q_0 = Q_1 = 0$. In this case, our indicial equation becomes simply $r^2 - r = 0 = r(r\text{-}1)$, so the roots are r=0 and r=1. I suspect that in this case, $g_1 = 0$. The two solutions are then just power series

$w_1(z) = z^1\ [\ 1 + \Sigma_{n=1}^{\infty}\ A^{(1)}{}_n z^n]$

$w_2(z) = z^0\ [\ 1 + \Sigma_{n=1}^{\infty}\ A^{(2)}{}_n z^n]$

Each solution is analytic in a disc about z = 0. So that is the main idea of an "ordinary point".

If on the other hand z = 0 is a regular singular point, then our various forms of "the theorem" apply, and we can have different cases of r_1 and r_2 and we get some cases which have a ln(z) as part of the solution. In these cases, apart from the ln(z) branch point, things are smooth near z = 0.

If z = 0 is an irregular singular point, then somehow all bets are off. Our "theorem" does not apply, things are not analytic, we don't have a general form for the solution. That is my guess.

Note Added: I think there are exceptions to the above theorem (g=0)

One exception is for **spherical Bessel** functions. The "indices" for the spherical Bessel ODE (I think) are ℓ and $-\ell-1$,and they differ by an integer assuming ℓ = integer, BUT the second solution is of the first form shown above for w_2 and does not require the logarithm! [Well, this is the g=0 case.] I quote from Sadri Hassani p 428 Google book:

http://books.google.com/books?id=BCMLOp6DyFIC

14.27. Theorem 14.2.6 states that under certain conditions, linearly independent solutions of SOLDE at regular singular points exist even though the difference between the characteristic exponents is an integer. An example is the case of Bessel functions of half-odd-integer orders. Evaluate the Wronskian of the two linearly independent solutions, J_ν and $J_{-\nu}$, of the Bessel equation and show that it vanishes *only if* ν *is an integer*. This shows, in particular, that $J_{n+1/2}$ and $J_{-n-1/2}$ are linearly independent. Hint: Consider the value of the Wronskian at $z = 0$, and use the formula $\Gamma(\nu)\Gamma(1-\nu) = \pi/\sin\nu\pi$.

Here is the theorem that the above refers to from page 409 of the same book

14.2.6. Theorem. *Suppose that the characteristic exponents of a SOLDE with a regular singular point at* $z = 0$ *are* ν_1 *and* ν_2. *Consider three cases:*

1. $\nu_1 - \nu_2$ *is not an integer.*
2. $\nu_2 = \nu_1 - n$ *where* n *is a nonnegative integer, and* ρ_n, *as defined in Equation (14.10), vanishes.*
3. $\nu_2 = \nu_1 - n$ *where* n *is a nonnegative integer, and* ρ_n, *as defined in Equation (14.10), does not vanish.*

Then, in the first two cases, there exists a basis of solutions $\{w_1, w_2\}$ *of the form*

$$w_i(z) = z^{\nu_i}\left(1 + \sum_{k=1}^{\infty} C_k^{(i)} z^k\right), \qquad i = 1, 2,$$

and in the third case, the basis of solutions takes the form

$$w_1(z) = z^{\nu_1}\left(1 + \sum_{k=1}^{\infty} a_k z^k\right), \quad w_2(z) = z^{\nu_2}\left(1 + \sum_{k=1}^{\infty} b_k z^k\right) + C w_1(z) \ln z,$$

where the power series are convergent in a neighborhood of $z = 0$.

The spherical Bessel functions must be in Case 2 above, and that is why j and n are independent solutions without a log deal.

Comments : (1) What does Hassini then say for n=0? He seems to say it could be either case 2 or 3 depending on his ρ_n which is WW $-sg_s$. But we know from WW that it is in fact case 3 and the ln(z) term is present. (2) If n > 0, we know from WW that we can get case 2 or 3 and I agree with the forms written for case 3. His form suggests that in case 2, the coefficients are just obtained from ladder equations.

Here is another statement of the above theorem, with some other stuff.

www4.ncsu.edu/~lkn/MA401/handouts/SummaryOfFrobeniusMethod.pdf

One issue might be what **form** you assume for your original ODE. Notice that the form here differs a little from the WW form which is $z^2D^2 + zP(z)\,D + Q(z)$. If we divide through we get WW saying $D^2 + (P(z)/z)\,D + (Q(z)/z^2)$. Then $p(z) = (P(z)/z)$ and $q(z) = (Q(z)/z^2)$ below. I guess as long as you examine things near but away from z=0, this division makes no difference. Here is the first quote of this new author:

DEF: A point x_0 is an **ordinary point** of the ODE $y'' + p(x)y' + q(x)y = 0$ if the functions $p(x)$ and $q(x)$ are analytic at x_0. Otherwise x_0 is a **singular point** of the ODE.

So this means that at an **ordinary point**, both P(z)/z must be analytic, and $Q(z)/z^2$ must be analytic. This really means that in this case, P(z) must have a zero at z=0 and Q(z) must have a double zero there. A *less* stringent condition on P(z) and Q(z) would be that *they* be analytic at z=0, not both having the required zeros. In this case, the point z=0 is a **regular singular point**. In all other cases of P(z) and Q(z) at z=0, the point z=0 is an **irregular singular** point. (Again, P = zp and $Q = z^2q$)

DEF: A singular point x_0 of the ODE $y'' + p(x)y' + q(x)y = 0$ is a **regular singular point** of the ODE if the functions $xp(x)$ and $x^2q(x)$ are analytic at x_0. Otherwise x_0 is an **irregular singular point** of the ODE.

Now here then is this author's offering on the nature of the two independent Frobenius solutions. He is allowing that z=0 is a regular singular point, but I think his theorem would also apply to the softer situation where z = 0 was an ordinary point. His basis is then that the WW functions P and Q are merely analytic at z = 0. He takes r_1 to be the larger root.

Theorem: Let $x_0 = 0$ be a regular singular point of the ODE $y'' + p(x)y' + q(x)y = 0$, and let r_1 and r_2 be the two roots of the indicial equation, with $r_1 \geq r_2$ when both are real. Then the ODE has two linearly independent solutions y_1 and y_2, where:

Case I: If $\boxed{r_1 - r_2 \neq \text{an integer}}$ then with $x > 0$

$$y_1 = x^{r_1}\sum_{n=0}^{\infty} a_n x^n,\ a_0 \neq 0 \quad \text{and} \quad y_2 = x^{r_2}\sum_{n=0}^{\infty} b_n x^n,\ b_0 \neq 0$$

Case II: If $\boxed{r = r_1 = r_2}$ then with $x > 0$

$$y_1 = x^{r}\sum_{n=0}^{\infty} a_n x^n,\ a_0 \neq 0 \quad \text{and} \quad y_2 = y_1(x)\ln(x) + x^{r}\sum_{n=1}^{\infty} b_n x^n$$

Case III: If $\boxed{r_1 - r_2 = \text{a positive integer}}$ then with $x > 0$

$$y_1 = x^{r_1}\sum_{n=0}^{\infty} a_n x^n,\ a_0 \neq 0 \quad \text{and} \quad y_2 = k y_1(x)\ln(x) + x^{r_2}\sum_{n=0}^{\infty} b_n x^n,\ b_0 \neq 0$$

This author breaks out the case $r_1 = r_2$ and the sum there starts at n=1 which agrees with WW. In the third case where the difference is a positive integer, then the sum starts at n = 0 which also agrees with WW. We could take b_0 = -1/s or 1 as we like, just scaling the whole result.

Laurent series

From Wikipedia, the free encyclopedia

In mathematics, the **Laurent series** of a complex function $f(z)$ is a representation of that function as a power series which includes terms of negative degree. It may be used to express complex functions in cases where a Taylor series expansion cannot be applied. The Laurent series was named after and first published by Pierre Alphonse Laurent in 1843. Karl Weierstrass may have discovered it first in 1841 but did not publish it at the time.

The Laurent series for a complex function $f(z)$ about a point c is given by:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-c)^n$$

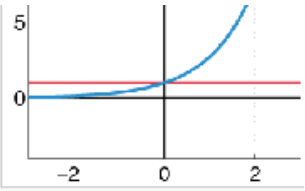
where the a_n are constants, defined by a line integral which is a generalization of Cauchy's integral formula:

$$a_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)\,dz}{(z-c)^{n+1}}.$$

Taylor Series, Maclaurin Series, definitions:

Definition [edit]

The Taylor series of a real or complex function $f(x)$ that is infinitely differentiable in a neighborhood of a real or complex number a, is the power series



The exponential function (in blue), and the sum of the first n+1 terms of its Taylor series at 0 (in red).

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots,$$

which in a more compact form can be written as

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

where $n!$ is the factorial of n and $f^{(n)}(a)$ denotes the nth derivative of f evaluated at the point a; the zeroth derivative of f is defined to be f itself and $(x-a)^0$ and 0! are both defined to be 1.

In the particular case where $a = 0$, the series is also called a **Maclaurin series**.

Regular solutions of an ODE.

Here is a discussion of the notion of a "regular solution" to an ODE.

2.4. Regular Solution. Regular Singularities

When the coefficients $p(z)$ and $q(z)$ of the equation

$$w'' + pw' + qw = 0 \qquad (1)$$

satisfy certain conditions, the Laurent series in (15), (16) or (17) of the last section happen to contain only a finite number of negative power terms. By extracting the lowest negative power factor and combining it in the factor $(z - z_0)^\rho$, we find

$$w_1(z) = (z - z_0)^{\rho_1} \sum_{n=0}^{\infty} c_n (z - z_0)^n \quad (c_0 \neq 0) , \qquad (2)$$

$$w_2(z) = g w_1(z) \ln(z - z_0) + (z - z_0)^{\rho_2} \sum_{n=0}^{\infty} d_n (z - z_0)^n \quad (d_n \neq 0) , \qquad (3)$$

where (3) includes the cases (16) and (17) of the last section. Solutions (2) and (3) are called *regular solutions*; they are characterized by the fact that the series in them are Taylor series. Substituting such solutions in (1), we obtain a sequence of recurrence relations [see (16) below] by which the undetermined coefficients in the series can be found one by one. The following is a discussion of the necessary and sufficient conditions for (1) to have regular solutions.

Theorem: The necessary and sufficient conditions for (1) to have two regular solutions in the vicinity $0 < |z - z_0| < R$ of its singular point z_0 are that

$$(z - z_0)p(z) \quad \text{and} \quad (z - z_0)^2 q(z) \quad \text{are analytic in} \quad |z - z_0| < R . \qquad (4)$$

A singular point satisfying conditions (4) is called a *regular singular point*, otherwise, *irregular singular point.*

They go on to prove this. So there is the answer. If a singular point z_o is "regular", then your ODE will have two solutions in a disk around this singularity. So I suspect that an irregular singular point "may not" have two solutions in such a disk. [except the $\ln(z\text{-}z_0)$ does have a well defined branch point at $z = z_0$]

Confluent defined (just in passing)

Now, let us give an explanation of what is meant by the confluent form of a differential equation. A differential equation derivable from another differential equation by making two or more singular points of the latter equation to coincide is called the confluent form of the latter equation. After confluence, the nature of the singularities is in general altered.

Euler Transform

This is an alternative to the Laplace Method, and allows the first coefficient function to be a quadratic. Laplace requires them all to be linear.

2.14. Euler Transform

The Euler transform

$$w(z) = \int_C (z-t)^{\mu} v(t)\,dt \quad (\mu \text{ to be determined}) \tag{1}$$

is suitable for solving equations of the following form:

$$L[w] \equiv p_0(z)w'' + p_1(z)w' + p_2(z)w(z) = 0\ , \tag{2}$$

where

$$p_0(z) = a_0 z^2 + b_0 z + c_0\ , \quad p_1(z) = b_1 z + c_1\ , \quad p_2(z) = c_2\ . \tag{3}$$

This is an equation of Fuchsian type having three regular singularities, including the point at infinity.

So there are a few comments from these two energetic guys.

I just found that Whittaker and Watson is available new for $23 on Amazon. // And I just ordered it along with the Iliad at 3.50 for total 26.00 to make free shipping. Finally I will have this book which I have been reading about for many years.

What is the significance of a Wronskian and Abel's Theorem?

Look at Stakgold I page 59. If a set of n functions are linearly dependent, then 1.44 is true for some c_i that are not all zero. This means you can express one of the functions as a lincomb of some others. Now, differentiate this equation to get a second equation that looks the same, then keep doing that. You get

$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$ // 1.44
$c_1 f_1' + c_2 f_2' + c_3 f_3' = 0$
$c_1 f_1'' + c_2 f_2'' + c_3 f_3'' = 0$

Write this in matrix form as

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 \end{pmatrix}\begin{pmatrix} f_1 & f_1' & f_1'' \\ f_2 & f_2' & f_2'' \\ f_3 & f_3' & f_3'' \end{pmatrix} = 0 \qquad \text{or} \qquad CW = 0$$

If the Wronskian matrix shown here is invertible (at some position x), we can compute W^{-1} and conclude that $C = 0\ W^{-1} = 0$, which says that the three c_i are zero, which contradicts our assumption that you could write the first equation above for some c_i not all zero. Thus we conclude that

f_i are linearly dependent => can write matrix equation for some $c_i \neq 0$
=> if detW ≠ 0 then C = 0 = contradiction, so must have detW = 0

So we find

f_i are linearly dependent => detW = 0 for all x in your "interval"	theorem
detW ≠ 0 => f_i are linearly independent	contrapositive

Stakgold notes the converse is not necessarily true and gives the example $f_1 = x^2$ and $f_2 = x|x|$ on -1,1. These are linearly independent, but W = 0. Here are some web comments:

The Wronskian can be used to determine whether a set of differentiable functions is linearly independent on a given interval:

- If the Wronskian is non-zero at some point in an interval, then the associated functions are *linearly independent* on the interval.

This is useful in many situations. For example, if we wish to verify that two solutions of a second-order differential equation are independent, we may use the Wronskian. Note that if the Wronskian is zero everywhere in the interval, the functions may or may not be linearly independent. A common misconception is that $W = 0$ everywhere implies linear dependence; the third example below shows that this is not true.

Now here is an interesting theorem

Abel's Theorem

If $y_1(t)$ and $y_2(t)$ are two solutions to

$$y'' + p(t)y' + q(t)y = 0$$

then the Wronskian of the two solutions is

$$W(y_1, y_2)(t) = W(y_1, y_2)(t_0)\, e^{-\int_{t_0}^{t} p(x)dx}$$

for some t_0.

Because we don't know the Wronskian and we don't know t_0 this won't do us a lot of good apparently. However, we can rewrite this as

$$W(y_1, y_2)(t) = c\, e^{-\int p(t)dt} \qquad (3)$$

where the original Wronskian sitting in front of the exponential is absorbed into the c and the evaluation of the integral at t_0 will put a constant in the exponential that can also be brought out and absorbed into the constant c. If you don't recall how

If you have an ODE with no linear term, then this says the Wronskian will be a constant ! The Coulomb Wave Function ODE of AS page 538 is a good example, and 14.2.4 then shows W = constant.