

## Chapter 17

### Dynamics of Planar Media

In this chapter, we continue our ascent of the dimensional ladder for linear systems. In Chapter 6, we embarked on our journey with equilibrium configurations of discrete systems — mass–spring chains, circuits, and structures — which are governed by certain linear algebraic systems. In Chapter 9, the dynamical behavior of such discrete systems was modeled by systems of linear ordinary differential equations. Chapter 11 began our treatment of continuous media with the boundary value problems that describe the equilibria of one-dimensional bars, strings and beams. Their dynamical motions formed the topic of Chapter 14, in the simplest case leading to two fundamental partial differential equations: the heat equation describing thermal diffusion, and the wave equation modeling vibrations. In Chapters 15 and 16, we focussed our attention on the boundary value problems describing equilibria of planar bodies — plates and membranes — with primary emphasis on solving the ubiquitous Laplace equation, both analytically or numerically. We now turn to the analysis of their dynamics, as governed by the two-dimensional<sup>†</sup> forms of the heat and wave equations. The heat equation describes diffusion of, say, heat energy, or population, or pollutants in a homogeneous two-dimensional domain. The wave equation models small vibrations of two-dimensional membranes such as a drum.

Although the increase in dimension does challenge our analytical prowess, we have, in fact, already mastered the key techniques: separation of variables and fundamental solutions. (Disappointingly, conformal mappings are not particularly helpful in the dynamical universe.) When applied to partial differential equation in higher dimensions, separation of variables often leads to new ordinary differential equations, whose solutions are no longer elementary functions. These so-called *special functions*, which include the Bessel functions appearing in the present chapter, and the Legendre functions, spherical harmonics, and spherical Bessel functions in three-dimensional problems, play a ubiquitous role in more advanced applications in physics, engineering and mathematics. Basic series solution techniques for ordinary differential equations, and key properties of the most important classes of special functions, can be found in Appendix C

In Appendix C, we collect together the required results about the most important classes of special functions, including a short presentation of the series approach for solving non-elementary ordinary differential equations.

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<sup>†</sup> Throughout, “dimension” refers to the number of space variables. In Newtonian dynamics, the time “dimension” is accorded a separate status, which distinguishes dynamics from equilibrium. Of course, in the more complicated relativistic universe, time and space must be regarded on an equal footing, and the dimension count modified accordingly.

Numerical methods for solving boundary value and initial value problems are, of course, essential in all but the simplest situations. The two basic methods — finite element and finite difference — have already appeared, and the only new aspect is the (substantial) complication of working in higher dimensions. Thus, in the interests of brevity, we defer the discussion of the numerical aspects of multi-dimensional partial differential equations to more advanced texts, e.g., [109], and student projects outlined in the exercises. However, the student should be assured that, without knowledge of the qualitative features based on direct analysis and explicit solutions, the design, implementation, and testing of numerical solution techniques would be severely hampered.

## 17.1. Diffusion in Planar Media.

As we learned in Chapter 15, the equilibrium temperature  $u(x, y)$  of a homogeneous plate is governed by the two-dimensional Laplace equation

$$\Delta u = u_{xx} + u_{yy} = 0.$$

In conformity with our general framework, the dynamical diffusion of such a plate will be modeled by the two-dimensional heat equation

$$u_t = \gamma \Delta u = \gamma (u_{xx} + u_{yy}), \quad (17.1)$$

where the diffusivity coefficient  $\gamma > 0$  measures the relative speed of diffusion of heat energy throughout the plate; its positivity is required on physical grounds, and also avoids ill-posedness of the dynamical system. In this simplest model of two-dimensional diffusion, we are assuming that there are no loss of heat or external heat sources in the plate's interior, which can be arranged by covering it with insulation.

The solution  $u(t, \mathbf{x}) = u(t, x, y)$  to (17.1) measures the temperature at time  $t$  at each point  $\mathbf{x} = (x, y) \in \Omega$  in the domain  $\Omega \subset \mathbb{R}^2$  occupied by the plate. To uniquely specify  $u(t, x, y)$  at each point  $(x, y) \in \Omega$  and each positive  $t > 0$ , we must impose both initial and boundary conditions. The most important are:

(a) *Dirichlet boundary conditions:*

$$u = h \quad \text{on} \quad \partial\Omega, \quad (17.2)$$

which fix the temperature on the boundary of the plate.

(b) *Neumann boundary conditions:*

$$\frac{\partial u}{\partial \mathbf{n}} = \frac{\partial u}{\partial \mathbf{n}} = k \quad \text{on} \quad \partial\Omega, \quad (17.3)$$

that prescribe the heat flux along the boundary; the case  $k = 0$  corresponds to an insulated boundary.

(c) *Mixed boundary conditions:* we impose Dirichlet conditions on part of the boundary  $D \subsetneq \partial\Omega$  and Neumann conditions on the remainder  $N = \partial\Omega \setminus D$ . For instance, the homogeneous mixed boundary conditions

$$u = 0 \quad \text{on} \quad D, \quad \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on} \quad N, \quad (17.4)$$

correspond to freezing part of the boundary and insulating the remainder. The initial conditions specify the temperature of the plate

$$u(0, x, y) = f(x, y), \quad (x, y) \in \Omega, \quad (17.5)$$

at an initial time, which for simplicity, we take as  $t_0 = 0$ . Under reasonable assumptions, e.g., the domain  $\Omega$  is bounded, its boundary is piecewise smooth, and the boundary data is piecewise continuous, say, a general theorem, [47], guarantees the existence of a unique solution  $u(t, x, y)$  to any of these initial-boundary value problems for all subsequent times  $t > 0$ . Our practical goal is to both compute and understand the behavior of the solution in specific situations.

### *Derivation of the Diffusion Equation*

This section is for those interested in understanding how the heat equation arises as a model for heat flow. The physical derivation of the two-dimensional (and three-dimensional) heat equation relies upon the same two basic thermodynamical laws that were used, in Section 14.1, to establish the one-dimensional version. The first principle is that heat energy flows from hot to cold as rapidly as possible. According to the multivariable calculus Theorem 19.38, the negative temperature gradient  $-\nabla u$  points in the direction of the steepest decrease in the function  $u$  at a point, and so heat energy will flow in that direction. Therefore, the heat flux vector  $\mathbf{w}$ , which measures the magnitude and direction of the flow of heat energy, should be proportional to the temperature gradient:

$$\mathbf{w}(t, x, y) = -\kappa(x, y) \nabla u. \quad (17.6)$$

The scalar quantity  $\kappa(x, y) > 0$  measures the *thermal conductivity* of the material, so (17.6) is the multi-dimensional form of *Fourier's Law of Cooling* (14.3). We are assuming that the thermal conductivity depends only on the position  $(x, y) \in \Omega$ , which means that the material in the plate

- (a) is not changing in time;
- (b) is *isotropic*, meaning that its conductivity is the same in all directions, and, further,
- (c) does not depend upon temperature.

Dropping either (b) or (c) would result in a much more complicated nonlinear diffusion equation.

The second thermodynamical principle is that, in the absence of external heat sources, heat can only enter any subregion  $D \subset \Omega$  through its boundary  $\partial D$ . (Keep in mind that the plate is insulated above and below.) In other words, the rate of change of the heat energy in  $D$  is prescribed by the heat flux across its boundary  $\partial D$ . Let  $\varepsilon(t, x, y)$  denote the heat energy density at each time and point in the domain, so that  $\iint_D \varepsilon(t, x, y) dx dy$  represents the total heat contained within the region  $D$  at time  $t$ . The amount of additional heat energy entering  $D$  at a boundary point  $\mathbf{x} \in \partial D$  is the inward normal component of the heat flux vector:  $-\mathbf{w} \cdot \mathbf{n}$ , where  $\mathbf{n}$  denotes the *outward* unit normal to  $\partial D$ . Thus, the total heat flux entering the region  $D$  is given by the flux line integral  $-\oint_{\partial D} \mathbf{w} \cdot \mathbf{n} ds$ ,

cf. (A.42). Equating the rate of change of heat energy to the heat flux yields

$$\frac{\partial}{\partial t} \iint_D \varepsilon(t, x, y) dx dy = - \oint_{\partial D} \mathbf{w} \cdot \mathbf{n} ds = - \iint_D \nabla \cdot \mathbf{w} dx dy,$$

where we applied the divergence form of Green's Theorem, (A.60), to convert the flux line integral into a double integral. We bring the time derivative inside the first integral and collect the terms, whence

$$\iint_D \left( \frac{\partial \varepsilon}{\partial t} + \nabla \cdot \mathbf{w} \right) dx dy = 0. \quad (17.7)$$

Keep in mind that this integral formula must hold for *any* subdomain  $D \subset \Omega$ . Now, the only way in which an integral of a continuous function can vanish for all subdomains is if the integrand is identically zero, cf. Exercise ■, and so

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot \mathbf{w} = 0. \quad (17.8)$$

In this way, we derive the basic *conservation law* relating heat energy  $\varepsilon$  and heat flux  $\mathbf{w}$ .

As in our one-dimensional model, (14.2), the heat energy  $\varepsilon(t, x, y)$  at each time and point in the domain is proportional to the temperature, so

$$\varepsilon(t, x, y) = \sigma(x, y) u(t, x, y), \quad \text{where} \quad \sigma(x, y) = \rho(x, y) \chi(x, y) \quad (17.9)$$

is the product of the *density* and the *heat capacity* of the material. Combining this with the Fourier Law (17.6) and the energy balance equation (17.9) leads to the general two-dimensional *diffusion equation*

$$\sigma \frac{\partial u}{\partial t} = \nabla \cdot (\kappa \nabla u) \quad (17.10)$$

governing the temperature dynamics of an isotropic medium in the absence of external heat sources. In full detail, this second order partial differential equation is

$$\sigma(x, y) \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( \kappa(x, y) \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( \kappa(x, y) \frac{\partial u}{\partial y} \right). \quad (17.11)$$

In particular, if the body is homogeneous, then both  $\sigma$  and  $\kappa$  are constant, and so general diffusion equation (17.10) reduces to the heat equation (17.1) with thermal diffusivity

$$\gamma = \frac{\kappa}{\sigma} = \frac{\kappa}{\rho \chi}. \quad (17.12)$$

The heat and diffusion equations are also used to model movements of populations, e.g., bacteria in a petri dish or wolves in the Canadian Rockies, [**biol**]. The solution  $u(t, x, y)$  represents the number of individuals near position  $(x, y)$  at time  $t$ , and diffuses over the domain due to random motions of the individuals. Similar diffusion processes model the mixing of chemical reagents in solutions, with the diffusion induced by the random Brownian motion from molecular collisions. Convection due to fluid motion and/or changes due to chemical reactions lead to the more general class of *reaction-diffusion and convective-diffusion equations*, [**chem**].

### Self-Adjoint Formulation

The general diffusion equation (17.11) can be readily fit into the self-adjoint framework of Section 14.7, taking the form

$$u_t = -K[u] = -\nabla^* \circ \nabla u. \quad (17.13)$$

The gradient operator  $\nabla$  maps scalar fields  $u$  to vector fields  $\mathbf{v} = \nabla u$ ; its adjoint  $\nabla^*$ , which goes in the reverse direction, is taken with respect to the weighted inner products

$$\langle u; \tilde{u} \rangle = \iint_{\Omega} u(x, y) \tilde{u}(x, y) \sigma(x, y) dx dy, \quad \langle \mathbf{v}; \tilde{\mathbf{v}} \rangle = \iint_{\Omega} \mathbf{v}(x, y) \cdot \tilde{\mathbf{v}}(x, y) \kappa(x, y) dx dy, \quad (17.14)$$

between, respectively, scalar and vector fields. A straightforward integration by parts argument, done in detail in Section 15.4, tells us that

$$\nabla^* \mathbf{v} = -\frac{1}{\sigma} \nabla \cdot (\kappa \mathbf{v}) = -\frac{1}{\sigma} \left[ \frac{\partial(\kappa v_1)}{\partial x} + \frac{\partial(\kappa v_2)}{\partial y} \right], \quad \text{when} \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \quad (17.15)$$

Therefore, the right hand side of (17.13) is equal to

$$-K[u] = -\nabla^* \circ \nabla u = \frac{1}{\sigma} \nabla \cdot (\kappa \nabla u), \quad (17.16)$$

which recovers (17.10). As always, we need to impose suitable homogeneous boundary conditions — Dirichlet, Neumann or mixed — to ensure the validity of the integration by parts argument used to establish the adjoint formula (17.15).

In particular, to obtain the heat equation, we take  $\sigma$  and  $\kappa$  to be constant, and so (17.14) reduce, up to a constant factor, to the usual  $L^2$  inner products between scalar and vector fields. In this case, the adjoint of the gradient is, up to a scale factor, minus the divergence:  $\nabla^* = -\gamma \nabla \cdot$ , where  $\gamma = \kappa/\sigma$ . In this scenario, (17.13) reduces to the two-dimensional heat equation (17.1).

The self-adjoint operator  $K = \nabla^* \circ \nabla$  is always positive semi-definite, and is positive definite if and only if  $\ker \nabla = \{0\}$ . As we saw in Section 15.4, positive definiteness holds in the Dirichlet and mixed cases, whereas the Neumann boundary conditions lead to only a positive semi-definite operator. Indeed, assuming  $\Omega$  is connected, the only functions in the kernel of the gradient operator are the constants. Moreover, only the zero constant function satisfies the Dirichlet or mixed boundary conditions, and hence the gradient's kernel is trivial, ensuring positive definiteness.

The heat and diffusion equations are examples of *parabolic* partial differential equations, the terminology being adapted from Definition 15.1 to apply to partial differential equations in more than two variables. As we will see, all of the basic qualitative features we learned when studying the one-dimensional heat equation carry over to solutions to parabolic partial differential equations in higher dimensions.

### Separation of Variables

In Section 14.1, we applied the method of separation of variables to express the solution to the one-dimensional heat equation as a series involving the eigenfunctions of an

associated boundary value problem. Section 14.7 argued that the eigenfunction series solution method carries over, essentially unchanged, to the general class of self-adjoint diffusion equations. In particular, the solutions to the two-dimensional heat and diffusion equations are expressed in series form based on the eigenfunctions of an associated two-dimensional boundary value problem.

As we know, the separable solutions to any diffusion equation (17.13) are of exponential form

$$u(t, x, y) = e^{-\lambda t} v(x, y). \quad (17.17)$$

Since the linear operator  $K$  only involves differentiation with respect to the spatial variables  $x, y$ , we find

$$\frac{\partial u}{\partial t} = -\lambda e^{-\lambda t} v(x, y), \quad \text{while} \quad K[u] = e^{-\lambda t} K[v].$$

Substituting back into the diffusion equation (17.13) and canceling the exponentials, we conclude that

$$K[v] = \lambda v, \quad (17.18)$$

Thus,  $v(x, y)$  must be an eigenfunction for the linear operator  $K$ , subject to the relevant homogeneous boundary conditions. In the case of the heat equation (17.1),

$$K[u] = -\gamma \Delta u,$$

and hence the eigenvalue equation (17.18) takes the form

$$\gamma \Delta v + \lambda v = 0, \quad \text{or, in detail,} \quad \gamma \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \lambda v = 0. \quad (17.19)$$

This generalization of the Laplace equation is known as the *Helmholtz equation*, and was briefly discussed in Example ■.

According to Theorem 14.18, the eigenvalues of the self-adjoint operator  $K = \nabla^* \circ \nabla$  are all real and non-negative:  $\lambda \geq 0$ . When  $K$  is positive definite, they are strictly positive:  $\lambda > 0$ . Let us index the eigenvalues in increasing order:

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \quad (17.20)$$

Under reasonable conditions on the underlying linear boundary value problem, it can be proved, [47; Chapter V], that there are, in fact, an infinite number of eigenvalues, and, moreover, their size increases without bound, so  $\lambda_k \rightarrow \infty$  as  $k \rightarrow \infty$ . Each eigenvalue is repeated according to the number (which is necessarily finite) of linearly independent eigenfunctions it admits. The problem has a zero eigenvalue,  $\lambda_0 = 0$ , corresponding to the constant null eigenfunction  $v(x, y) \equiv 1$ , if and only if  $K$  is not positive definite, i.e., only in the case of pure Neumann boundary conditions.

Each eigenvalue and eigenfunction pair will produce a separable solution

$$u_k(t, x, y) = e^{-\lambda_k t} v_k(x, y)$$

to the diffusion equation (17.13). The solutions corresponding to positive eigenvalues are exponentially decaying in time, while a zero eigenvalue, which only occurs in the Neumann

problem, produces a constant solution. The general solution to the homogeneous boundary value problem can then be built up as a linear combination of these basic solutions, in the form of an *eigenfunction series*

$$u(t, x, y) = \sum_{k=1}^{\infty} c_k u_k(t, x, y) = \sum_{k=1}^{\infty} c_k e^{-\lambda_k t} v_k(x, y), \quad (17.21)$$

which is a form of generalized Fourier series. The eigenfunction coefficients  $c_k$  are prescribed by the initial conditions, which require

$$\sum_{k=1}^{\infty} c_k v_k(x, y) = f(x, y). \quad (17.22)$$

Theorem 14.19 guarantees orthogonality of the eigenfunctions, and hence the coefficients are given by our standard orthogonality formula

$$c_k = \frac{\langle f; v_k \rangle}{\|v_k\|^2} = \frac{\iint_{\Omega} f(x, y) v_k(x, y) \sigma(x, y) dx dy}{\iint_{\Omega} v_k(x, y)^2 \sigma(x, y) dx dy}, \quad (17.23)$$

where the weighting function  $\sigma(x, y)$  was defined in (17.9). In the case of the heat equation,  $\sigma$  is constant and so can be canceled from both numerator and denominator, leaving the simpler formula

$$c_k = \frac{\iint_{\Omega} f(x, y) v_k(x, y) dx dy}{\iint_{\Omega} v_k(x, y)^2 dx dy}. \quad (17.24)$$

Under fairly general hypotheses, it can be shown that the eigenfunctions form a *complete system*, which means that the eigenfunction series (17.22) will converge (at least in norm) to the function  $f(x, y)$ , provided it is not too bizarre. Moreover, the resulting series (17.21) converges to the solution to the initial-boundary value problem for the diffusion equation. See [47; p. 369] for a precise statement and proof of the general theorem.

### *Qualitative Properties*

Before tackling simple examples where we are able to construct explicit formulae for the eigenfunctions and eigenvalues, let us see what the eigenfunction series solution (17.21) can tell us about general diffusion processes. Based on our experience with the case of a one-dimensional bar, the final conclusions will not be especially surprising. Indeed, they also apply, word for word, to diffusion processes in three-dimensional solid bodies. A reader who prefers to see explicit solution formulae may wish to skip ahead to the following section, returning here later.

Keep in mind that we are still dealing with the solution to the homogeneous boundary value problem. The first observation is that all terms in the series solution (17.21), with the possible exception of a null eigenfunction term that appears in the semi-definite case,

are tending to zero exponentially fast. Since most eigenvalues are large, all the higher order terms in the series become almost instantaneously negligible, and hence the solution can be accurately approximated by a finite sum over the first few eigenfunction modes. As time goes on, more and more of the modes can be neglected, and the solution decays to thermal equilibrium at an exponentially fast rate. The rate of convergence to thermal equilibrium is, for most initial data, governed by the smallest positive eigenvalue  $\lambda_1 > 0$  for the Helmholtz boundary value problem on the domain.

In the positive definite cases of homogeneous Dirichlet or mixed boundary conditions, thermal equilibrium is  $u(t, x, y) \rightarrow u_*(x, y) \equiv 0$ . Thus, in these cases, the equilibrium temperature is equal to the boundary temperature — even if this temperature is only fixed on a small part of the boundary. The initial heat is eventually dissipated away through the non-insulated part of the boundary. In the semi-definite Neumann case, corresponding to a completely insulated plate, In this case, the general solution has the form

$$u(t, x, y) = c_0 + \sum_{k=1}^{\infty} c_k e^{-\lambda_k t} v_k(x, y), \quad (17.25)$$

where the sum is over the positive eigenmodes,  $\lambda_k > 0$ . Since all the summands are exponentially decaying, the final equilibrium temperature  $u_* = c_0$  is the same as the constant term in the eigenfunction expansion. We evaluate this term using the orthogonality formula (17.23), and so, as  $t \rightarrow \infty$ ,

$$u(t, x, y) \longrightarrow c_0 = \frac{\langle f; 1 \rangle}{\|1\|^2} = \frac{\iint_{\Omega} f(x, y) \sigma(x, y) dx dy}{\iint_{\Omega} \sigma(x, y) dx dy},$$

representing a suitably weighted average of the initial temperature over the domain. In particular, in the case of the heat equation, the weighting function  $\sigma$  is constant, and so the equilibrium temperature

$$u(t, x, y) \longrightarrow c_0 = \frac{1}{\text{area } \Omega} \iint_{\Omega} f(x, y) dx dy \quad (17.26)$$

equals the average initial temperature distribution. In this case, the heat energy is not allowed to escape through the boundary, and thus redistributes itself in a uniform manner over the domain.

Diffusion has a smoothing effect on the initial temperature distribution  $f(x, y)$ . Assume that the eigenfunction coefficients are uniformly bounded, so  $|c_k| \leq M$  for some constant  $M$ . This will certainly be the case if  $f(x, y)$  is piecewise continuous, but even holds for quite rough initial data, including delta functions. Then, at any time  $t > 0$  after the initial instant, the coefficients  $c_k e^{-\lambda_k t}$  in the eigenfunction series solution (17.21) are exponentially small as  $k \rightarrow \infty$ , which is enough to ensure smoothness<sup>†</sup> of the solution  $u(t, x, y)$  for each  $t > 0$ . Therefore, the diffusion process serves to immediately smooth out

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<sup>†</sup> For a general diffusion equation, this requires that the functions  $\sigma(x, y)$  and  $\kappa(x, y)$  be smooth.



**Figure 17.1.** Smoothing a Grey Scale Image.

jumps, corners and other discontinuities in the initial data. As time progresses, the local variations in the solution become less and less pronounced, as it asymptotically reaches a constant equilibrium state.

As a result, diffusion processes can be effectively applied to clean and denoise planar images. The initial data  $f(x, y)$  represents the grey-scale value of the image at position  $(x, y)$ , so that  $0 \leq f(x, y) \leq 1$  with 0 representing black, and 1 representing white. As time progresses, the solution  $u(t, x, y)$  represents a more and more smoothed version of the image. Although this has the effect of removing unwanted noise from the image, there is also a gradual blurring of the actual features. Thus, the “time” or “multiscale” parameter  $t$  needs to be chosen to optimally balance between the two effects — the larger  $t$  is the more noise is removed, but the more noticeable the blurring. A representative illustration appears in Figure 17.1. To further suppress undesirable blurring effects, recent image processing filters are based on anisotropic (and thus *nonlinear*) diffusion equations. See Sapiro, [162], for a survey of recent progress in this active field.

Since the forward heat equation effectively blurs the features in an image, we might be tempted to try going backwards in time to sharpen the image. However, the argument presented in Section 14.1 tells us that the backwards heat equation is ill-posed, and hence cannot be used directly for this purpose. Various “regularization” strategies have been devised to circumvent this mathematical barrier, and thereby design effective image enhancement algorithms, [reg].

### *Inhomogeneous Boundary Conditions and Forcing*

Finally, let us briefly mention how to incorporate inhomogeneous boundary conditions and external heat sources into the problem. Consider, as a specific example, the forced heat equation

$$u_t = \gamma \Delta u + F(x, y), \quad \text{for} \quad (x, y) \in \Omega, \quad (17.27)$$

where  $F(x, y)$  represents an unvarying external heat source, subject to inhomogeneous Dirichlet boundary conditions

$$u = h \quad \text{for} \quad (x, y) \in \partial\Omega, \quad (17.28)$$

that fixes the temperature of the plate on its boundary. When the external forcing is fixed for all  $t$ , we expect the solution to eventually settle down to an equilibrium configuration:  $u(t, x, y) \rightarrow u_\star(x, y)$  as  $t \rightarrow \infty$ , which will be justified below.

The time-independent equilibrium temperature  $u = u_*(x, y)$  satisfies the equation obtained by setting  $u_t = 0$  in the differential equation (17.27), namely Poisson equation

$$-\gamma \Delta u_* = F, \quad \text{for} \quad (x, y) \in \Omega, \quad (17.29)$$

and subject to the same inhomogeneous Dirichlet boundary conditions (17.28). Positive definiteness of the Dirichlet boundary value problem implies that there is a unique equilibrium solution.

Once we have determined the equilibrium solution — usually through a numerical approximation — we set

$$v(t, x, y) = u(t, x, y) - u_*(x, y),$$

so that  $v$  measures the deviation of the solution  $u$  from its eventual equilibrium. By linearity  $v(t, x, y)$  satisfies the unforced heat equation subject to homogeneous boundary conditions:

$$v_t = \gamma \Delta v, \quad (x, y) \in \Omega, \quad u = 0, \quad (x, y) \in \partial\Omega. \quad (17.30)$$

Therefore,  $v$  can be expanded in an eigenfunction series (17.21), and will decay to zero,  $v(t, x, y) \rightarrow 0$ , at a exponentially fast rate prescribed by the smallest eigenvalue  $\lambda_1$  of the associated homogeneous Helmholtz boundary value problem. Consequently, the solution to the forced, inhomogeneous problem

$$u(t, x, y) = v(t, x, y) + u_*(x, y) \longrightarrow u_*(x, y)$$

will approach thermal equilibrium, namely  $u_*(x, y)$ , at exactly the same exponential rate as its homogeneous counterpart.

## 17.2. Explicit Solutions for the Heat Equation.

Thus, solving the two-dimensional heat equation in series form requires knowing the eigenfunctions for the associated Helmholtz boundary value problem. Unfortunately, as with any partial differential equation, explicit solution formulae are few and far between. In this section, we discuss two specific cases where the required eigenfunctions can be found in closed form. The calculations rely on separation of variables, which, as we know, works in only a very limited class of domains. Nevertheless, interesting solution features can be gleaned from these particular cases.

### *Heating of a Rectangle*

A homogeneous rectangular plate

$$R = \{ 0 < x < a, 0 < y < b \}$$

is heated to a prescribed initial temperature

$$u(0, x, y) = f(x, y), \quad \text{for} \quad (x, y) \in R, \quad (17.31)$$

and then insulated. The sides of the plate are held at zero temperature. Our task is to determine how fast the plate returns to thermal equilibrium.

The temperature  $u(t, x, y)$  evolves according to the two-dimensional heat equation

$$u_t = \gamma(u_{xx} + u_{yy}), \quad \text{for} \quad (x, y) \in R, \quad t > 0, \quad (17.32)$$

subject to homogeneous Dirichlet conditions

$$u(0, y) = u(a, y) = 0 = u(x, 0) = u(x, b), \quad 0 < x < a, \quad 0 < y < b, \quad (17.33)$$

along the boundary of the rectangle. As in (17.17), the eigensolutions to the heat equation are obtained from the usual exponential ansatz  $u(t, x, y) = e^{-\lambda t} v(x, y)$ . Substituting this expression into the heat equation, we conclude that the function  $v(x, y)$  solves the Helmholtz eigenvalue problem

$$\gamma(v_{xx} + v_{yy}) + \lambda v = 0, \quad (x, y) \in R, \quad (17.34)$$

subject to the same homogeneous Dirichlet boundary conditions

$$v(x, y) = 0, \quad (x, y) \in \partial R. \quad (17.35)$$

To solve the rectangular Helmholtz eigenvalue problem (17.34–35), we shall, as in (15.13), introduce a further separation of variables, writing

$$v(x, y) = p(x) q(y)$$

as the product of functions depending upon the individual Cartesian coordinates. Substituting this ansatz into the Helmholtz equation (17.34), we find

$$\gamma p''(x) q(y) + \gamma p(x) q''(y) + \lambda p(x) q(y) = 0.$$

To effect the variable separation, we collect all terms involving  $x$  on one side and all terms involving  $y$  on the other side of the equation. This is accomplished by dividing by  $v = pq$  and rearranging the terms; the result is

$$\gamma \frac{p''(x)}{p(x)} = -\gamma \frac{q''(y)}{q(y)} - \lambda \equiv -\mu.$$

The left hand side of this equation only depends on  $x$ , whereas the right hand side only depends on  $y$ . As argued in Section 15.2, the only way this can occur is if the two sides equal a common *separation constant*, denoted by  $-\mu$ . (The minus sign is for later convenience.) In this manner, we reduce our partial differential equation to a pair of one-dimensional eigenvalue problems

$$\gamma p'' + \mu p = 0, \quad \gamma q'' + (\lambda - \mu) q = 0,$$

each of which is subject to homogeneous Dirichlet boundary conditions

$$p(0) = p(a) = 0, \quad q(0) = q(b) = 0,$$

stemming from the boundary conditions (17.35). To obtain a nontrivial solution to the Helmholtz equation, we seek nonzero solutions to these two supplementary eigenvalue problems. The fact that we are dealing with a rectangular domain is critical to the success of this procedure.

We have already solved these particular two boundary value problems many times; see, for instance, (14.17). The eigenfunctions are, respectively,

$$p_m(x) = \sin \frac{m\pi x}{a}, \quad m = 1, 2, 3, \dots, \quad q_n(y) = \sin \frac{n\pi y}{b}, \quad n = 1, 2, 3, \dots,$$

with

$$\mu = \frac{m^2 \pi^2 \gamma}{a^2}, \quad \lambda - \mu = \frac{n^2 \pi^2 \gamma}{b^2}, \quad \text{so that} \quad \lambda = \frac{m^2 \pi^2 \gamma}{a^2} + \frac{n^2 \pi^2 \gamma}{b^2}.$$

Therefore, the separable eigenfunction solutions to the Helmholtz boundary value problem (17.33–34) have the doubly trigonometric form

$$v_{m,n}(x, y) = \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \quad (17.36)$$

with associated eigenvalues

$$\lambda_{m,n} = \frac{m^2 \pi^2 \gamma}{a^2} + \frac{n^2 \pi^2 \gamma}{b^2} = \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \pi^2 \gamma. \quad (17.37)$$

Each of these corresponds to an exponentially decaying, separable solution

$$u_{m,n}(t, x, y) = e^{-\lambda_{m,n} t} v_{m,n}(x, y) = \exp \left[ - \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \pi^2 \gamma t \right] \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (17.38)$$

to the original rectangular boundary value problem for the heat equation.

Using the fact that the univariate sine functions form a complete system, it is not hard to prove, [187], that the separable eigenfunction solutions (17.38) are complete, which implies that there are no non-separable eigenfunctions. As a consequence, the general solution to the initial-boundary value problem can be expressed as a linear combination

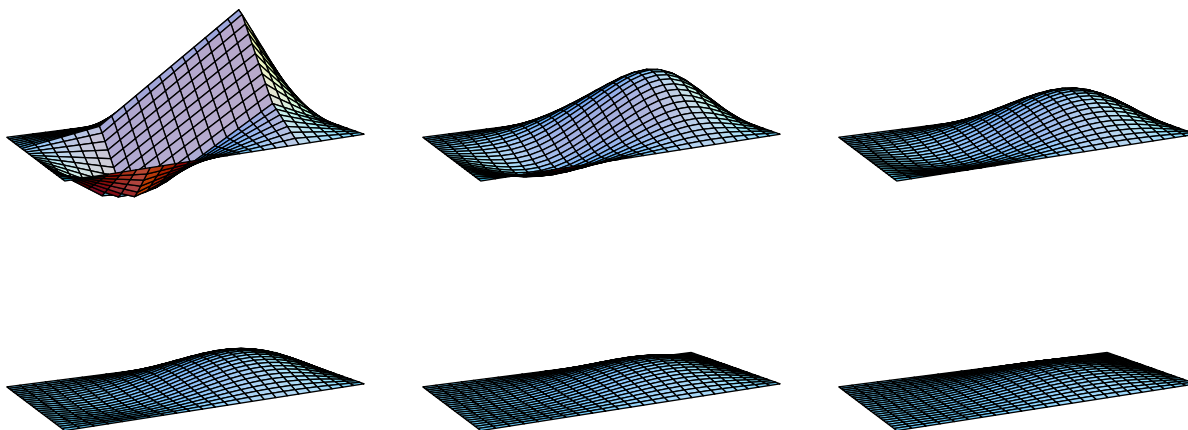
$$u(t, x, y) = \sum_{m,n=1}^{\infty} c_{m,n} u_{m,n}(t, x, y) = \sum_{m,n=1}^{\infty} c_{m,n} e^{-\lambda_{m,n} t} v_{m,n}(x, y) \quad (17.39)$$

of our eigenfunction modes. The coefficients  $c_{m,n}$  are prescribed by the initial conditions, which take the form of a double Fourier sine series

$$f(x, y) = u(0, x, y) = \sum_{m,n=1}^{\infty} c_{m,n} v_{m,n}(x, y) = \sum_{m,n=1}^{\infty} c_{m,n} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}.$$

Self-adjointness of the Laplacian coupled with the boundary conditions implies that the eigenfunctions  $v_{m,n}(x, y)$  are orthogonal with respect to the  $L^2$  inner product on the rectangle:

$$\langle v_{k,l}; v_{m,n} \rangle = \int_0^b \int_0^a v_{k,l}(x, y) v_{m,n}(x, y) dx dy = 0 \quad \text{unless} \quad k = m \quad \text{and} \quad l = n.$$



**Figure 17.2.** Heat Diffusion in a Rectangle.

(The skeptical reader can verify the orthogonality relations directly from the eigenfunction formulae (17.36).) Thus, we can appeal to our usual orthogonality formula (17.24) to evaluate the coefficients

$$c_{m,n} = \frac{\langle f; v_{m,n} \rangle}{\|v_{m,n}\|^2} = \frac{4}{ab} \int_0^b \int_0^a f(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy, \quad (17.40)$$

where the formula for the norms of the eigenfunctions

$$\|v_{m,n}\|^2 = \int_0^b \int_0^a v_{m,n}(x,y)^2 dx dy = \int_0^b \int_0^a \sin^2 \frac{m\pi x}{a} \sin^2 \frac{n\pi y}{b} dx dy = \frac{1}{4} ab. \quad (17.41)$$

follows from a direct evaluation of the double integral. (Unfortunately, while the orthogonality is automatic, the computation of the norm must inevitably be done “by hand”.)

The rectangle approaches thermal equilibrium at the rate equal to the smallest eigenvalue:

$$\lambda_{1,1} = \left( \frac{1}{a^2} + \frac{1}{b^2} \right) \pi^2 \gamma, \quad (17.42)$$

i.e., the sum of the reciprocals of the squared lengths of its sides multiplied by the diffusion coefficient. The larger the rectangle, or the smaller the diffusion coefficient, the smaller  $\lambda_{1,1}$ , and hence slower the return to thermal equilibrium. The exponentially fast decay rate of the Fourier series implies that the solution immediately smooths out any initial discontinuities in the initial temperature profile. Indeed, the higher modes, with  $m$  and  $n$  large, decay to zero almost instantaneously, and so the solution immediately behaves like a finite sum over a few low order modes. Assuming that  $c_{1,1} \neq 0$ , the slowest decaying mode in the Fourier series (17.39) is

$$c_{1,1} u_{1,1}(t, x, y) = c_{1,1} \exp \left[ - \left( \frac{1}{a^2} + \frac{1}{b^2} \right) \pi^2 \gamma t \right] \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}. \quad (17.43)$$

Thus, in the long run, the temperature is of one sign — either positive or negative depending upon the sign of  $c_{1,1}$  — throughout the rectangle. This observation is, in fact,

indicative of the general phenomenon that the eigenfunction associated with the smallest positive eigenvalue of a self-adjoint elliptic operator is of one sign throughout the domain. A typical solution is plotted in Figure 17.2.

### *Heating of a Disk*

Let us perform a similar analysis of the thermodynamics of a circular disk. For simplicity (or by choice of suitable physical units), we will assume that the disk

$$D = \{ x^2 + y^2 \leq 1 \} \subset \mathbb{R}^2$$

has unit radius and unit diffusion coefficient  $\gamma = 1$ . We shall solve the heat equation on  $D$  subject to homogeneous Dirichlet boundary values of zero temperature at the circular edge

$$\partial D = C = \{ x^2 + y^2 = 1 \}.$$

Thus, the full initial-boundary value problem is

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, & x^2 + y^2 < 1, & t > 0, \\ u(t, x, y) &= 0, & x^2 + y^2 = 1, \\ u(0, x, y) &= f(x, y), & x^2 + y^2 \leq 1. \end{aligned} \tag{17.44}$$

We remark that a simple rescaling of space and time, as outlined in Exercise ■, can be used to recover the solution for an arbitrary diffusion coefficient and a disk of arbitrary radius from this particular case.

Since we are working in a circular domain, we instinctively pass to polar coordinates  $(r, \theta)$ . In view of the polar coordinate formula (15.29) for the Laplace operator, the heat equation and boundary conditions assume the form

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}, & 0 \leq r < 1, & t > 0, \\ u(t, 1, \theta) &= 0, & u(0, r, \theta) &= f(r, \theta), & r \leq 1, \end{aligned} \tag{17.45}$$

where the solution  $u(t, r, \theta)$  is defined for all  $0 \leq r \leq 1$  and  $t \geq 0$ . To ensure that the solution represents a single-valued function on the entire disk, it is required to be a  $2\pi$  periodic function of the angular variable:

$$u(t, r, \theta + 2\pi) = u(t, r, \theta)$$

To obtain the separable solutions

$$u(t, r, \theta) = e^{-\lambda t} v(r, \theta), \tag{17.46}$$

we need to solve the polar coordinate form of the Helmholtz equation

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} + \lambda v = 0, \quad \begin{aligned} 0 \leq r < 1, \\ 0 \leq \theta \leq 2\pi, \end{aligned} \tag{17.47}$$

subject to the boundary conditions

$$v(1, \theta) = 0, \quad v(r, \theta + 2\pi) = v(r, \theta). \quad (17.48)$$

We invoke a further separation of variables by writing

$$v(r, \theta) = p(r) q(\theta). \quad (17.49)$$

Substituting this ansatz into the polar Helmholtz equation (17.47), and then collecting together all terms involving  $r$  and all terms involving  $\theta$ , we are led to the pair of ordinary differential equations

$$r^2 p'' + r p' + (\lambda r^2 - \mu) p = 0, \quad q'' + \mu q = 0, \quad (17.50)$$

where  $\lambda$  is the Helmholtz eigenvalue, and  $\mu$  the separation constant.

Let us start with the equation for  $q(\theta)$ . The periodicity condition (17.48) requires that  $q(\theta)$  be  $2\pi$  periodic. Therefore, the required solutions are the elementary trigonometric functions

$$q(\theta) = \cos m\theta \quad \text{or} \quad \sin m\theta, \quad \text{where} \quad \mu = m^2, \quad (17.51)$$

with  $m = 0, 1, 2, \dots$  a non-negative integer.

Substituting the formula,  $\mu = m^2$ , for the separation constant, the differential equation for  $p(r)$  takes the form

$$r^2 \frac{d^2 p}{dr^2} + r \frac{dp}{dr} + (\lambda r^2 - m^2) p = 0, \quad 0 \leq r \leq 1. \quad (17.52)$$

Ordinarily, one imposes two boundary conditions in order to pin down a solution to such a second order ordinary differential equation. But our Dirichlet condition, namely  $p(1) = 0$ , only specifies its value at one of the endpoints. The other endpoint is a *singular point* for the ordinary differential equation, because the coefficient of the highest order derivative, namely  $r^2$ , vanishes at  $r = 0$ . This situation might remind you of our solution to the Euler differential equation (15.35) in the context of separable solutions to the Laplace equation on the disk. As there, we only require the solution to be bounded at  $r = 0$ :

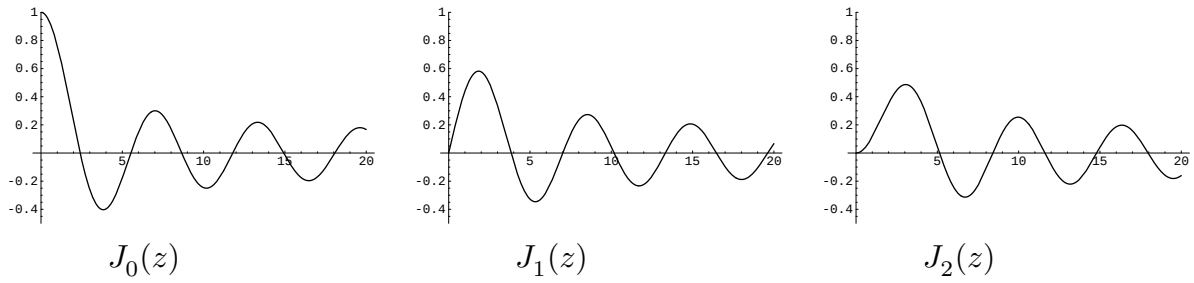
$$|p(0)| < \infty, \quad p(1) = 0. \quad (17.53)$$

As we now show, this pair of boundary conditions suffices to distinguish the relevant eigenfunction solutions to (17.52).

Although the ordinary differential equation (17.52) appears in a variety of applications, this may be the first time that you have encountered it. It is not an elementary equation; indeed, most solutions cannot be written in terms of the elementary functions you see in first year calculus. Nevertheless, owing to their significance in a wide range of physical applications, the solutions have been extensively studied and tabulated, and so are, in a sense, well-known. After some preparatory manipulations, we shall summarize their relevant properties, relegating details and proofs to Appendix C.

To simplify the analysis, we make a preliminary rescaling of the independent variable, replacing  $r$  by

$$z = \sqrt{\lambda} r.$$



**Figure 17.3.** Bessel Functions.

Note that, by the chain rule,

$$\frac{dp}{dr} = \sqrt{\lambda} \frac{dp}{dz}, \quad \frac{d^2p}{dr^2} = \lambda \frac{d^2p}{dz^2},$$

and hence

$$r \frac{dp}{dr} = z \frac{dp}{dz}, \quad r^2 \frac{d^2p}{dr^2} = z^2 \frac{d^2p}{dz^2}.$$

The net effect is to eliminate the eigenvalue parameter  $\lambda$  (or, rather, hide it in the change of variables), so that (17.52) assumes the slightly simpler form

$$z^2 \frac{d^2p}{dz^2} + z \frac{dp}{dz} + (z^2 - m^2)p = 0. \quad (17.54)$$

The ordinary differential equation (17.54) is known as *Bessel's equation*, named after the early 19<sup>th</sup> century astronomer Wilhelm Bessel, who first used its solutions to analyze planetary orbits. The solutions to Bessel's equation have become an indispensable tool in applied mathematics, physics, and engineering.

To begin, the one thing we know for sure is that, as with any second order ordinary differential equation, there are two linearly independent solutions. However, it turns out that, up to a constant multiple, only one solution remains bounded as  $z \rightarrow 0$ . This solution is known as the *Bessel function of order  $m$* , and is denoted by  $J_m(z)$ . Applying the general systematic method for finding power series solutions to linear ordinary differential equations presented in Appendix C, it can be shown that the Bessel function of order  $m$  has the Taylor expansion

$$\begin{aligned} J_m(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k z^{m+2k}}{2^{m+2k} k! (m+k)!} \\ &= \frac{z^m}{2^m m!} \left[ 1 - \frac{z^2}{4(m+1)} + \frac{z^4}{32(m+1)(m+2)} - \frac{z^6}{384(m+1)(m+2)(m+3)} + \cdots \right]. \end{aligned} \quad (17.55)$$

A simple application of the ratio test tells us that the power series converges for all (complex) values of  $z$ , and hence  $J_m(z)$  is everywhere analytic. Indeed, the convergence is quite rapid when  $z$  is of moderate size, and so summing the series is a reasonably effective method for computing the Bessel function  $J_m(z)$  — although in serious applications one adopts more sophisticated numerical techniques based on asymptotic expansions and integral formulae, [3, 144]. Moreover, verification that it indeed solves the Bessel equation (17.54) is a straightforward computation. Figure 17.3 displays graphs of the first three

Bessel functions for  $z > 0$ . Most software packages, both symbolic and numerical, include routines for accurately evaluating and graphing Bessel functions.

Reverting back to our original radial coordinate  $r = z/\sqrt{\lambda}$ , we conclude that every solution to the radial equation (17.52) which is bounded at  $r = 0$  is a constant multiple

$$p(r) = J_m(\sqrt{\lambda} r) \quad (17.56)$$

of the rescaled Bessel function of order  $m$ . So far, we have only dealt with the boundary condition at the singular point  $r = 0$ . The Dirichlet condition at the other end requires

$$p(1) = J_m(\sqrt{\lambda}) = 0.$$

Therefore, in order that  $\lambda$  be a legitimate eigenvalue,  $\sqrt{\lambda}$  must be a *root* of the  $m^{\text{th}}$  order Bessel function  $J_m$ .

*Remark:* We already know, thanks to the positive definiteness of the Dirichlet boundary value problem, that the Helmholtz eigenvalues  $\lambda > 0$  must be positive, and so there is no problem taking the square root. Indeed, it can be proved, [186], that the Bessel functions do not have any negative roots!

The graphs of  $J_m(z)$  strongly indicate, and, indeed, it can be rigorously proved, that each Bessel function oscillates between positive and negative values as  $z$  increases above 0, with slowly decreasing amplitude. In fact, it can be proved that, asymptotically,

$$J_m(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \left(\frac{1}{2}m + \frac{1}{4}\right)\pi\right) \quad \text{as} \quad z \longrightarrow \infty, \quad (17.57)$$

and so the oscillations become essentially the same as a (phase-shifted) cosine whose amplitude decreases, relatively slowly, like  $z^{-1/2}$ . As a consequence, there exists an infinite sequence of *Bessel roots*, which we number in the order in which they appear:

$$J_m(\zeta_{m,n}) = 0, \quad \text{where} \quad 0 < \zeta_{m,1} < \zeta_{m,2} < \zeta_{m,3} < \cdots \quad \text{with} \quad \zeta_{m,n} \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty. \quad (17.58)$$

It is worth noting that the Bessel functions are *not* periodic, and their roots are not initially evenly spaced.

Owing to their physical importance in a wide range of problems, the Bessel roots have been extensively tabulated in the literature, cf. [3, 145]. The accompanying table displays all Bessel roots that are  $< 12$  in magnitude. The columns of the table are indexed by  $m$ , the order of the Bessel function, and the rows by  $n$ , the root number.

Table of Bessel Roots  $\zeta_{m,n}$

| $n \backslash m$ | 0        | 1        | 2        | 3        | 4        | 5        | 6        | 7        | ... |
|------------------|----------|----------|----------|----------|----------|----------|----------|----------|-----|
| 1                | 2.4048   | 3.8317   | 5.1356   | 6.3802   | 7.5883   | 8.7715   | 9.9361   | 11.0860  | ... |
| 2                | 5.5201   | 7.0156   | 8.4172   | 9.761    | 11.0650  | $\vdots$ | $\vdots$ | $\vdots$ |     |
| 3                | 8.6537   | 10.1730  | 11.6200  | $\vdots$ | $\vdots$ |          |          |          |     |
| 4                | 11.7920  | $\vdots$ | $\vdots$ |          |          |          |          |          |     |
| $\vdots$         | $\vdots$ |          |          |          |          |          |          |          |     |

*Remark:* According to (17.55),

$$J_m(0) = 0 \quad \text{for} \quad m > 0, \quad \text{while} \quad J_0(0) = 1.$$

However, we do not count 0 as a *bona fide* Bessel root, since it does not lead to a valid eigenfunction for the Helmholtz boundary value problem.

Summarizing our progress, the eigenvalues

$$\lambda_{m,n} = \zeta_{m,n}^2, \quad n = 1, 2, 3, \dots, \quad m = 0, 1, 2, \dots, \quad (17.59)$$

of the Bessel boundary value problem (17.52–53) are the squares of the roots of the Bessel function of order  $m$ . The corresponding eigenfunctions are

$$w_{m,n}(r) = J_m(\zeta_{m,n} r), \quad n = 1, 2, 3, \dots, \quad m = 0, 1, 2, \dots, \quad (17.60)$$

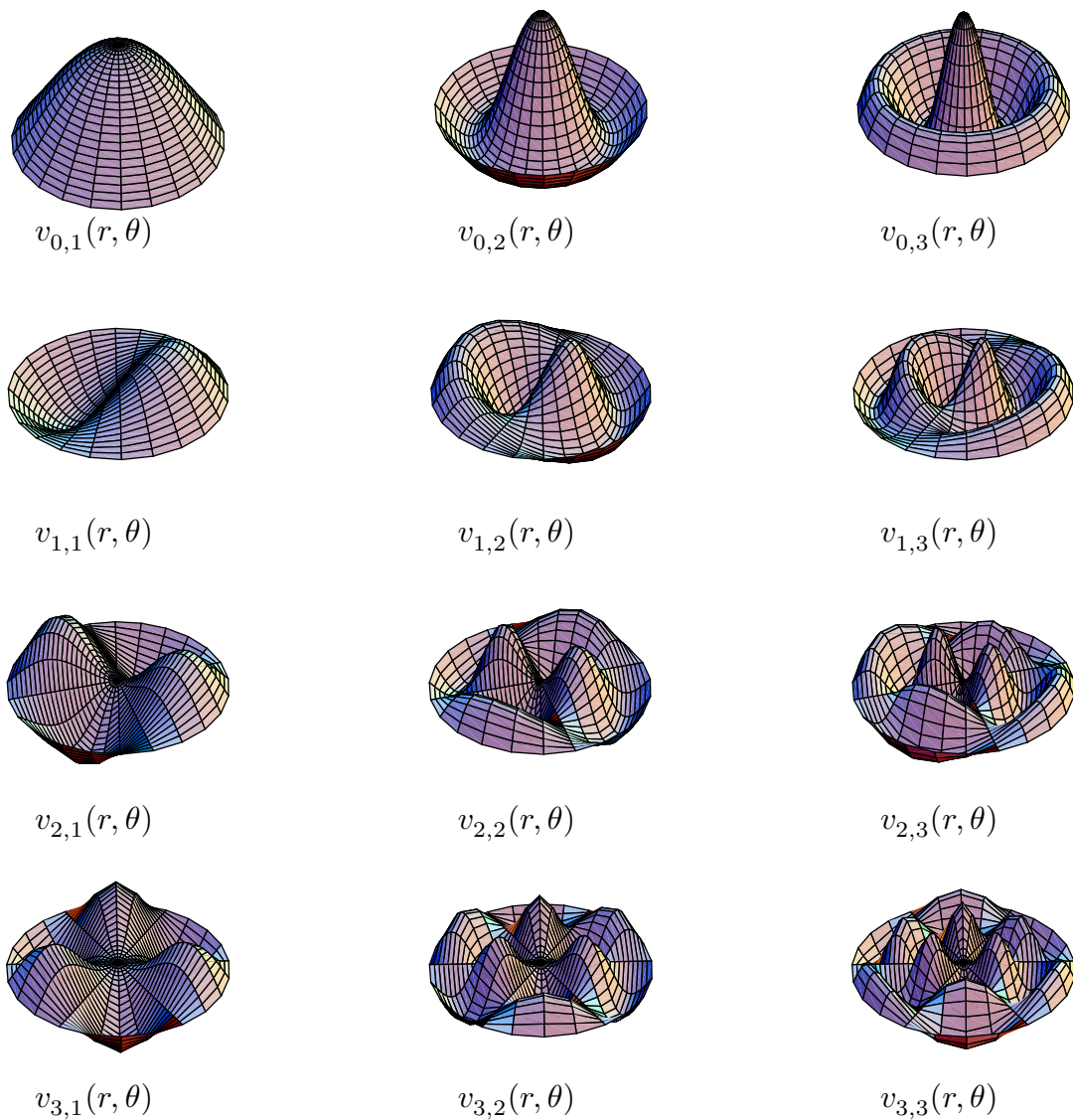
defined for  $0 \leq r \leq 1$ . Combining (17.60) with the formula (17.51) for the angular components, we conclude that the separable solutions (17.49) to the polar Helmholtz boundary value problem (17.47) are

$$\begin{aligned} v_{0,n}(r, \theta) &= J_0(\zeta_{0,n} r), \\ v_{m,n}(r, \theta) &= J_m(\zeta_{m,n} r) \cos m\theta, \\ \widehat{v}_{m,n}(r, \theta) &= J_m(\zeta_{m,n} r) \sin m\theta, \end{aligned} \quad \text{where} \quad \begin{aligned} n &= 1, 2, 3, \dots, \\ m &= 0, 1, 2, \dots \end{aligned} \quad (17.61)$$

These solutions define the so-called *normal modes* for the unit disk, and Figure 17.4 plots the first few of them. The eigenvalues  $\lambda_{0,n}$  are simple, and contribute radially symmetric eigenfunctions, whereas the eigenvalues  $\lambda_{m,n}$  for  $m > 0$  are double, and produce two linearly independent separable eigenfunctions, with trigonometric dependence on the angular variable.

We have at last produced the basic separable solutions

$$\begin{aligned} u_{0,n}(t, r) &= e^{-\zeta_{0,n}^2 t} J_0(\zeta_{0,n} r), \\ u_{m,n}(t, r, \theta) &= e^{-\zeta_{m,n}^2 t} J_m(\zeta_{m,n} r) \cos m\theta, \\ \widehat{u}_{m,n}(t, r, \theta) &= e^{-\zeta_{m,n}^2 t} J_m(\zeta_{m,n} r) \sin m\theta, \end{aligned} \quad \begin{aligned} n &= 1, 2, 3, \dots, \\ m &= 1, 2, \dots \end{aligned} \quad (17.62)$$



**Figure 17.4.** Normal Modes for a Disk.

to the homogeneous Dirichlet boundary value problem for the heat equation on the unit disk (17.45). The general solution is a linear superposition, in the form of an infinite series

$$u(t, r, \theta) = \frac{1}{2} \sum_{n=1}^{\infty} a_{0,n} u_{0,n}(t, r) + \sum_{m,n=1}^{\infty} [a_{m,n} u_{m,n}(t, r, \theta) + b_{m,n} \hat{u}_{m,n}(t, r, \theta)], \quad (17.63)$$

where the initial factor of  $\frac{1}{2}$  is included, as with ordinary Fourier series, for later convenience. As usual, the coefficients  $a_{m,n}, b_{m,n}$  are determined by the initial condition,

Norms of the Fourier–Bessel Eigenfunctions  $\|v_{m,n}\| = \|\widehat{v}_{m,n}\|$

| $n \backslash m$ | 0     | 1     | 2     | 3     | 4     | 5     | 6     | 7     |
|------------------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1                | .9202 | .5048 | .4257 | .3738 | .3363 | .3076 | .2847 | .2658 |
| 2                | .6031 | .3761 | .3401 | .3126 | .2906 | .2725 | .2572 | .2441 |
| 3                | .4811 | .3130 | .2913 | .2736 | .2586 | .2458 | .2347 | .2249 |
| 4                | .4120 | .2737 | .2589 | .2462 | .2352 | .2255 | .2169 | .2092 |
| 5                | .3661 | .2462 | .2353 | .2257 | .2171 | .2095 | .2025 | .1962 |

so

$$u(0, r, \theta) = \frac{1}{2} \sum_{n=1}^{\infty} a_{0,n} v_{0,n}(r) + \sum_{m,n=1}^{\infty} [a_{m,n} v_{m,n}(r, \theta) + b_{m,n} \widehat{v}_{m,n}(r, \theta)] = f(r, \theta). \quad (17.64)$$

Thus, we must expand the initial data into a *Fourier–Bessel series* in the eigenfunctions. As in the rectangular case, it is possible to prove, [47], that the separable eigenfunctions are *complete* — there are no other eigenfunctions — and, moreover, every (reasonable) function defined on the unit disk can be written as a convergent series in the Bessel eigenfunctions.

Theorem 14.19 guarantees that the eigenfunctions are orthogonal<sup>†</sup> with respect to the standard  $L^2$  inner product

$$\langle u; v \rangle = \iint_D u(x, y) v(x, y) dx dy = \int_0^1 \int_0^{2\pi} u(r, \theta) v(r, \theta) r d\theta dr$$

on the unit disk. (Note the extra factor of  $r$  coming from the polar coordinate form (A.51) of the infinitesimal element of area  $dx dy = r dr d\theta$ .) The  $L^2$  norms of the Fourier–Bessel functions are given by the interesting formulae

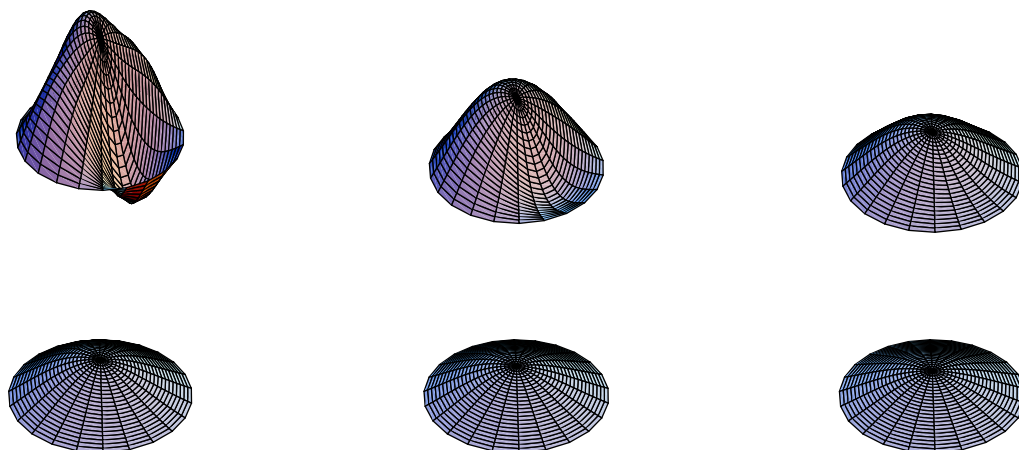
$$\|v_{0,n}\| = \sqrt{\pi} |J_1(\zeta_{0,n})|, \quad \|v_{m,n}\| = \|\widehat{v}_{m,n}\| = \sqrt{\frac{\pi}{2}} |J_{m+1}(\zeta_{m,n})|, \quad (17.65)$$

that involves the value of the Bessel function of the next higher order at the appropriate Bessel root; numerical values appear in the accompanying table. A proof of this formula can be found in C.3.65–66.

Orthogonality of the eigenfunctions implies that the coefficients in the Fourier–Bessel series

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<sup>†</sup> For the two eigenfunctions corresponding to one of the double eigenvalues, orthogonality must be verified by hand.



**Figure 17.5.** Heat Diffusion in a Disk.

(17.64) are given by

$$\begin{aligned}
 a_{0,n} &= 2 \frac{\langle f; v_{0,n} \rangle}{\|v_{0,n}\|^2} = \frac{2}{\pi J_1(\zeta_{0,n})^2} \int_0^1 \int_{-\pi}^{\pi} f(r, \theta) J_0(\zeta_{0,n} r) r d\theta dr, \\
 a_{m,n} &= \frac{\langle f; v_{m,n} \rangle}{\|v_{m,n}\|^2} = \frac{2}{\pi J_{m+1}(\zeta_{m,n})^2} \int_0^1 \int_{-\pi}^{\pi} f(r, \theta) J_m(\zeta_{m,n} r) r \cos m\theta d\theta dr, \\
 b_{m,n} &= \frac{\langle f; \hat{v}_{m,n} \rangle}{\|\hat{v}_{m,n}\|^2} = \frac{2}{\pi J_{m+1}(\zeta_{m,n})^2} \int_0^1 \int_{-\pi}^{\pi} f(r, \theta) J_m(\zeta_{m,n} r) r \sin m\theta d\theta dr.
 \end{aligned} \tag{17.66}$$

In accordance with the general theory, each individual separable solution (17.62) to the heat equation decays exponentially fast, at a rate  $\lambda_{m,n} = \zeta_{m,n}^2$  prescribed by the square of the corresponding Bessel root. In particular, the dominant mode, meaning the one that persists the longest, is

$$u_{0,1}(t, r, \theta) = e^{-\zeta_{0,1}^2 t} J_0(\zeta_{0,1} r). \tag{17.67}$$

Its decay rate

$$\zeta_{0,1}^2 \approx 5.783 \tag{17.68}$$

is the square of the smallest non-zero root of the Bessel function  $J_0(z)$ . The dominant eigenfunction  $v_{0,1}(r, \theta) = J_0(\zeta_{0,1} r) > 0$  is strictly positive within the entire disk and radially symmetric. Consequently, for most initial conditions (specifically those for which  $a_{0,1} \neq 0$ ), the disk's temperature distribution eventually becomes entirely of one sign and radially symmetric, while decaying exponentially fast to zero at the rate given by (17.68). See Figure 17.5 for a plot of a typical solution, displayed as successive times  $t = 0, .04, .08, .12, .16, .2$ . Note how, in accordance with the theory, the solution almost immediately acquires a radial symmetry, followed by a fairly rapid decay to thermal equilibrium.

### 17.3. The Fundamental Solution.

As we learned in Section 14.1, the *fundamental solution* to the heat equation measures the temperature distribution resulting from a concentrated initial heat source, e.g., a hot soldering iron applied instantaneously at one point. The physical problem is modeled mathematically by imposing a delta function as the initial condition for the heat equation, along with the chosen homogeneous boundary conditions. Knowledge of the fundamental solution enables one to use linear superposition to recover the solution for any other initial data.

As in our one-dimensional analysis, we shall concentrate on the most tractable case, when the domain is the entire plane:  $\Omega = \mathbb{R}^2$ . Our first goal will be to solve the initial value problem

$$u_t = \gamma \Delta u, \quad u(0, x, y) = \delta(x - \xi, y - \eta) = \delta(x - \xi) \delta(y - \eta), \quad (17.69)$$

for  $t > 0$  and  $(x, y) \in \mathbb{R}^2$ . The initial data is a delta function representing a concentrated unit heat source placed at position  $(\xi, \eta)$ . The resulting solution  $u = F(t, x, y; \xi, \eta)$  is called the *fundamental solution* for the heat equation on  $\mathbb{R}^2$ .

The quickest route to the desired solution relies on the following simple lemma that combines solutions of the one-dimensional heat equation to produce solutions of the two-dimensional version.

**Lemma 17.1.** *If  $v(t, x)$  and  $w(t, x)$  are any two solutions to the one-dimensional heat equation  $u_t = \gamma u_{xx}$ , then the product*

$$u(t, x, y) = v(t, x) w(t, y) \quad (17.70)$$

*is a solution to the two-dimensional heat equation  $u_t = \gamma(u_{xx} + u_{yy})$ .*

*Proof:* Our assumptions imply that  $v_t = \gamma v_{xx}$ , while  $w_t = \gamma w_{yy}$  when we write  $w(t, y)$  as a function of  $t$  and  $y$ . Therefore, differentiating (17.70), we find

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} w + v \frac{\partial w}{\partial t} = \gamma \frac{\partial^2 v}{\partial x^2} w + \gamma v \frac{\partial^2 w}{\partial y^2} = \gamma \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),$$

and hence  $u(t, x, y)$  solves the heat equation. Q.E.D.

For example, if

$$v(t, x) = e^{-\gamma \omega^2 t} \sin \omega x, \quad w(t, y) = e^{-\gamma \nu^2 t} \sin \nu y,$$

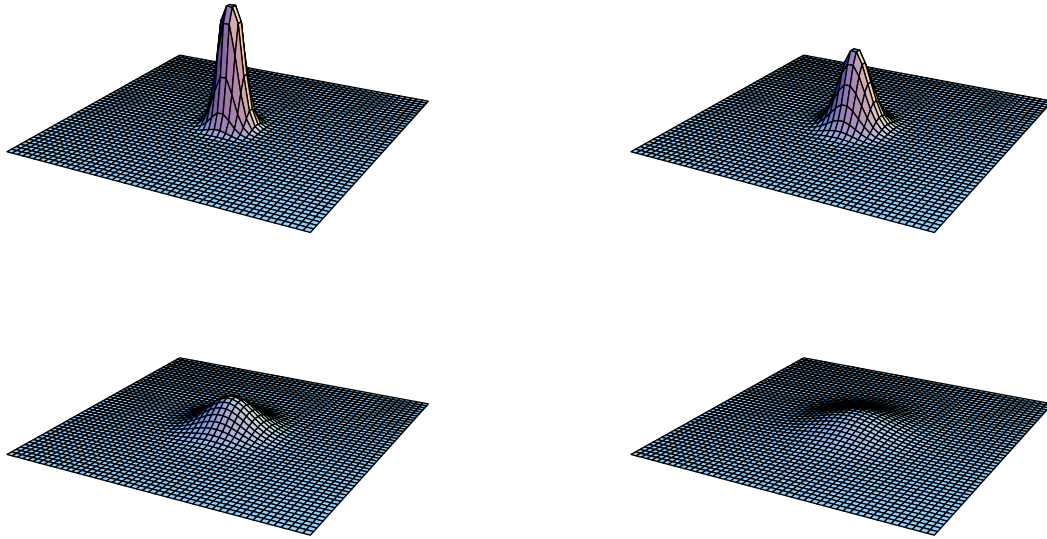
are separable solutions of the one-dimensional heat equation, then

$$u(t, x, y) = e^{-\gamma(\omega^2 + \nu^2)t} \sin \omega x \sin \nu y$$

are the separable solutions we used to solve the heat equation on a rectangle. A more interesting case is to chose

$$v(t, x) = \frac{1}{2\sqrt{\pi\gamma t}} e^{-(x-\xi)^2/(4\gamma t)}, \quad w(t, y) = \frac{1}{2\sqrt{\pi\gamma t}} e^{-(y-\eta)^2/(4\gamma t)}, \quad (17.71)$$

to be the fundamental solutions (14.59) to the one-dimensional heat equation at respective locations  $x = \xi$  and  $y = \eta$ . Multiplying these two solutions together produces the fundamental solution for the two-dimensional problem.



**Figure 17.6.** Fundamental Solution to the Planar Heat Equation.

**Proposition 17.2.** *The fundamental solution to the heat equation  $u_t = \gamma \Delta u$  corresponding to a unit delta function placed at position  $(\xi, \eta) \in \mathbb{R}^2$  at the initial time  $t_0 = 0$  is*

$$F(t, x, y; \xi, \eta) = \frac{1}{4\pi\gamma t} e^{-[(x-\xi)^2 + (y-\eta)^2]/(4\gamma t)}. \quad (17.72)$$

*Proof:* Since we already know that both function (17.71) are solutions to the one-dimensional heat equation, Lemma 17.1 guarantees that their product  $u(t, x, y) = v(t, x) w(t, y)$ , which equals (17.72), solves the two-dimensional heat equation for  $t > 0$ . Moreover, at the initial time

$$u(0, x, y) = v(0, x) w(0, y) = \delta(x - \xi) \delta(y - \eta)$$

is a product of delta functions, and hence the result follows. Indeed, the total heat

$$\iint u(t, x, y) dx dy = \int_{-\infty}^{\infty} v(t, x) dx \int_{-\infty}^{\infty} w(t, y) dy = 1, \quad t \geq 0,$$

remains constant, while

$$\lim_{t \rightarrow 0^+} u(t, x, y) = \begin{cases} \infty, & (x, y) = (\xi, \eta), \\ 0, & \text{otherwise.} \end{cases}$$

has the standard delta function limit at the initial time instant.

*Q.E.D.*

Figure 17.6 depicts the evolution of the fundamental solution when  $\gamma = 1$  at successive times  $t = .01, .02, .05, .1$ . Observe that the initially concentrated heat spreads out in a radially symmetric manner. The total amount of heat remains constant. At any individual point  $(x, y) \neq (0, 0)$ , the initially zero temperature rises slightly at first, but then decays monotonically back to zero at a rate proportional to  $1/t$ .

Both the one- and two-dimensional fundamental solutions have a bell-shaped profile known as a *Gaussian* function. The most important difference is the initial factor. In a one-dimensional medium, the fundamental solution decays in proportion to  $1/\sqrt{t}$ , whereas in the plane the decay is more rapid, being proportional to  $1/t$ . The physical explanation is that the energy is able to spread out in two independent directions, and hence diffuses away from its initial source more rapidly. As we shall see, the decay in three-dimensional space is more rapid still, being proportional to  $t^{-3/2}$  for similar reasons; see (18.103).

The principal purpose of the fundamental solution is to solve the general initial value problem. We express the initial temperature distribution as a superposition of delta function sources,

$$u(0, x, y) = f(x, y) = \iint f(\xi, \eta) \delta(x - \xi, y - \eta) d\xi d\eta,$$

where, at the point  $(\xi, \eta) \in \mathbb{R}^2$ , the source has magnitude  $f(\xi, \eta)$ . Linearity implies that the solution is then given by the same superposition of fundamental solutions.

**Theorem 17.3.** *The solution to the initial value problem*

$$u_t = \gamma \Delta u, \quad u(t, x, y) = f(x, y), \quad (x, y) \in \mathbb{R}^2,$$

for the planar heat equation is given by the linear superposition formula

$$u(t, x, y) = \frac{1}{4\pi\gamma t} \iint f(\xi, \eta) e^{-[(x-\xi)^2 + (y-\eta)^2]/(4\gamma t)} d\xi d\eta. \quad (17.73)$$

We can interpret the solution formula (17.73) as a two-dimensional *convolution*

$$u(t, x, y) = F(t, x, y) * f(x, y) \quad (17.74)$$

of the initial data with a one-parameter family of progressively wider and shorter Gaussian filters

$$F(t, x, y) = F(t, x, y; 0, 0) = \frac{1}{4\pi\gamma t} e^{-[x^2 + y^2]/(4\gamma t)}. \quad (17.75)$$

As in (13.128), such a convolution can be interpreted as a weighted averaging of the function, which has the effect of smoothing out the initial signal  $f(x, y)$ .

**Example 17.4.** If our initial temperature distribution is constant on a circular region, say

$$u(0, x, y) = \begin{cases} 1 & x^2 + y^2 < 1, \\ 0, & \text{otherwise,} \end{cases}$$

Then the solution can be evaluated using (17.73), as follows:

$$u(t, x, y) = \frac{1}{4\pi t} \iint_D e^{-[(x-\xi)^2 + (y-\eta)^2]/(4t)} d\xi d\eta,$$

where the integral is over the unit disk  $D = \{\xi^2 + \eta^2 \leq 1\}$ . Unfortunately, the integral cannot be expressed in terms of elementary functions. On the other hand, numerical evaluation of the integral is straightforward. A plot of the resulting radially symmetric solution appears in Figure h2disk■.

For more general configurations, when analytical formulas are no longer available, one turns to numerical approximation methods. The most popular are based on a two-dimensional variant of the Crank–Nicholson scheme (14.159), relying on either finite differences or finite elements to discretize the space coordinates. We do not have space to develop the details, and refer the interested reader to [35, 153, 109].

## 17.4. The Planar Wave Equation.

The second important class of dynamical equations are those governing vibrational motions. The simplest planar system of this type is the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad (17.76)$$

which models the free (unforced) vibrations of a uniform two-dimensional membrane, e.g., a drum. Here  $u(t, x, y)$  represents the displacement of the membrane at time  $t$  and position  $(x, y) \in \Omega$ , where the domain  $\Omega \subset \mathbb{R}^2$  represents the undeformed shape of the membrane. The constant  $c^2 > 0$  encapsulates the membrane's physical properties — density, tension, stiffness, etc.; its square root  $c$  is called, as in the one-dimensional case, the *wave speed*, since it is the speed at which localized signals propagate through the membrane.

*Remark:* In this simplified model, we are only allowing small, transverse (vertical) displacements of the membrane. Large elastic vibrations lead to the nonlinear partial differential equations of elastodynamics, [8]. The bending vibrations of a flexible plate, which can be viewed as the two-dimensional version of a beam, are governed by a more complicated fourth order partial differential equation; see Exercise ■.

The solution  $u(t, x, y)$  to the wave equation will be uniquely specified once we impose suitable boundary and initial conditions. The Dirichlet conditions

$$u = h \quad \text{on} \quad \partial\Omega, \quad (17.77)$$

correspond to gluing our membrane to a fixed boundary — a rim. On the other hand, the homogeneous Neumann conditions

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial\Omega, \quad (17.78)$$

represent a free boundary where the membrane is not attached to any support. Mixed boundary conditions attach part of the boundary and leave the remaining portion free to vibrate:

$$u = h \quad \text{on} \quad D, \quad \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on} \quad N, \quad (17.79)$$

where  $\partial\Omega = D \cup N$  with  $D$  and  $N$  non-overlapping. Since the wave equation is second order in time, we also need to impose two initial conditions:

$$u(0, x, y) = f(x, y), \quad \frac{\partial u}{\partial t}(0, x, y) = g(x, y), \quad (x, y) \in \Omega. \quad (17.80)$$

The first specifies the initial displacement of the membrane, while the second prescribes its initial velocity.

The wave equation is the simplest example of a general second order system of Newtonian form

$$\frac{\partial^2 u}{\partial t^2} = -K[u] = -\nabla^* \circ \nabla u, \quad (17.81)$$

as presented in Section 14.7. As in (17.15), adopting general weighted inner products

$$\langle u; \tilde{u} \rangle = \iint_{\Omega} u(x, y) \tilde{u}(x, y) \rho(x, y) dx dy, \quad \langle \mathbf{v}; \tilde{\mathbf{v}} \rangle = \iint_{\Omega} \mathbf{v}(x, y) \cdot \tilde{\mathbf{v}}(x, y) \kappa(x, y) dx dy, \quad (17.82)$$

on, respectively, the spaces of scalar and vector fields, the adjoint to the gradient is a rescaled version of the divergence operator

$$\nabla^* \mathbf{v} = -\frac{1}{\rho} \nabla \cdot (\kappa \mathbf{v}).$$

Therefore, the system (17.81) assumes the self-adjoint form

$$u_{tt} = -K[u] = \frac{1}{\rho} \nabla \cdot (\kappa \mathbf{v}),$$

or, in full detail,

$$\rho(x, y) \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} \left( \kappa(x, y) \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( \kappa(x, y) \frac{\partial u}{\partial y} \right). \quad (17.83)$$

The resulting hyperbolic partial differential equation models the small transverse vibrations of a nonuniform membrane, in which  $\rho(x, y) > 0$  represents the density of the membrane at the point  $(x, y) \in \Omega$ , while  $\kappa(x, y) > 0$  represents its stiffness — in direct analogy with the one-dimensional version (14.192). In particular, if the material is homogeneous, then both  $\rho$  and  $\kappa$  are constant, and (17.83) reduces to the two-dimensional wave equation (17.76) with wave speed

$$c = \sqrt{\frac{\kappa}{\rho}}. \quad (17.84)$$

### *Separation of Variables*

According to the general framework established in Section 14.7, the separable solutions to the vibration equation (17.81) have the trigonometric form

$$u_k(t, x, y) = \cos \omega_k t v_k(x, y) \quad \text{and} \quad \tilde{u}_k(t, x, y) = \cos \omega_k t v_k(x, y), \quad (17.85)$$

in which  $v_k(x, y)$  is an eigenfunction of the associated boundary value problem

$$K[v] = \omega_k^2 v = \lambda_k v. \quad (17.86)$$

The eigenvalue  $\lambda_k = \omega_k^2$  equals the square of the vibrational frequency. As in (17.20), there are an infinite number of such *normal modes*, having progressively faster and faster vibrational frequencies:  $\omega_k \rightarrow \infty$  as  $k \rightarrow \infty$ . In addition, in the positive semi-definite case

— which occurs under homogeneous Neumann boundary conditions — there is a single constant null eigenfunction, leading to the additional separable solutions

$$u_0(t, x, y) = 1 \quad \text{and} \quad \tilde{u}_0(t, x, y) = t. \quad (17.87)$$

The first represents a stationary membrane that has been displaced by a fixed amount in the vertical direction, while the second represents a membrane that is moving off in the vertical direction with constant unit speed. (Think of the membrane moving in outer space unaffected by any external gravitational force.) Specializing to the wave equation (17.76), the required eigenvalue problem (17.86) reduces to the Helmholtz eigenvalue problem

$$c^2 \Delta v + \lambda v = c^2 (v_{xx} + v_{yy}) + \lambda v = 0 \quad (17.88)$$

that we analyzed earlier in this chapter.

In the stable cases (Dirichlet or mixed), the general solution to the initial value problem can be built up as a quasi-periodic *eigenfunction series*

$$u(t, x, y) = \sum_{k=1}^{\infty} a_k u_k(t, x, y) + b_k \tilde{u}_k(t, x, y) = \sum_{k=1}^{\infty} (a_k \cos \omega_k t + b_k \sin \omega_k t) v_k(x, y) \quad (17.89)$$

in the fundamental vibrational modes. The coefficients  $a_k, b_k$  are prescribed by the initial conditions:

$$\sum_{k=1}^{\infty} a_k v_k(x, y) = f(x, y), \quad \sum_{k=1}^{\infty} \omega_k b_k v_k(x, y) = g(x, y), \quad (17.90)$$

whence, by orthogonality of the eigenfunctions,

$$a_k = \frac{\langle f; v_k \rangle}{\|v_k\|^2} = \frac{\iint_{\Omega} f v_k \rho \, dx \, dy}{\iint_{\Omega} v_k^2 \rho \, dx \, dy}, \quad b_k = \frac{1}{\omega_k} \frac{\langle g; v_k \rangle}{\|v_k\|^2} = \frac{\iint_{\Omega} g v_k \rho \, dx \, dy}{\omega_k \iint_{\Omega} v_k^2 \rho \, dx \, dy}. \quad (17.91)$$

In the case of the wave equation, the density  $\rho$  is constant, and hence can be canceled from the numerator and denominator of the orthogonality formulae (17.90).

For the Neumann boundary value problem, the eigenfunction series solution takes an amended form

$$u(t, x, y) = a_0 + b_0 t + \sum_{k=1}^{\infty} (a_k \cos \omega_k t + b_k \sin \omega_k t) v_k(x, y). \quad (17.92)$$

The coefficients  $a_k, b_k$  for  $k > 0$  are given by the same orthogonality formulae (17.91). The only unstable, non-periodic mode is the linearly growing term  $b_0 t$ ; its coefficient

$$b_0 = \frac{\langle g; 1 \rangle}{\|1\|^2} = \frac{\iint_{\Omega} g \rho \, dx \, dy}{\iint_{\Omega} \rho \, dx \, dy},$$

is a weighted average of the initial velocity  $g(x, y) = u_t(0, x, y)$  over the domain. In the case of the wave equation, the density  $\rho$  is constant, and hence

$$b_0 = \frac{1}{\text{area } \Omega} \iint_{\Omega} g(x, y) dx dy$$

equals the average initial velocity. If the (weighted) average initial velocity  $b_0 \neq 0$  is nonzero, then the membrane will move off at an average vertical speed  $b_0$  — while quasiperiodically vibrating in any of the normal modes that have been excited by the initial displacement and/or initial velocity. Again, this is a two-dimensional formulation of our observations of a free, vibrating bar — which in turn was the continuum version of an unsupported mass–spring chain.

*Remark:* An interesting question is whether two different drums can have identical vibrational frequencies. Or, more descriptively, can one hear the shape of a drum? The answer turns out to be “no”, but for quite subtle reasons. See [drum] for a discussion.

## 17.5. Analytical Solutions of the Wave Equation.

The previous section summarized the general, qualitative features exhibited by solutions to two-dimensional vibration and wave equations. Exact analytical formulas are, of course, harder to come by. In this section, we analyze the two most important special cases — rectangular and circular membranes.

### *Vibration of a Rectangular Drum*

Let us first consider the vibrations of a membrane in the shape of a rectangle

$$R = \{ 0 < x < a, 0 < y < b \}$$

with side lengths  $a$  and  $b$ , whose sides are fixed to the  $(x, y)$ –plane. Thus, we seek to solve the wave equation

$$u_{tt} = c^2 \Delta u = c^2(u_{xx} + u_{yy}), \quad 0 < x < a, \quad 0 < y < b, \quad (17.93)$$

subject to the initial and boundary conditions

$$\begin{aligned} u(t, 0, y) = v(t, a, y) = 0 = v(t, x, 0) = v(t, x, b), & \quad 0 < x < a, \\ u(0, x, y) = f(x, y), \quad u_t(0, x, y) = g(x, y), & \quad 0 < y < b. \end{aligned} \quad (17.94)$$

As we saw in Section 17.2 the eigenvalues and eigenfunctions for the associated Helmholtz equation

$$c^2(v_{xx} + v_{yy}) + \lambda v = 0, \quad (x, y) \in R, \quad (17.95)$$

on a rectangle, subject to the homogeneous Dirichlet boundary conditions

$$v(0, y) = v(a, y) = 0 = v(x, 0) = v(x, b), \quad 0 < x < a, \quad 0 < y < b, \quad (17.96)$$

are

$$v_{m,n}(x, y) = \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \quad \text{where} \quad \lambda_{m,n} = \pi^2 c^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right), \quad (17.97)$$

with  $m, n = 1, 2, \dots$ . The fundamental frequencies of vibration are the square roots of the eigenvalues, so

$$\omega_{m,n} = \sqrt{\lambda_{m,n}} = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}. \quad (17.98)$$

The frequencies will depend upon the underlying geometry — meaning the side lengths — of the rectangle, as well as the wave speed  $c$ , which in turn is a function of the membrane's density and stiffness, (17.84). The higher the wave speed, or the smaller the rectangle, the faster the vibrations. In layman's terms, (17.98) quantifies the observation that smaller, stiffer drums made of less dense material vibrate faster.

According to (17.85), the normal modes of vibration of our rectangle are

$$\begin{aligned} u_{m,n}(t, x, y) &= \cos \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} t \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \\ \tilde{u}_{m,n}(t, x, y) &= \sin \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} t \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \end{aligned} \quad (17.99)$$

The general solution can then be written as a double Fourier series

$$u(t, x, y) = \sum_{m,n=1}^{\infty} [a_{m,n} u_{m,n}(t, x, y) + b_{m,n} \tilde{u}_{m,n}(t, x, y)].$$

in the normal modes. The coefficients  $a_{m,n}, b_{m,n}$  are fixed by the initial displacement  $u(0, x, y) = f(x, y)$  and the initial velocity  $u_t(0, x, y) = g(x, y)$ , as in (17.90). The usual orthogonality relations among the eigenfunctions imply

$$\begin{aligned} a_{m,n} &= \frac{\langle v_{m,n}; f \rangle}{\|v_{m,n}\|^2} = \frac{4}{ab} \int_0^b \int_0^a f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy, \\ b_{m,n} &= \frac{\langle v_{m,n}; g \rangle}{\omega_{m,n} \|v_{m,n}\|^2} = \frac{4}{\pi c \sqrt{m^2 b^2 + n^2 a^2}} \int_0^b \int_0^a g(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy. \end{aligned} \quad (17.100)$$

Since the fundamental frequencies are not rational multiples of each other, the general solution is a genuinely quasi-periodic superposition of the various normal modes.

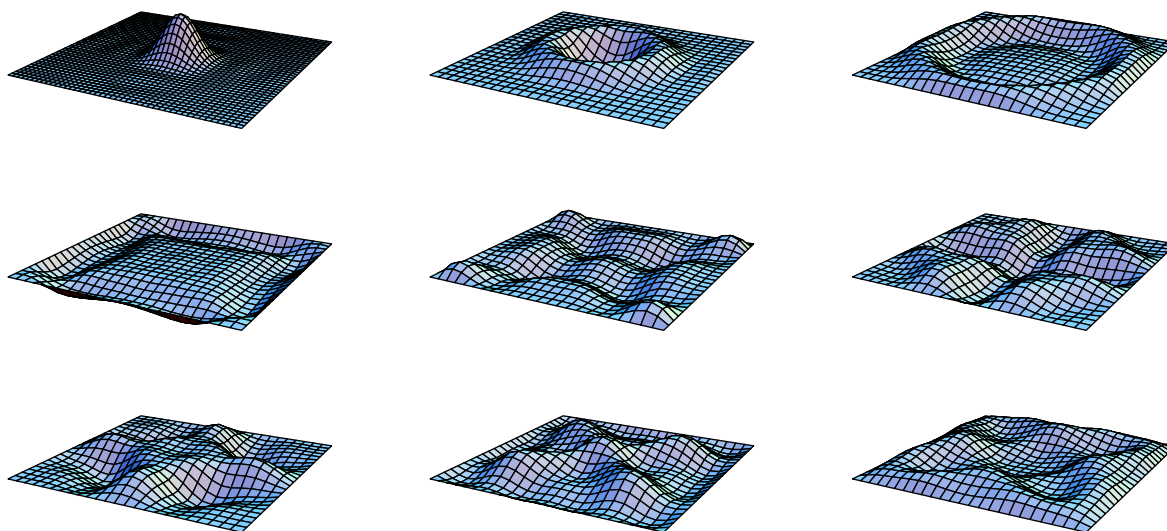
In Figure 17.7, we plot the solution resulting from the initially concentrated displacement<sup>†</sup>

$$u(0, x, y) = f(x, y) = e^{-100[(x-.5)^2 + (y-.5)^2]}$$

at the center of a unit square, so  $a = b = 1$  and the wave speed  $c = 1$ . The plots are at successive times 0, .02, .04, ..., 1.6. Note that, unlike a one-dimensional string where a

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<sup>†</sup> The alert reader may object that the initial displacement  $f(x, y)$  does not exactly satisfy the Dirichlet boundary conditions on the edges of the rectangle. But this does not prevent the existence of a well-defined solution to the initial value problem, whose initial boundary discontinuities will subsequently propagate inside the rectangle. However, they are so tiny as to be unnoticeable in the solution graphs.



**Figure 17.7.** Vibrations of a Square.

concentrated displacement remains concentrated at all subsequent times and periodically repeats, the initial displacement spreads out in a radially symmetric manner and propagates to the edges of the rectangle, where it reflects and then interacts with itself. However, owing to the quasi-periodicity of the solution, the displacement of the drum never exactly repeats itself, and the initial concentrated signal never quite reforms in the rectangle's center.

### *Vibration of a Circular Drum*

Let us next analyze the vibrations of a circular membrane. As always, we build up the solution as a quasi-periodic linear combination of the normal modes, which, by (17.85), are specified by the eigenfunctions for the associated Helmholtz boundary value problem.

As we saw in Section 17.2, the eigenfunctions of the Helmholtz equation on a disk of radius 1, say, subject to homogeneous Dirichlet boundary conditions, are products of trigonometric and Bessel functions:

$$\begin{aligned} v_{m,n}(r, \theta) &= J_m(\zeta_{m,n} r) \cos m\theta, & m &= 0, 1, 2, \dots, \\ \tilde{v}_{m,n}(r, \theta) &= J_m(\zeta_{m,n} r) \sin m\theta, & n &= 1, 2, 3, \dots \end{aligned} \quad (17.101)$$

Here  $r, \theta$  are the usual polar coordinates, while  $\zeta_{m,n} > 0$  denotes the  $n^{\text{th}}$  (positive) root of the  $m^{\text{th}}$  order Bessel function  $J_m(z)$ , cf. (17.58). The corresponding eigenvalue is its square,  $\lambda_{m,n} = \zeta_{m,n}^2$ , and hence the natural frequencies of vibration are equal to the Bessel roots, scaled by the wave speed:

$$\omega_{m,n} = c \sqrt{\lambda_{m,n}} = c \zeta_{m,n}. \quad (17.102)$$

A table of their values (for the case  $c = 1$ ) can be found in the preceding section. The Bessel roots do not follow any easily discernible pattern, and are certainly not rational multiples of each other. Thus, the vibrations of a circular drum are also truly quasi-periodic.

The frequencies  $\omega_{0,n} = c\zeta_{0,n}$  correspond to simple eigenvalues, with a single radially symmetric eigenfunction  $J_0(\zeta_{0,n}r)$ , while the “angular modes”  $\omega_{m,n}$ , for  $m > 0$ , are double, each possessing two linearly independent eigenfunctions (17.101). According to the general formula (17.85), each eigenfunction engenders two independent normal modes of vibration, having the explicit forms

$$\begin{aligned} \cos c\zeta_{m,n}t \cos m\theta J_m(\zeta_{m,n}r), & \quad \cos c\zeta_{m,n}t \sin m\theta J_m(\zeta_{m,n}r), \\ \sin c\zeta_{m,n}t \cos m\theta J_m(\zeta_{m,n}r), & \quad \sin c\zeta_{m,n}t \sin m\theta J_m(\zeta_{m,n}r). \end{aligned} \quad (17.103)$$

The general solution is written as a Fourier–Bessel series:

$$\begin{aligned} u(t, r, \theta) = \frac{1}{2} \sum_{n=1}^{\infty} [a_{0,n} \cos c\zeta_{0,n}t + c_{0,n} \sin c\zeta_{0,n}t] J_m(\zeta_{m,n}r) \\ + \sum_{m,n=1}^{\infty} [(a_{m,n} \cos c\zeta_{m,n}t + c_{m,n} \sin c\zeta_{m,n}t) \cos m\theta \\ + (b_{m,n} \cos c\zeta_{m,n}t + d_{m,n} \sin c\zeta_{m,n}t) \sin m\theta] J_m(\zeta_{m,n}r), \end{aligned} \quad (17.104)$$

whose coefficients  $a_{m,n}, b_{m,n}, c_{m,n}, d_{m,n}$  are determined, as usual, by the initial displacement and velocity of the membrane. In Figure [vdisk](#), the vibrations due to an initially concentrated displacement are displayed. Again, the motion is only quasi-periodic and never quite returns to the original configuration.

*Remark:* As we learned in Section 14.4, the natural frequencies of vibration a (homogeneous) one-dimensional medium, e.g., a violin string or a column of air in a flute, are integer multiples of each other. This has the important consequence that any resulting vibration is periodic in time. Musically, the overtones in such a one-dimensional instrument are integer multiples of each other, and so the music sounds harmonic to our ear. On the other hand, the natural frequencies of circular and rectangular drums are irrationally related, and the vibrations are only quasi-periodic. As a result, we hear a percussive sound! Thus, for some reason, our appreciation of music is psychologically attuned to the differences between rationally related/periodic and irrationally related/quasi-periodic vibrations.

### Scaling and Symmetry

Symmetry methods can be effectively employed in the analysis of the wave equation. Let us consider the simultaneous rescaling

$$t \mapsto \alpha t, \quad x \mapsto \beta x, \quad y \mapsto \beta y, \quad (17.105)$$

of time and space, whose effect is to change the function  $u(t, x, y)$  into a rescaled version

$$U(t, x, y) = u(\alpha t, \beta x, \beta y). \quad (17.106)$$

The chain rule is employed to relate their derivatives:

$$\frac{\partial^2 U}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial t^2}, \quad \frac{\partial^2 U}{\partial x^2} = \beta^2 \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial^2 U}{\partial y^2} = \beta^2 \frac{\partial^2 u}{\partial y^2}.$$

Therefore, if  $u$  satisfies the wave equation

$$u_{tt} = c^2 \Delta u,$$

then  $U$  satisfies the rescaled wave equation

$$U_{tt} = \frac{\alpha^2 c^2}{\beta^2} \Delta U = \tilde{c}^2 \Delta U, \quad \text{where the rescaled wave speed is } \tilde{c} = \frac{\alpha c}{\beta}. \quad (17.107)$$

In particular, rescaling only time by setting  $\alpha = 1/c$ ,  $\beta = 1$ , results in a unit wave speed  $\tilde{c} = 1$ . In other words, we are free to choose our unit of time measurement so as to fix the wave speed equal to 1.

If we set  $\alpha = \beta$ , scaling space and time in the same proportion, then the wave speed does not change,  $\tilde{c} = c$ , and so

$$t \mapsto \beta t, \quad x \mapsto \beta x, \quad y \mapsto \beta y, \quad (17.108)$$

defines a *symmetry transformation* for the wave equation: If  $u(t, x, y)$  is any solution to the wave equation, then so is its rescaled version

$$U(t, x, y) = u(\beta t, \beta x, \beta y) \quad (17.109)$$

for any choice of scale parameter  $\beta \neq 0$ . Observe that if  $u(t, x, y)$  is defined on a domain  $\Omega$ , then the rescaled solution  $U(t, x, y)$  will be defined on the rescaled domain

$$\tilde{\Omega} = \frac{1}{\beta} \Omega = \left\{ \left( \frac{x}{\beta}, \frac{y}{\beta} \right) \mid (x, y) \in \Omega \right\} = \{ (x, y) \mid (\beta x, \beta y) \in \Omega \}. \quad (17.110)$$

For instance, the scaling parameter  $\beta = 2$  has the effect of halving the size of the domain. The normal modes for the rescaled domain have the form

$$\begin{aligned} U_n(t, x, y) &= u_n(\beta t, \beta x, \beta y) = \cos(\beta \omega_n t) v_n(\beta x, \beta y), \\ \tilde{U}_n(t, x, y) &= \tilde{u}_n(\beta t, \beta x, \beta y) = \sin(\beta \omega_n t) v_n(\beta x, \beta y), \end{aligned}$$

and hence the vibrational frequencies  $\tilde{\omega}_n = \beta \omega_n$  are scaled by the same overall factor. Thus, when  $\beta < 1$ , the rescaled membrane is larger by a factor  $1/\beta$ , and its vibrations are slowed down by the same factor  $\beta$ . For instance, a drum that is twice as large will vibrate twice as slowly, and hence have an octave lower overall tone. Musically, this means that all drums of a similar shape have the same pattern of overtones, differing only in their overall pitch, which is a function of their size, tautness and density.

In particular, choosing  $\beta = 1/R$  will rescale the unit disk into a disk of radius  $R$ . The fundamental frequencies of the rescaled disk are

$$\tilde{\omega}_{m,n} = \beta \omega_{m,n} = \frac{c}{R} \zeta_{m,n}, \quad (17.111)$$

where  $c$  is the wave speed and  $\zeta_{m,n}$  are the Bessel roots, defined in (17.58). Observe that the ratios  $\omega_{m,n}/\omega_{m',n'}$  between vibrational frequencies remain the same, independent of the size of the disk  $R$  and the wave speed  $c$ . We define the *relative vibrational frequencies*

$$\rho_{m,n} = \frac{\omega_{m,n}}{\omega_{0,1}} = \frac{\zeta_{m,n}}{\zeta_{0,1}}, \quad \text{in proportion to} \quad \omega_{0,1} = \frac{c \zeta_{0,1}}{R} \approx 2.4 \frac{c}{R}, \quad (17.112)$$

which is the drum's dominant or lowest vibrational frequency. The relative frequencies  $\rho_{m,n}$  are independent of the size, stiffness or composition of the drum membrane. In the following table, we display a list of all relative vibrational frequencies (17.112) that are  $< 6$ . Once the lowest frequency  $\omega_{0,1}$  has been determined — either theoretically, numerically or experimentally — all the higher overtones  $\omega_{m,n} = \rho_{m,n} \omega_{0,1}$  are obtained by rescaling.

Relative Vibrational Frequencies of a Circular Disk

| $n \backslash m$ | 0        | 1        | 2        | 3        | 4        | 5        | 6        | 7        | 8        | 9        | ... |
|------------------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|-----|
| 1                | 1.000    | 1.593    | 2.136    | 2.653    | 3.155    | 3.647    | 4.132    | 4.610    | 5.084    | 5.553    | ... |
| 2                | 2.295    | 2.917    | 3.500    | 4.059    | 4.601    | 5.131    | 5.651    | $\vdots$ | $\vdots$ | $\vdots$ |     |
| 3                | 3.598    | 4.230    | 4.832    | 5.412    | 5.977    | $\vdots$ | $\vdots$ |          |          |          |     |
| 4                | 4.903    | 5.540    | $\vdots$ | $\vdots$ | $\vdots$ |          |          |          |          |          |     |
| $\vdots$         | $\vdots$ | $\vdots$ |          |          |          |          |          |          |          |          |     |

## 17.6. Nodal Curves.

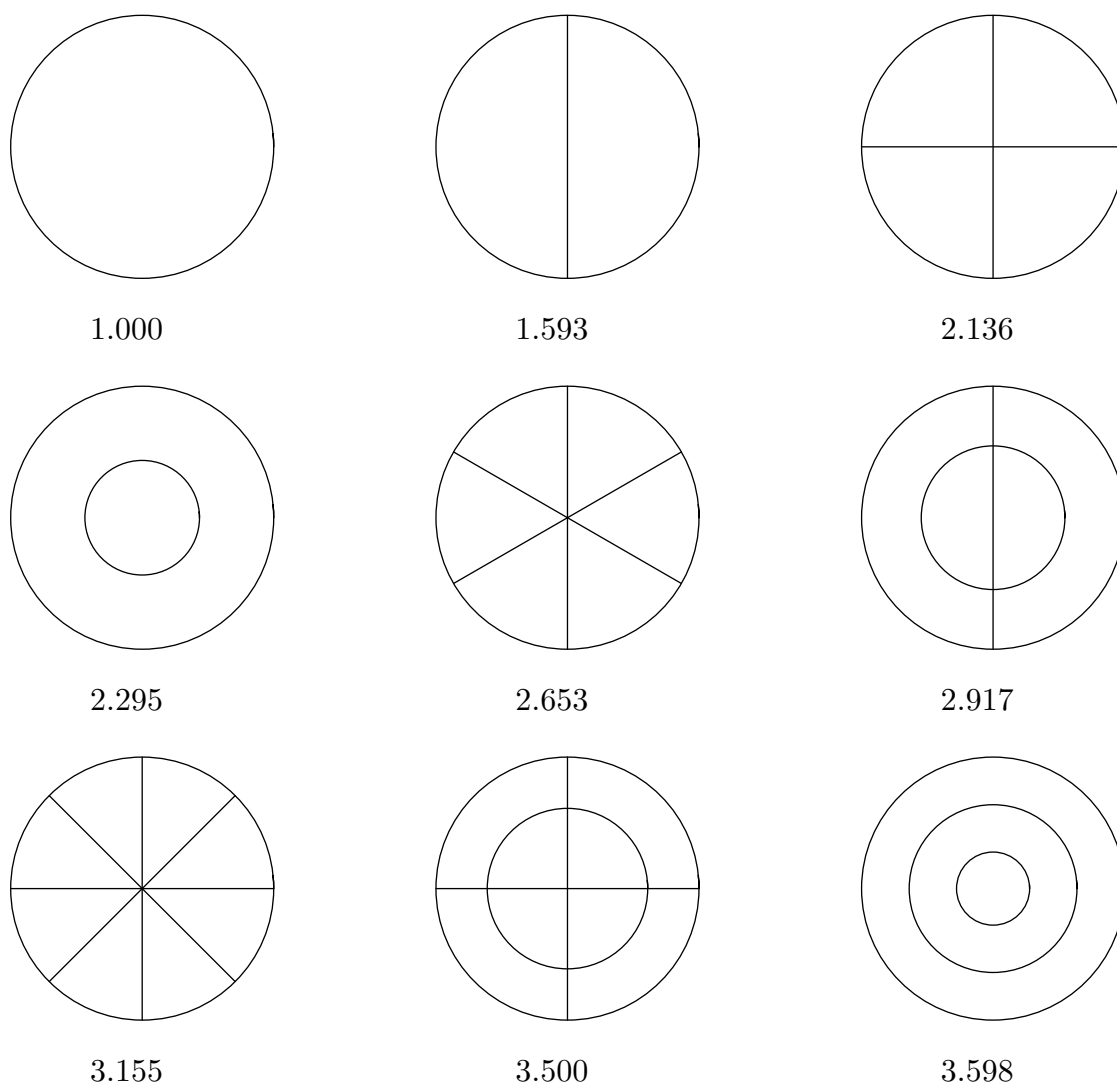
When a membrane vibrates, the individual points move up and down in a quasi-periodic manner. As such, correlations between the motions of different points are not immediately evident. However, if the membrane is set to vibrate in a pure eigenmode, say

$$u_n(t, x, y) = \cos(\omega_n t) v_n(x, y),$$

then all points move up and down at a common frequency  $\omega_n = \sqrt{\lambda_n}$ , which is the square root of the eigenvalue corresponding to the eigenfunction  $v_n(x, y)$ . The exceptions are the points where the eigenfunction vanishes:

$$v_n(x, y) = 0. \quad (17.113)$$

Such points remain stationary. The set of all points  $(x, y) \in \Omega$  that satisfy (17.113) is known as the  $n^{\text{th}}$  *nodal set* of the domain  $\Omega$ . Scattering small particles (e.g., fine sand) over the membrane performing such a pure vibration, will enable us to see the nodal set, because the particles will, though random movement over the oscillating regions of the membrane, tend to accumulate along the stationary nodal curves.



**Figure 17.8.** Nodal Curves and Relative Vibrational Frequencies of a Circular Membrane.

It can be shown that, in general, each nodal set consists of a finite system of *nodal curves*. The nodal curves intersect at critical points of the eigenfunction, where  $\nabla v_n = \mathbf{0}$ , and thereby partition the membrane into *nodal regions*. Points lying in a common nodal region all vibrate in tandem, so that all points in a common nodal region are either up or down, except, momentarily, when the *entire* membrane has zero displacement. Adjacent nodal regions, lying on the opposite sides of a nodal curve, vibrate in opposing directions — when one side is up, the other is down, and then, as the membrane becomes momentarily flat, simultaneously switch directions.

**Example 17.5.** *Circular Drums.* Since the eigenfunctions (17.101) for a disk are products of trigonometric functions in the angular variable and Bessel functions of the radius, the nodal curves for the normal modes of vibrations of a circular membrane are rays emanating from and circles centered at the origin. Thus, the nodal regions are annu-

lar sectors. Pictures of the nodal curves for the first nine normal modes indexed by their relative frequencies are plotted in Figure 17.8. Representative displacements of the membrane in each of the first twelve modes can be found in Figure 17.4. The dominant (lowest frequency) mode is the only one that has no nodal curves; it has the form of a radially symmetric bump where the entire membrane flexes up and down. Every other mode has at least one nodal curve. For instance, the next lowest modes vibrate proportionally faster at a relative frequency  $\rho_{1,1} \approx 1.593$ . The most general solution with this vibrational frequency is a linear combination  $\alpha u_{1,1} + \beta \tilde{u}_{1,1}$  of the two eigensolutions. Each combination has a single diameter as a nodal curve, whose slope depends upon the coefficients  $\alpha, \beta$ . Here, the two semicircular halves of the drum vibrate in opposing directions — when the top half is up, the bottom half is down and vice versa. The next set of modes have two perpendicular diameters as nodal curves; the four quadrants of the drum vibrate in tandem, with opposite quadrants having the same displacements. Next in increasing order of vibrational frequency is a single mode, with a circular nodal curve whose (relative) radius  $\zeta_{0,2}/\zeta_{0,1} \approx .43565$  equals the ratio of the first two roots of the order zero Bessel function; see Exercise ■ for a justification. In this case, the inner disk and the outer annulus vibrate in opposing directions.

**Example 17.6. Rectangular Drums.** For a general rectangular drum, the nodal curves are relatively uninteresting. Since the normal modes (17.99) are separable products of trigonometric functions in the coordinate variables  $x, y$ , the nodal curves are regularly equi-spaced straight lines parallel to the sides of the rectangle. The internodal regions are small rectangles, all of the same size and shape, with adjacent rectangles vibrating in opposite directions.

A more interesting collection of nodal curves occurs when the rectangle admits multiple eigenvalues — so-called *accidental degeneracies*. Two of the eigenvalues (17.97) coincide,  $\lambda_{m,n} = \lambda_{k,l}$ , if and only if

$$\frac{m^2}{a^2} + \frac{n^2}{b^2} = \frac{k^2}{a^2} + \frac{l^2}{b^2} \quad (17.114)$$

where  $(m, n) \neq (k, l)$  are distinct pairs of positive integers. In such situations, the distinct eigenmodes happen to vibrate with a common frequency  $\omega = \omega_{m,n} = \omega_{k,l}$ . Consequently, any linear combination of the eigenmodes, e.g.,

$$\cos \omega t \left( \alpha \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} + \beta \sin \frac{k\pi x}{a} \sin \frac{l\pi y}{b} \right), \quad \alpha, \beta \in \mathbb{R},$$

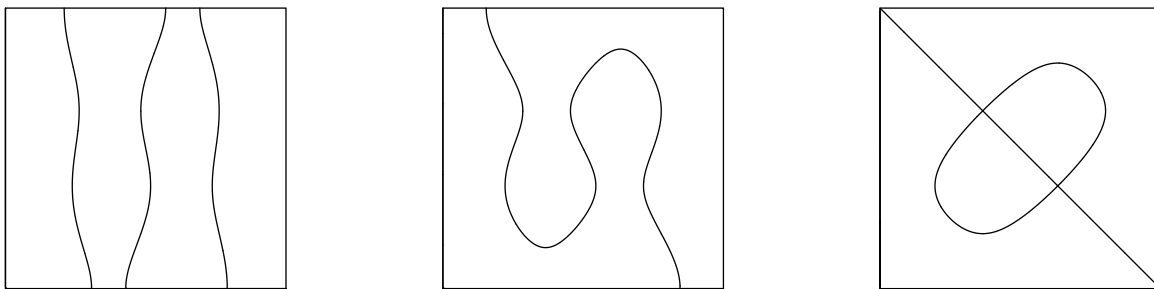
is a pure vibration, and hence also qualifies as a normal mode. The associated nodal curves

$$\alpha \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} + \beta \sin \frac{k\pi x}{a} \sin \frac{l\pi y}{b} = 0 \quad (17.115)$$

have a more intriguing geometry, which can change dramatically as  $\alpha, \beta$  vary.

For example, on a unit square  $R = \{0 < x, y < 1\}$ , an accidental degeneracy occurs whenever

$$m^2 + n^2 = k^2 + l^2 \quad (17.116)$$



**Figure 17.9.** Some Nodal Curves for a Square Membrane.

for distinct pairs of positive integers  $(m, n) \neq (k, l)$ . The simplest possibility arises whenever  $m \neq n$ , in which case we can merely reverse the order, setting  $k = n$ ,  $l = m$ . In Figure 17.9 we plot three sample nodal curves

$$\sin 4\pi x \sin \pi y + \beta \sin \pi x \sin 4\pi y = 0, \quad \text{with} \quad \beta = .2, .5, 1,$$

corresponding to the three different linear combinations of the eigenfunctions with  $m = l = 4$ ,  $n = k = 1$ . The associated vibrational frequency is  $\omega_{4,1} = \pi c \sqrt{17}$ .

*Remark:* Classifying such accidental degeneracies takes us into the realm of number theory, [10, 40]. The simplest (square) case (17.116) asks one to determine all integer points that lie on a common circle.

*Remark:* An interesting question is whether a circular disk has accidental degeneracies, which would occur if two different Bessel roots were to coincide. However, it is known, [186; p. 129], that  $\zeta_{m,n} \neq \zeta_{k,l}$  whenever  $(m, n) \neq (k, l)$ . Thus, a disk has no such degeneracies, and all nodal curves are circles and rays around the origin.