ODEs with a k^2 type parameter. PhL 10.11.10

The purpose of this document is to investigate the implications of having a parameter appear in an ODE in a squared form only. The classic case is $f''(x) = k^2 f(x)$ which has solutions $[e^{kx}, e^{-kx}]$ or [sh(kx), ch(kx)]. In the first pair we find that f(x; -k) gives "the other solution" and is independent of f(x; k), while for the second pair we find that f(x; -k) gives a multiple of f(x; k). Another issue is the possibility of having solutions which are even or odd in x or in k. Other cases of quadratic parameters are the m^2 in the associated Legendre ODE as seen in $P_n^m(z)$, the n^2 of the Bessel equation as in $J_n(x)$, and the a^2 and b^2 which appear in the Lamé equation, this last case being the motivation for this digression.

In Part A we show for $f''(x) = k^2 f(x)$ that the indicial equation gives r = 0 and r = 1 (as for any ordinary point) with corresponding Frobenius solutions ch(kz) and sh(kz)/k and we note that both solutions are even in k when written in this manner. In Part B we convince ourselves that you can always find two independent solutions of an ODE containing k^2 which are even in k. So these two solutions are solutions "of definite Frobenius index", whereas the $e^{\pm bx}$ solutions are not.

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A. The official Frobenius Method for $f''(z) = k^2 f(z)$

One of the main ODE tools we have is the Frobenius Method, so here I want to just follow it through for our prototype case. We try a solution of the form (where r and the a_n are to be determined)

$$f(z) = z^r \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n z^{n+r}$$

$$f'(z) = \sum_{n=0}^{\infty} (n+r)\, a_n z^{n+r-1}$$

$$f''(z) = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n z^{n+r-2}$$

Aside: We are going to insist that $a_0 \neq 0$, and will take it as 1. But suppose $a_0 = 0$ and $a_1 \neq 0$. Then we would write, using n' = n-1,

$$f(z) = \sum_{n=1}^{\infty} a_n z^{n+r} = \sum_{n'=0}^{\infty} a_{n'+1} z^{n'+r+1} = \sum_{n=0}^{\infty} a_{n+1} z^{n+r+1} = z^s \sum_{n=0}^{\infty} b_n z^n$$

where s= r+1 and $b_n = a_{n-1}$. So by letting $a_0 = 0$, we end up with the *exact same starting form* for our assumed power series solution (just new names), and we would then assume $b_0 \neq 0$. So the point is that we might as well assume $a_0 \neq 0$ "without loss of generality".

So, insert the above forms into the ODE to get

$$\Sigma_{n=0}^{\infty} (n+r)(n+r-1)\, a_n z^{n+r-2} = k^2 \Sigma_{n=0}^{\infty} a_n z^{n+r}$$

Multiply both sides by z^2 (this really takes the ODE to standard Frobenius form which we should have done at the start)

$$\Sigma_{n=0}^{\infty} (n+r)(n+r-1)\, a_n z^{n+r} = k^2 \Sigma_{n=0}^{\infty} a_n z^{n+r+2}$$

Divide both sides by z^r

$$\Sigma_{n=0}^{\infty} (n+r)(n+r-1)\, a_n z^{n} = k^2 \Sigma_{n=0}^{\infty} a_n z^{n+2}$$

Let n+2 = n' (then rename n' to be n) to write the second sum as

$$\{ \Sigma_{n=0}^{\infty} a_n z^{n+2}\} = \{ \Sigma_{n'=2}^{\infty} a_{n'-2} z^{n'-2+2}\} = \{ \Sigma_{n'=2}^{\infty} a_{n'-2} z^{n'}\} = \{ \Sigma_{n=2}^{\infty} a_{n-2} z^{n}\}$$

Break out the n=0 and n=1 terms in the first sum so we then have

$$(0+r)(0+r-1)\, a_0 + (1+r)(1+r-1)\, a_1 z + \Sigma_{n=2}^{\infty} (n+r)(n+r-1)\, a_n z^n - k^2\{ \Sigma_{n=2}^{\infty} a_{n-2} z^n\} = 0$$

Now combine the two sums together

$$r(r-1)a_0 + r(1+r)\, a_1 z + \Sigma_{n=2}^{\infty} [(n+r)(n+r-1)\, a_n - k^2\, a_{n-2}]\, z^n = 0$$

If we set z = 0 and require $a_0 \neq 0$, the first term is all that is left and we have r(r-1) = 0 which is the **indicial equation** telling us there are only two choices, r = 0 and r = 1. For such special values of r we have

$$r(1+r)\, a_1 z + \Sigma_{n=2}^{\infty} [(n+r)(n+r-1)\, a_n - k^2\, a_{n-2}]\, z^n = 0$$

Divide by z

$$r(1+r)\, a_1 + \Sigma_{n=2}^{\infty} [(n+r)(n+r-1)\, a_n - k^2\, a_{n-2}]\, z^{n-1} = 0$$

We must have the first term vanish AND the bracket [] in the second term must vanish for all n:

r = 0:	a_1 = any	$[n(n-1)\, a_n - k^2\, a_{n-2}] = 0$	n=2...
r = 1:	$a_1 = 0$	$[n(n+1)\, a_n - k^2\, a_{n-2}] = 0$	n=2...

For the r = 0 case we arbitrarily assume $a_1 = 0$ which kills off all odd a_n (we are certainly allowed to do that searching for a solution) and for the even n the recursion relation says

$2*1\ a_2 = k^2\ a_0 = k^2 \quad \Rightarrow \quad a_2 = k^2/2!$

$4*3\ a_4 = k^2 a_2 = k^4/2! \quad \Rightarrow \quad a_4 = k^4/\ 4!$

$6*5 a_6 = k^2 a_4 = k^6/4! \quad \Rightarrow \quad a_6 = k^6/6!$

$\Rightarrow \quad a_n = k^n/n! \quad n = 2,4,.... \text{ and } a_0 = 1$

Our solution is then

$f(z) = \Sigma_{n=0}^{\infty} a_n z^n = \Sigma_{n=0,2,4} (kz)^n/n! = 1 + (kz)^2/2! + (kz)^4/4! = ch(kz)$ // r=0 sol

For the r = 1 case, the recursion relation says

$2*3\ a_2 = k^2\ a_0 = k^2 \quad \Rightarrow \quad a_2 = k^2/3!$

$4*5\ a_4 = k^2 a_2 = k^4/3! \quad \Rightarrow \quad a_4 = k^4/\ 5!$

$6*7 a_6 = k^2 a_4 = k^6/5! \quad \Rightarrow \quad a_6 = k^6/7!$

$\Rightarrow \quad a_n = k^n/(n+1)! \quad n = 2,4,.... \text{ and } a_0 = 1$

Our solution is then (remember r = 1 now)

$f(z) = \Sigma_{n=0}^{\infty} a_n z^{n+1}$

Let n' = n+1 and write this as

$f(z) = \Sigma_{n'=1}^{\infty} a_{n'-1} z^{n'-1+1} = \Sigma_{n'=1}^{\infty} a_{n'-1} z^{n'} = \Sigma_{n=1}^{\infty} a_{n-1} z^n$

where from above we then have $a_{n-1} = k^{n-1}\ /n!$ so we get

$f(z) = \Sigma_{n=1}^{\infty} (k^{n-1}\ /n!) z^n = (1/k)\ \Sigma_{n=1}^{\infty} (kz)\ /n! = (1/k)\ [\ kz\ + (kz)^3/3! + ...] = (1/k)\ sh(kz)$

So with the assumption that $a_0 = 1$ we find these two Frobenius solutions to our ODE:

r = 0 : $f_0(z) = ch(kz)$
r = 1 : $f_1(z) = (1/k)\ sh(kz)$

In both solutions, we get only even a_n coefficients so $\Sigma_{n=0}^{\infty} a_n z^n$ is even. Our first solution then is even, and our second solution is z times an even function, which is of course an odd function. We could think of this as f(z) = z [(1/(kz) sh(z)] where the bracket is even. [We are talking even and off in z here.]

Note that both solutions as shown are even in k. So for either of these solutions we have

$$f_i(z; -k) = f_i(z; k)$$

So in this example, we might note that the solutions [e^{kx}, e^{-kx}] are *not* Frobenius solutions, but are linear combinations of same.

B. Question: Is it always possible, for an ODE which contains only k^2, to select a pair of independent solutions which are both even in k?

My tentative answer to this question is "yes", but I don't have a bulletproof proof.

Short answer: (added after the long answer was written)

Let $f_i(x; k)$ be two independent solutions. Then construct:

$$F_1(x; k) = f_1(x; k) + f_1(x; -k)$$

Whether or not $f_1(x; -k)$ is independent of $f_1(x; k)$, this is an even solution. If $f_1(x; -k) = - f_1(x; k)$, this solution would vanish. In this case, construct this alternative solution

$$F_{1alt}(x; k) = k\, f_1(x; k)$$

which is again even in k. So here is our suggestion:

$F_1(x; k) = k\, f_1(x; k)$ if $f_1(x; k)$ is odd in k
$= f_1(x; k) + f_1(x; -k)$ otherwise

$F_2(x; k) = k\, f_2(x; k)$ if $f_1(x; k)$ is odd in k
$= f_2(x; k) + f_2(x; -k)$ otherwise

How do we know that F_1 and F_2 so selected will be independent? They might be the same, as in the case where $f_1 = e^{kx}$ and $f_2 = e^{-kx}$. In this case, use

$$F_2(x; k) = k\, [f_2(x; k) - f_2(x; -k)]$$

as the second even function.

Long answer:

This seems to be a pretty simple question. For our case above we can pick ch(kz) and k sh(kz) which are both even in k. But showing this for a general ODE is not easy. We begin:

Suppose we have some solution $f_1(z; k)$ to an ODE in which only k^2 appears. We know that $f_1(z; -k)$ is therefore also a solution:

$$L(k^2)\, f_1(z; k) = 0 \quad \Rightarrow \quad L([-k]^2)\, f_1(z; -k) = 0 \quad \Rightarrow L(k^2)\, f_1(z; -k) = 0$$

Either $f_1(z; -k) = C(k)\, f_1(z;k)$ in which case $f_1(z; -k)$ is not an independent solution along with $f_1(z; k)$, or $f_1(z; -k)$ can be taken as an independent solution.

1. The case that $f_1(z; -k) = C(k)\, f_1(z;k)$

In the first case $f_1(z;k)$ is *almost* even in k to begin with. Now, if it is true that $f_1(z; -k) = C(k)\, f_1(z;k)$, then if we apply this twice we find that

$$C(k)\, C(-k) = 1$$

Side Question: If someone tells you that the real function f(x) has the property f(x)f(-x) = 1, what can you say about the solution f(x) ? I had to go off and write a whole doc to answer this question! The answer is this: pick any $f_{odd}(z)$ you like, then and most general solution is $f(z) = f_{odd}(z) + \sqrt{1 + [f_{odd}(z)]^2}$

Thus, C(k) cannot be "just any function", it must be one of this form

$$C(k) = C_{odd}(k) + \sqrt{1 + [C_{odd}(k)]^2}$$

where $C_{odd}(k)$ is an arbitrary odd function of k. Now, our goal is to define

$$F_1(z; k) \equiv A(k)\, f_1(z;k)$$

such that F_1 is even in k. This will then require that

$$A(k)\, f_1(z;k) = A(-k)\, f_1(z;-k) = A(-k)\, C(k)\, f_1(z;k)$$

We must then select an A(k) such that this is true:

$$A(k) = A(-k)\, C(k) \qquad \text{or} \qquad A(k)/A(-k) = C(k)$$

In order to find a suitable A(k), decompose it into even and odd parts, so

$$[A_{even}(k) + A_{odd}(k)] = [A_{even}(k) - A_{odd}(k)]C(k)$$

$$= [A_{even}(k) - A_{odd}(k)]\,[C_{odd}(k) + \sqrt{1 + [C_{odd}(k)]^2}\ ]$$

Thus we have this requirement:

$$[A_{even}(k) + A_{odd}(k)] = [A_{even}(k) - A_{odd}(k)]\,[C_{odd}(k) + \sqrt{1 + [C_{odd}(k)]^2}\ ]$$

$$= A_{even}(k)\,C_{odd}(k) - A_{odd}(k)\,C_{odd}(k) + A_{even}(k)\sqrt{1 + [C_{odd}(k)]^2} - A_{odd}(k)\sqrt{1 + [C_{odd}(k)]^2}$$

We can imagine decomposing each side of this equation into its even piece and its odd piece:

$$A_{even}(k) = -A_{odd}(k)\,C_{odd}(k) + A_{even}(k)\sqrt{1 + [C_{odd}(k)]^2} \qquad \text{// even}$$
$$A_{odd}(k) = A_{even}(k)\,C_{odd}(k) - A_{odd}(k)\sqrt{1 + [C_{odd}(k)]^2} \qquad \text{// odd}$$

$$A_{even}(k)\,[\,1 - \sqrt{1 + [C_{odd}(k)]^2}\ ] = -A_{odd}(k)\,C_{odd}(k)$$
$$A_{odd}(k)\,[\,1 + \sqrt{1 + [C_{odd}(k)]^2}\ ] = A_{even}(k)\,C_{odd}(k) \qquad (*)$$

Insert the first into the second and you find

$$A_{odd}(k)\,[\,1 + \sqrt{1 + [C_{odd}(k)]^2}\ ] = \{A_{even}(k)\}\,C_{odd}(k)$$

$$= \{-A_{odd}(k)\,C_{odd}(k)/\,[\,1 - \sqrt{1 + [C_{odd}(k)]^2}\ ]\}\,C_{odd}(k)$$

$$= -A_{odd}(k)\,C^2_{odd}(k)/\,[\,1 - \sqrt{1 + [C_{odd}(k)]^2}\ ]\}$$

But this will be true for any A_{odd} we select, because cancelling it we get

$$[\,1 + \sqrt{1 + [C_{odd}(k)]^2}\ ] = -C^2_{odd}(k)/\,[\,1 - \sqrt{1 + [C_{odd}(k)]^2}\ ]$$

$$[\,1 + \sqrt{1 + [C_{odd}(k)]^2}\ ]\,[\,1 - \sqrt{1 + [C_{odd}(k)]^2}\ ] = -C^2_{odd}(k)$$

$$1 - \{\,1 + [C_{odd}(k)]^2\} = -C^2_{odd}(k)$$

which is true. This seems to say that any old $A_{odd}(k)$ is OK. So imagine that we select some $A_{odd}(k)$. Then we need to have (from (*))

$$A_{even}(k) = A_{odd}(k)\,[\,1 + \sqrt{1 + [C_{odd}(k)]^2}\ ]/\,C_{odd}(k)$$

The simplest $A_{odd}(k)$ I can think of is $A_{odd}(k) = k$, so we have

$$A_{even}(k) = k\,[\,1 + \sqrt{1 + [C_{odd}(k)]^2}\ ]/\,C_{odd}(k)$$

Then we have

$$A(k) = k\{ 1 + [1 + \sqrt{1 + [C_{odd}(k)]^2}]/ C_{odd}(k) \}$$

$$A(-k) = k\{ -1 + [1 + \sqrt{1 + [C_{odd}(k)]^2}]/ C_{odd}(k) \}$$

$$A(k)/A(-k) = \{ 1 + [1 + \sqrt{1 + [C_{odd}(k)]^2}]/ C_{odd}(k) \} / \{ -1 + [1 + \sqrt{1 + [C_{odd}(k)]^2}]/ C_{odd}(k)\}$$

$$= \{ C_{odd}(k) + [1 + \sqrt{1 + [C_{odd}(k)]^2}] \} / \{ - C_{odd}(k) + [1 + \sqrt{1 + [C_{odd}(k)]^2}]\}$$

$$= \{ 1 + C_{odd}(k) + \sqrt{1 + [C_{odd}(k)]^2}] \} / \{ 1 - C_{odd}(k) + \sqrt{1 + [C_{odd}(k)]^2}]\}$$

We want to show this is equal to $C(k) = [C_{odd}(k) + \sqrt{1 + [C_{odd}(k)]^2}]$. So we want to show this is true":

$$[C_{odd}(k) + \sqrt{1 + [C_{odd}(k)]^2}]$$

$$= \{ 1 + C_{odd}(k) + \sqrt{1 + [C_{odd}(k)]^2}] \} / \{ 1 - C_{odd}(k) + \sqrt{1 + [C_{odd}(k)]^2}]\}$$

which means

$$[C_{odd}(k) + \sqrt{1 + [C_{odd}(k)]^2}]\{ 1 - C_{odd}(k) + \sqrt{1 + [C_{odd}(k)]^2}]\}$$

$$= \{ 1 + C_{odd}(k) + \sqrt{1 + [C_{odd}(k)]^2}] \}$$

The LHS we can multiply out to get

$$LHS = C_{odd}(k) - C_{odd}^2(k) + C_{odd}(k) \sqrt{1 + [C_{odd}(k)]^2}$$
$$+ \sqrt{1 + [C_{odd}(k)]^2} - \sqrt{1 + [C_{odd}(k)]^2} \; C_{odd}(k) + [1 + C_{odd}^2(k)]$$

$$= C_{odd}(k) + C_{odd}(k) \sqrt{1 + [C_{odd}(k)]^2}$$
$$+ \sqrt{1 + [C_{odd}(k)]^2} - C_{odd}(k)\sqrt{1 + [C_{odd}(k)]^2} + 1$$

$$= C_{odd}(k) + \sqrt{1 + [C_{odd}(k)]^2} + 1$$

and this is indeed the RHS. So I have verified my solution for A(k) shown above.

So going way back, we were considering the situation where $f_1(z; -k) = C(k)\, f_1(z;k)$. We have shown several things:

(1) C(k) cannot be arbitrary, but must have the form

$$C(k) = C_{odd}(k) + \sqrt{1 + [C_{odd}(k)]^2}$$

(2) The function

$$F_1(z; k) \equiv A(k)\, f_1(z;k)$$

will then be an even function in k, provided we make this selection for A(k) :

$$A(k) = k\{\, 1 + [\, 1 + \sqrt{1 + [C_{odd}(k)]^2}\,]/\, C_{odd}(k)\, \}$$

So all this mess was just to show that we can find multiple of f_1 which is even in k in this case.

2. The case that $f_1(z; -k)$ = independent of $f_1(z;k)$

In this case We can construct

$$F_2(z; k) = f_1(z; k) + f_1(z; -k)$$

as our second solution. We have ruled out $f_1(z; -k) = - f_1(z; k)$ because we have said $f_1(z; -k)$ is independent, so $F_2 \neq 0$ and is of course even in k.

3. Summary

We have an ODE in which only k^2 appears, and we have two independent solutions $f_i(z; k)$. We know that $f_i(z; -k)$ are both solutions of the ODE. We first work with f_1 :

If $f_1(z; -k)$ is *not* independent of $f_1(z; k)$, then we have $f_1(z; -k) = C(k)\, f_1(z;k)$ where C(k) is some multiple which must have a certain special form $C(k) = C_{odd}(k) + \sqrt{1 + [C_{odd}(k)]^2}$. In this case, we can construct an even solution function $F_{11}(z; k) \equiv A(k)\, f_1(z;k)$ where A(k) has the form shown above.

If $f_1(z; -k)$ *is* independent of $f_1(z; k)$, then we can construct an even solution function

$$F_{12}(z; k) = f_1(z; k) + f_1(z; -k)$$

Whichever case it is, we have obtained an even solution function. So

$$F_{11}(z; k) \equiv A(k)\, f_1(z;k)$$
$$F_{12}(z; k) = f_1(z; k) + f_1(z; -k)$$

Now we can repeat this entire whizbang for solution $f_2(z; k)$ and we will end up with an even solution function which is one of these

$$F_{21}(z; k) \equiv A(k)\, f_2(z;k)$$
$$F_{22}(z; k) = f_2(z; k) + f_2(z; -k)$$

If it happens that both solutions have independent $f_i(z; k)$ so we get F_{12} and F_{22} as our even functions, and if it happens that $F_{12} = F_{22}$, then we take as our second even function

$$F_{23} = k[f_2(z; k) - f_2(z; -k)]$$

4. Examples

Example 1: Suppose $f_1(z; k) = ch(kz)$. Then $C(k) = 1$. We then have $C_{odd}(k) = 0$. And we then have $A(k) = k\{ 1 + [1 + 1]/ C_{odd}(k) \} \approx 2 k / C_{odd}(k)$ before final limit. But $A(-k) = A(k)$ so we then have $A(k)/A(-k) = 1 = C(k)$. Our even solution is then $F_{11}(z; k) \equiv A(k) f_1(z;k) = [2 k / C_{odd}(k)]$ ch(kz) but we can just take it then to be ch(kz). [$C_{odd}(k) = [C(k)^2 - 1] / [2 C(k)] = (1-1)/2*1 = 0$]

Example 2: Suppose $f_1(z; k) = sh(kz)$ sp $C(k) = -1$. Then $C_{odd}(k) = 0$ again, but we take the other sign of the square root to get $C(k) = -1$. In this case we find that $A(k) = k\{ 1 + [1 - 1]/ C_{odd}(k) \} = k$ and then we have $A(k)/A(-k) = k / (-k) = -1 = C(k)$. Our even solution is then $F_{11}(z; k) \equiv A(k) f_1(z;k) = k$ sh(kz).

Example 3: Suppose $f_1(z; k) = e^{zk}$. Then $F_{12} = e^{kx} + e^{-kz} = 2$ ch(kz) as our even function.

Example 4: Suppose in Example 1, we also had $f_2(z; k) = e^{zk}$. By example 3, this would lead us to $F_{22} =$ 2 ch(kz) which is the same basically as F_{11}, so we use the escape hatch to get $F_{23} = k$ 2 sh(kz) .

So our method for generating two even-in-k independent solutions seems to work for our prototype case, but what about some more general case?

Example 5: How about associated Legendres? In the case that m ≠ integer, we have P_n^{-m} not a multiple of P_n^m and then our one solution would be just $P_n^{-m} + P_n^m$. Maybe this same thing with Q's would be the second even-in-m solution.

Example 6: In the case that m is an integer, we know that $P_n^{-m} = f(n,-m) P_n^m$ so $C(m) = f(n,-m)$. As expected, we then have $C(m)C(-m) = 1$. But what is A(m) in this case? First, what is $C_{odd}(m)$?

$$C(m) = f(n,-m) = \Gamma(n-m+1)/ \Gamma(n+m+1)$$

Then

$$C_{odd}(m) = [C(m)^2 - 1] / [2 C(m)] = [f^2(n,-m) - 1] / [2 f(n,-m)]$$

$$= f(n,-m)[f(n,-m) - f(n,m)] / [2 f(n,-m)] = [f(n,-m) - f(n,m)]/2 = \text{odd, yup}$$

$$A(m) = m\{ 1 + [1 + \sqrt{1 + [C_{odd}(m)]^2}]/ C_{odd}(m) \}$$

$$= m\{ 1 + [1 + \sqrt{1 + [[f(n,-m) - f(n,m)]/2]^2}]/ [f(n,-m) - f(n,m)]/2 \}$$

$$= m\{ 1 + [1 + \sqrt{1 + [[f(n,-m) - f(n,m)]/2]^2}]/ [f(n,-m) - f(n,m)]/2 \}$$

$$= m \{ 1 + \frac{1 + \sqrt{1 + [[f(n,-m) - f(n,m)]/2]^2}}{[f(n,-m) - f(n,m)]/2} \}$$

$$= m \{ 1 + \frac{1 + \sqrt{4 + [f(n,-m) - f(n,m)]^2}}{[f(n,-m) - f(n,m)]} \}$$

Notice that the second term in {} is odd in m and the first even, so A(m) has neither symmetry. But presumably this is our resulting even-in-m solution

$$p_n^m(z) = m \{ 1 + \frac{1 + \sqrt{4 + [f(n,-m) - f(n,m)]^2}}{[f(n,-m) - f(n,m)]} \} P_n^m(z)$$

I am too lazy to check it.