

Chapter 18

Partial Differential Equations in Three-Dimensional Space

At last we have ascended the dimensional ladder to its ultimate rung (at least for those of us living in a three-dimensional universe): partial differential equations in physical space. As in the one and two-dimensional settings developed in the preceding chapters, the three key examples are the three-dimensional Laplace equation, modeling equilibrium configurations of solid bodies, the three-dimensional wave equation, governing vibrations of solids, liquids, gasses, and electromagnetic waves, and the three-dimensional heat equation, modeling basic spatial diffusion processes.

Fortunately, almost everything of importance has already appeared in the one- and two-dimensional situations, and appending a third dimension is, for the most part, simply a matter of appropriately adapting the constructions. We have already seen the basic underlying solution techniques: separation of variables and Green's functions or fundamental solutions. (Unfortunately, the most powerful of our planar tools, conformal mapping, does *not* carry over to higher dimensions.) In three-dimensional problems, separation of variables is applicable in rectangular, cylindrical and spherical coordinates. The first two do not produce anything fundamentally new, and are therefore relegated to the exercises. Separation in spherical coordinates leads to spherical harmonics and spherical Bessel functions, whose properties are investigated in some detail. These new special functions play important roles in a number of physical systems, including the quantum theory of atomic structure that underlies the spectral and chemical properties of atoms.

The Green's function for the three-dimensional Poisson equation in space can be identified as the classic Newtonian (and Coulomb) $1/r$ potential. The fundamental solution for the three-dimensional heat equation can be easily guessed from its one- and two-dimensional versions. The three-dimensional wave equation, surprisingly, has an explicit, although more intricate, solution formula of d'Alembert form, due to Poisson. Paradoxically, the best way to treat the two-dimensional version is by "descending" from the simpler three-dimensional formula. This result highlights a remarkable difference between waves in planar and spacial media. In three-dimensions, Huygens' principle states that waves emanating from a localized initial disturbance remain localized as they propagate through space. In contrast, in two dimensions, initially concentrated pulses leave a slowly decaying remnant that never entirely disappears.

18.1. The Laplace and Poisson Equations.

We begin our investigations, as usual, with systems in equilibrium, deferring dynamics

until later. The prototypical equilibrium system is the three-dimensional Laplace equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0, \quad (18.1)$$

in which $\mathbf{x} = (x, y, z)^T$ represents rectangular coordinates on $\mathbb{R}^3$. The solutions $u(x, y, z)$ continue to be known as *harmonic functions*. The Laplace equation models unforced equilibria; *Poisson's equation* is the inhomogeneous version

$$-\Delta u = f(x, y, z), \quad (18.2)$$

where the inhomogeneity f represents some form of external forcing.

The basic boundary value problem for the Laplace or the Poisson equation seeks a solution inside a bounded domain $\Omega \subset \mathbb{R}^3$ subject to either *Dirichlet boundary conditions*, prescribing the function values

$$u = h \quad \text{on} \quad \partial\Omega, \quad (18.3)$$

or *Neumann boundary conditions* prescribing its normal derivative or flux

$$\frac{\partial u}{\partial \mathbf{n}} = k \quad \text{on} \quad \partial\Omega, \quad (18.4)$$

or *mixed boundary conditions* in which one imposes Dirichlet conditions on part of the boundary and Neumann conditions on the remainder. Keep in mind that the boundary of the solid domain Ω consists of one or more piecewise smooth closed surfaces, which will be oriented by the outwards unit normal $\mathbf{n}$;

The boundary value problems for the three-dimensional Laplace and Poisson equations govern a wide variety of physical systems, including:

- (a) *Heat conduction*: In this application, u represents the equilibrium temperature in a solid body. Dirichlet conditions correspond to fixing the temperature on the bounding surface(s), whereas homogeneous Neumann conditions correspond to an insulated boundary, i.e., one which does not allow any heat flux. The inhomogeneity f represents some form of internal heat source.
- (b) *Ideal fluid flow*: Here u represents the velocity potential for an incompressible, irrotational steady state fluid flow inside a container, Ω , with velocity vector field $\mathbf{v} = \nabla u$. Homogeneous Neumann boundary conditions correspond to a solid boundary which the fluid cannot penetrate.
- (c) *Elasticity*: In certain restricted situations, u represents an equilibrium deformation of a solid body, e.g., the radial deformation of a solid ball. Fully three-dimensional elasticity is governed by a more complicated system of partial differential equations, which can be found in Example 21.9.
- (d) *Electrostatics*: In applications to electromagnetism, u represents the electric potential in a conducting medium; its gradient ∇u prescribes the electromotive force on a charged particle. The inhomogeneity represents an electrostatic force field.

- (e) *Gravitation*: The Newtonian gravitational potential in flat empty space is also prescribed by the Laplace equation. (In contrast, general relativity requires a vastly more complicated nonlinear system of partial differential equations, [132].)

Self-Adjoint Formulation and Minimum Principle

The Laplace and Poisson equations naturally fit into the general self-adjoint equilibrium framework summarized in Section 14.7. The construction is a straightforward adaptation of the planar version of Section 15.4. We introduce the L^2 inner products

$$\begin{aligned}\langle u; \tilde{u} \rangle &= \iiint_{\Omega} u(x, y, z) \tilde{u}(x, y, z) dx dy dz, \\ \langle \mathbf{v}; \tilde{\mathbf{v}} \rangle &= \iiint_{\Omega} \mathbf{v}(x, y, z) \cdot \tilde{\mathbf{v}}(x, y, z) dx dy dz,\end{aligned}\tag{18.5}$$

between scalar fields $u, \tilde{u}$, and between vector fields $\mathbf{v}, \tilde{\mathbf{v}}$ defined on the domain $\Omega \subset \mathbb{R}^3$. We assume that the functions in question are sufficiently nice that these inner products are well-defined; if Ω is unbounded, this requires that, at large distances, they decay reasonably rapidly to zero.

When subject to suitable homogeneous boundary conditions, the three-dimensional Laplace equation can be placed in our standard self-adjoint form

$$-\Delta u = -\nabla \cdot \nabla u = \nabla^* \circ \nabla u.\tag{18.6}$$

This relies on the fact that the adjoint of the gradient operator with respect to the L^2 inner products (18.5) is minus the divergence operator:

$$\nabla^* \mathbf{v} = -\nabla \cdot \mathbf{v}.\tag{18.7}$$

As usual, the determination of the adjoint rests on an integration by parts formula, which, in three-dimensional space, follows from the Divergence Theorem B.35. The first step is to establish the three-dimensional analog of Green's formula (15.88). We apply the divergence identity (B.85) to the product $u\mathbf{v}$ of a scalar field u and a vector field $\mathbf{v}$, leading to

$$\iiint_{\Omega} (u \nabla \cdot \mathbf{v} + \nabla u \cdot \mathbf{v}) dx dy dz = \iiint_{\Omega} \nabla \cdot (u \mathbf{v}) dx dy dz = \iint_{\partial\Omega} u (\mathbf{v} \cdot \mathbf{n}) dS.\tag{18.8}$$

Rearranging the terms produces the desired integration by parts formula for triple integrals:

$$\iiint_{\Omega} (\nabla u \cdot \mathbf{v}) dx dy dz = \iint_{\partial\Omega} u (\mathbf{v} \cdot \mathbf{n}) dS - \iiint_{\Omega} u (\nabla \cdot \mathbf{v}) dx dy dz.\tag{18.9}$$

The boundary integral will vanish provided either $u = 0$ or $\mathbf{v} \cdot \mathbf{n} = \mathbf{0}$ at each point on $\partial\Omega$. When $u = 0$ on all of $\partial\Omega$, we have homogeneous Dirichlet conditions. Setting $\mathbf{v} \cdot \mathbf{n} = \mathbf{0}$ everywhere on $\partial\Omega$ results in the homogeneous Neumann boundary value problem; see Section 15.4 for a detailed explanation. Finally, when $u = 0$ on part of $\partial\Omega$ and $\mathbf{v} \cdot \mathbf{n} = \mathbf{0}$ on the rest leads to the mixed boundary value problem. Thus, subject to one of these choices, the integration by parts formula (18.9) reduces to

$$\langle \nabla u; \mathbf{v} \rangle = \langle u; -\nabla \cdot \mathbf{v} \rangle,\tag{18.10}$$

which suffices to prove the adjoint formula (18.7).

Remark: Adopting more general weighted inner products results in a more general elliptic boundary value problem. See Exercise ■ for details.

According to the abstract Theorem 7.60, the self-adjoint formulation (18.6) implies positive semi-definiteness of the boundary value problem, and positive definiteness provided $\ker \nabla = \{0\}$. Since, on a connected domain, only constant functions are annihilated by the gradient operator — see Theorem B.28 — both the Dirichlet and mixed boundary value problems are positive definite, while the Neumann boundary value problem is only semi-definite.

Finally, in the positive definite cases, the solution can be characterized by the three-dimensional version of the Dirichlet minimization principle (15.103).

Theorem 18.1. *The solution $u(x, y, z)$ to the Poisson equation (18.2) subject to homogeneous Dirichlet or mixed boundary conditions (18.3) is characterized as the unique function that minimizes the Dirichlet integral*

$$\frac{1}{2} \|\nabla u\|^2 - \langle u; f \rangle = \iiint_{\Omega} \left[\frac{1}{2} (u_x^2 + u_y^2 + u_z^2) - f u \right] dx dy dz \quad (18.11)$$

among all C^1 functions that satisfy the prescribed boundary conditions.

As in the two-dimensional version discussed in Section 15.4, the minimization principle continues to hold without modification in the case of the inhomogeneous Dirichlet boundary value problem. Modifications for the inhomogeneous mixed boundary value problem are discussed in Exercise ■. The three-dimensional finite element method for constructing numerical solutions to such boundary value problems rests on the associated minimization principle; see [153, 109] for details.

18.2. Separation of Variables.

With conformal mapping no longer a viable option in three dimensional space, separation of variables reasserts its primacy for generating explicit solutions to the Laplace equation. As always, its applicability is unfortunately restricted to rather special, but important, geometrical configurations. In three-dimensional space, the simplest separable cases are problems formulated on rectangular, cylindrical or spherical domains. Since the first two are straightforward extensions of their two-dimensional counterparts, we will only discuss spherically separable solutions in any detail. The simplest domain to which the separation of variables method applies is a rectangular box:

$$B = \{ 0 < x < a, 0 < y < b, 0 < z < c \}.$$

For functions of three variables, one begins the separation process by splitting off one of them, by setting $u(x, y, z) = v(x) w(y, z)$, say. The function $v(x)$ satisfies a simple second order ordinary differential equation, while $w(y, z)$ solves the two-dimensional Helmholtz equation, which is then separated by writing $w(y, z) = p(y) q(z)$. The resulting fully separated solutions $u(x, y, z) = v(x) p(y) q(z)$ are (mostly) products of trigonometric and hyperbolic functions. Complete details of the technique and the resulting series solution are relegated to Exercise ■.

In the case when the domain is a cylinder, one passes to cylindrical coordinates r, θ, z to effect the separation. The resulting separable solutions $u(r, \theta, z) = v(r, \theta), w(z) = p(r)q(\theta), w(z)$ are products of Bessel functions of the cylindrical radius r , trigonometric functions of the polar angle θ , and hyperbolic functions of z . Details are outlined in Exercise ■.

The most interesting case is that of spherical coordinates, which we proceed to analyze in detail in the following subsection.

Remark: Beyond these three well-known cases, there are, in fact, a total of eleven different coordinate systems in which the three-dimensional Laplace equation separates. See [131, 134, 136] for details on the more exotic types of separation, including ellipsoidal, toroidal, and parabolic spheroidal coordinates. The resulting separable solutions lead to new classes of special functions.

Laplace's Equation in a Ball

Suppose a solid ball (e.g., the earth), is subject a specified steady temperature distribution on its spherical boundary. Our task is to determine the equilibrium temperature within the ball. To simplify matters, we assume that the body is composed of an isotropic, homogeneous medium, and shall choose units in which its radius equals 1. Then, to find the equilibrium temperature within the ball, we must solve the Dirichlet boundary value problem

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} &= 0, & x^2 + y^2 + z^2 &< 1, \\ u(x, y, z) &= h(x, y, z), & x^2 + y^2 + z^2 &= 1. \end{aligned} \quad (18.12)$$

Problems in spherical geometries tend to simplify when re-expressed in terms of spherical coordinates r, φ, θ , as defined by the usual formulae

$$x = r \sin \varphi \cos \theta, \quad y = r \sin \varphi \sin \theta, \quad z = r \cos \varphi. \quad (18.13)$$

Here $0 \leq \theta < 2\pi$ measures the *azimuthal angle* or *longitude*, while $0 \leq \varphi \leq \pi$ measures the *zenith angle* or *latitude*.

Warning: We use the mathematician's convention for spherical coordinates. Physicists often interchange the notation for the azimuthal and zenith angles; see Example B.8 for a detailed discussion.

In spherical coordinates, the Laplace equation for[†] $u(r, \varphi, \theta)$ takes the form

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} + \frac{\cos \varphi}{r^2 \sin \varphi} \frac{\partial u}{\partial \varphi} + \frac{1}{r^2 \sin^2 \varphi} \frac{\partial^2 u}{\partial \theta^2} = 0. \quad (18.14)$$

This important formula is the final result of a fairly nasty chain rule computation, whose details are left to the motivated reader. (Set aside lots of paper and keep an eraser handy!)

[†] *Warning:* See Section 15.2 for our convention on rewriting functions in new coordinates.

To construct separable solutions to the spherical coordinate form (18.14) of the Laplace equation, we begin by separating off the radial part of the solution, setting

$$u(r, \varphi, \theta) = v(r) w(\varphi, \theta). \quad (18.15)$$

Substituting this ansatz into (18.14), multiplying the resulting equation through by $\frac{r^2}{v w}$, and then placing all the terms involving r on one side yields

$$\frac{1}{v} \left(r^2 \frac{d^2 v}{dr^2} + 2r \frac{dv}{dr} \right) = -\frac{1}{w} \Delta_S w = \mu, \quad (18.16)$$

where μ is the separation constant, and

$$\Delta_S w = \frac{\partial^2 w}{\partial \varphi^2} + \frac{\cos \varphi}{\sin \varphi} \frac{\partial w}{\partial \varphi} + \frac{1}{\sin^2 \varphi} \frac{\partial^2 w}{\partial \theta^2}. \quad (18.17)$$

The second order differential operator Δ_S , which contains only the angular components of the full Laplacian operator Δ , is of particular significance. It is known as the *spherical Laplacian*, and governs the equilibrium and dynamics of thin spherical shells, cf. Example 18.13.

Returning to equation (18.16), our usual separation argument applies. The left hand side depends only on r , while the right hand side depends only on the angles φ, θ . This can only occur when both sides are equal to a common separation constant, denoted by μ . As a consequence, the radial component $v(r)$ satisfies the ordinary differential equation

$$r^2 v'' + 2r v' - \mu v = 0, \quad (18.18)$$

which is of Euler type (7.51), and hence can be readily solved. We will put this equation aside for the time being, and concentrate our efforts on the more complicated part.

The angular components in (18.16) assume the form

$$\Delta_S[w] + \mu w = \frac{\partial^2 w}{\partial \varphi^2} + \frac{\cos \varphi}{\sin \varphi} \frac{\partial w}{\partial \varphi} + \frac{1}{\sin^2 \varphi} \frac{\partial^2 w}{\partial \theta^2} + \mu w = 0. \quad (18.19)$$

This second order partial differential equation can be regarded as the eigenvalue equation for the spherical Laplacian operator Δ_S , and is known as the *spherical Helmholtz equation*. To solve it, we adopt a further separation of angular variables,

$$w(\varphi, \theta) = p(\varphi) q(\theta), \quad (18.20)$$

which we substitute into (18.19). Dividing the result by the product $w = p q$, multiplying by $\sin^2 \varphi$, and then rearranging terms, we are led to the separated system

$$\frac{\sin^2 \varphi}{p} \frac{d^2 p}{d\varphi^2} + \frac{\cos \varphi \sin \varphi}{p} \frac{dp}{d\varphi} + \mu \sin^2 \varphi = -\frac{1}{q} \frac{d^2 q}{d\theta^2} = \nu.$$

The left hand side depends only on the zenith coordinate φ while the right hand side depends only on the azimuthal coordinate θ . Since these angles are independent, the only way this could hold is when the two sides equal a common separation constant, denoted

by ν . The spherical Helmholtz equation thereby splits into a pair of ordinary differential equations

$$\sin^2 \varphi \frac{d^2 p}{d\varphi^2} + \cos \varphi \sin \varphi \frac{dp}{d\varphi} + (\mu \sin^2 \varphi - \nu) p = 0, \quad \frac{d^2 q}{d\theta^2} + \nu q = 0.$$

The equation for $q(\theta)$ is easy to solve. As one circumnavigates the sphere from west to east, the azimuthal angle θ increases from 0 to 2π , so $q(\theta)$ must be a 2π periodic function. Thus, $q(\theta)$ satisfies the well-studied periodic boundary value problem treated, for instance, in (15.33). Up to constant multiple, non-zero periodic solutions occur only when the separation constant assumes one of the values $\nu = m^2$, where $m = 0, 1, 2, \dots$ is an integer, with

$$q(\theta) = \cos m\theta \quad \text{or} \quad \sin m\theta, \quad m = 0, 1, 2, \dots \quad (18.21)$$

Each positive $\nu = m^2 > 0$ admits two linearly independent 2π periodic solutions, while when $\nu = 0$, only the constant solutions are periodic.

With this information, we endeavor to solve the zenith equation

$$\sin^2 \varphi \frac{d^2 p}{d\varphi^2} + \cos \varphi \sin \varphi \frac{dp}{d\varphi} + (\mu \sin^2 \varphi - m^2) p = 0. \quad (18.22)$$

This is not easy, and constructing analytic formulas for its solutions requires some effort. The motivation behind the following steps will not be immediately apparent to the reader, since they are the result of a long, detailed study of this important differential equation by mathematicians over the last 200 years.

As an initial simplification, we will eliminate the trigonometric functions. To this end, we invoke the change of variables

$$t = \cos \varphi, \quad \text{with} \quad p(\varphi) = P(\cos \varphi) = P(t). \quad (18.23)$$

Since

$$0 \leq \varphi \leq \pi, \quad \text{we have} \quad 0 \leq \sqrt{1 - t^2} = \sin \varphi \leq 1.$$

According to the chain rule,

$$\begin{aligned} \frac{dp}{d\varphi} &= -\sin \varphi \frac{dP}{dt} = -\sqrt{1 - t^2} \frac{dP}{dt}, \\ \frac{d^2 p}{d\varphi^2} &= \sin^2 \varphi \frac{d^2 P}{dt^2} - \cos \varphi \frac{dP}{dt} = (1 - t^2) \frac{d^2 P}{dt^2} - t \frac{dP}{dt}. \end{aligned}$$

Substituting these expressions into (18.22), we conclude that $P(t)$ must satisfy

$$(1 - t^2)^2 \frac{d^2 P}{dt^2} - 2t(1 - t^2) \frac{dP}{dt} + [\mu(1 - t^2) - m^2] P = 0. \quad (18.24)$$

Unfortunately, the resulting differential equation is still not so easy to solve, but at least its coefficients are polynomials. Equation (18.24) is known as the *Legendre differential equation* of order m , and its solutions are known as *Legendre functions*, having first been employed by Legendre to study the gravitational attraction of ellipsoidal bodies.

Power series solutions to the Legendre equation can be constructed by the standard techniques presented in Appendix C. The most general solution is a new type of special function, known as a *Legendre function*, [3, 145]. However, the solutions we are actually interested in can all be written in terms of elementary algebraic functions. First of all, since $t = \cos \varphi$, the solution only needs to be defined on the interval $-1 \leq t \leq 1$. The endpoints of this interval, $t = \pm 1$, correspond to the sphere's north pole, $\varphi = 0$, and south pole, $\varphi = \pi$. Both endpoints are singular points for the Legendre equation since the coefficient $(1 - t^2)^2$ of the leading order derivative vanishes when $t = \pm 1$. In fact, both are regular singular points, as shown in Exercise C.3.44. Since ultimately we need the separable solution (18.15) to be a well-defined function of x, y, z (even at points where the spherical coordinates degenerate, i.e., on the z axis), we need $p(\varphi)$ to be well-defined at $\varphi = 0$ and π , and this requires $P(t)$ to be bounded at the singular points:

$$|P(-1)| < \infty, \quad |P(+1)| < \infty. \quad (18.25)$$

The combined boundary value problem (18.24–25) takes the form of an eigensystem, in which the separation constant μ is the eigenvalue and the non-zero solutions $P(t) \not\equiv 0$ are the associated eigenfunctions.

Mathematical justifications of the following statements can be found in Appendix C. Consider first the case $m = 0$, which assumes the simpler form

$$(1 - t^2) \frac{d^2 P}{dt^2} - 2t \frac{dP}{dt} + \mu P = 0. \quad (18.26)$$

In this case, it turns out that the eigenfunctions, i.e., solutions to the Legendre boundary value problem (18.26, 25), are the *Legendre polynomials*

$$P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n} (t^2 - 1)^n \quad (18.27)$$

that first arose in Chapter 5 as our simplest example of orthogonal polynomials. Indeed, we can now finally comprehend the reason for the orthogonality of the Legendre polynomials: they are the common eigenfunctions of a self-adjoint boundary value problem! Explicit formulas for the first few Legendre polynomials appear in (5.46).

When $m > 0$, the eigenfunctions of the Legendre boundary value problem (18.24–25) are not always polynomials. They are known as the *associated Legendre functions*, and have the explicit formula[†]

$$\begin{aligned} P_n^m(t) &= (1 - t^2)^{m/2} \frac{d^m}{dt^m} P_n(t) \\ &= (-1)^n \frac{(1 - t^2)^{m/2}}{2^n n!} \frac{d^{n+m}}{dt^{n+m}} (1 - t^2)^n, \end{aligned} \quad n = m, m + 1, \dots, \quad (18.28)$$

[†] *Warning:* Some authors include $(-1)^m$ in the formula, resulting in the opposite sign when m is odd. Another source of confusion is that many tables define the associated Legendre functions using the alternative initial factor $(t^2 - 1)^{m/2}$. But we are solely interested in values of t lying between $-1 \leq t \leq 1$, which would result in complex values for odd m .

which generalizes the Rodrigues formula (5.48) for the classical Legendre polynomials. Its proof is similar, and done in Exercise ■. Here is a list of the first few Legendre polynomials and associated Legendre functions:

$$\begin{aligned}
P_0^0(t) &= 1, & P_1^0(t) &= t, & P_1^1(t) &= \sqrt{1-t^2}, \\
P_2^0(t) &= -\frac{1}{2} + \frac{3}{2}t^2, & P_2^1(t) &= 3t\sqrt{1-t^2}, & P_2^2(t) &= 3(1-t^2), \\
P_3^0(t) &= -\frac{3}{2}t + \frac{5}{2}t^3, & P_3^1(t) &= \left(-\frac{3}{2} + \frac{15}{2}t^2\right)\sqrt{1-t^2}, & & \\
P_3^2(t) &= 15t(1-t^2), & P_3^3(t) &= 15(1-t^2)^{3/2}, & & (18.29) \\
P_4^0(t) &= \frac{3}{8} - \frac{15}{4}t^2 + \frac{35}{8}t^4, & P_4^1(t) &= \left(-\frac{15}{2} + \frac{35}{2}t^2\right)\sqrt{1-t^2}, & & \\
P_4^2(t) &= \left(-\frac{15}{2} + \frac{105}{2}t^2\right)(1-t^2), & P_4^3(t) &= 105t(1-t^2)^{3/2}, & P_4^4(t) &= 105(1-t^2)^2.
\end{aligned}$$

When $m = 2k \leq n$ is an even integer, $P_n^m(t)$ is a polynomial function, while when $m = 2k + 1 \leq n$ is odd, there is an extra factor of $\sqrt{1-t^2}$. Keep in mind that the square root is real and positive since we are restricting our attention to the interval $-1 \leq t \leq 1$. If $m > n$, formula (18.28) reduces to the zero function, and is not needed in the final tally.

Warning: Even though half of the associated Legendre functions are polynomials, only those with $m = 0$, i.e., $P_n(t) = P_n^0(t)$, are called *Legendre polynomials*.

Graphs of the first few Legendre polynomials can be found in Figure 5.4. In addition, Figure 18.1 displays the graphs of the associated Legendre functions $P_n^m(t)$ for $1 \leq m \leq n \leq 4$. Pay particular attention to the fact that, owing to the choice of normalization factor, their graphs have very different vertical scales.

The following result states that the Legendre polynomials and associated Legendre functions are a complete list of solutions to the Legendre boundary value problem (18.24–25). A proof can be found in [26].

Theorem 18.2. *Let $m \geq 0$ be a non-negative integer. Then the m^{th} order Legendre boundary value problem prescribed by (18.24–25) has eigenvalues $\mu_n = n(n+1)$ for $n = 0, 1, 2, \dots$, and associated eigenfunctions $P_n^m(t)$ where $m = 0, \dots, n$.*

Returning to the original variable φ via (18.23), Theorem 18.2 implies that our original boundary value problem

$$\sin^2 \varphi \frac{d^2 p}{d\varphi^2} + \cos \varphi \sin \varphi \frac{dp}{d\varphi} + (\mu \sin^2 \varphi - m^2) p = 0, \quad |p(0)|, |p(\pi)| < \infty, \quad (18.30)$$

has its eigenvalues and eigenfunctions expressed in terms of the Legendre functions:

$$\mu_n = n(n+1), \quad p_n^m(\varphi) = P_n^m(\cos \varphi), \quad \text{for } 0 \leq m \leq n. \quad (18.31)$$

The eigenfunction $p_n^m(\varphi)$ is, in fact, a trigonometric polynomial of degree n ; here are the

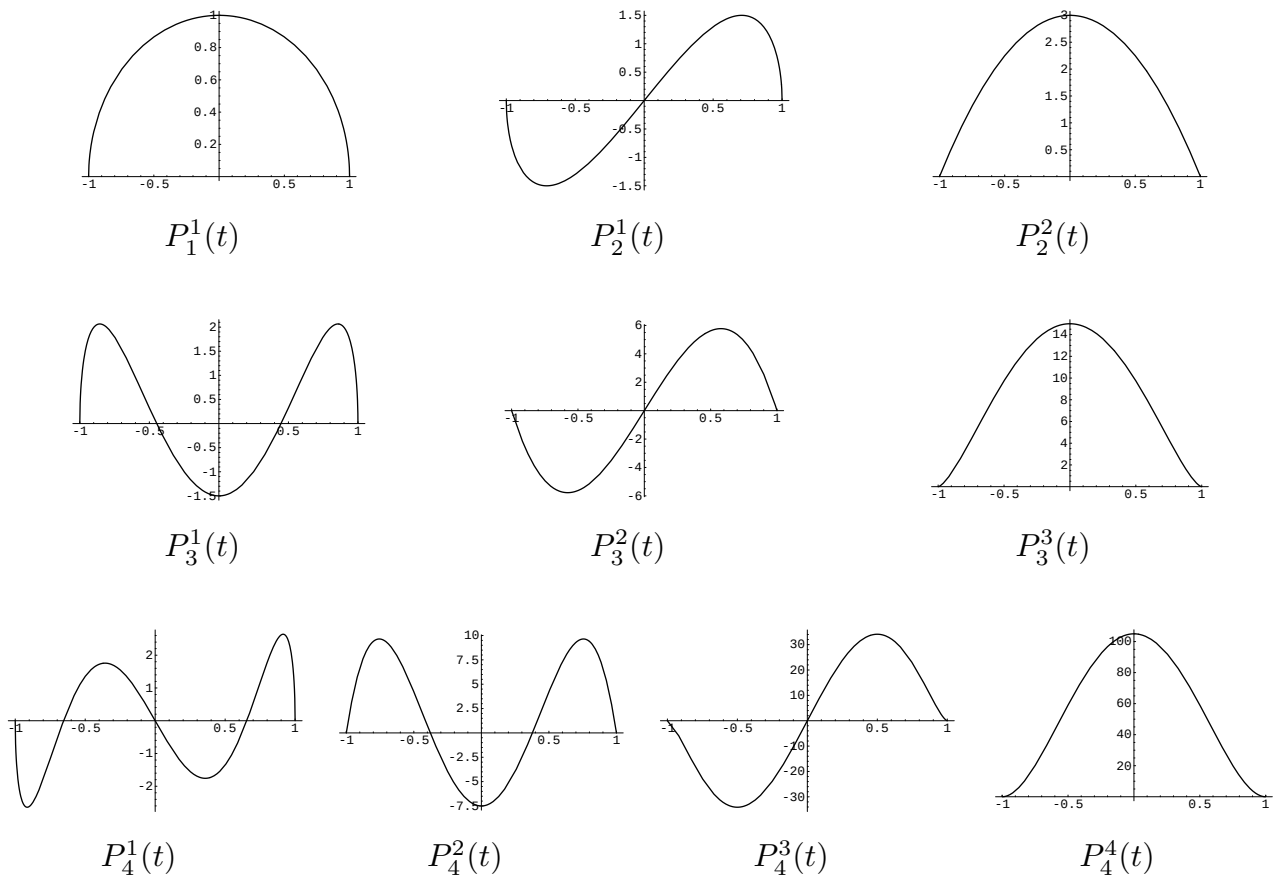


Figure 18.1. Associated Legendre Functions.

first few, written in Fourier form:

$$\begin{aligned}
 p_0^0(\varphi) &= 1, & p_1^0(\varphi) &= \cos \varphi, & p_1^1(\varphi) &= \sin \varphi, \\
 p_2^0(\varphi) &= \frac{1}{4} + \frac{3}{4} \cos 2\varphi, & p_2^1(\varphi) &= \frac{3}{2} \sin 2\varphi, & p_2^2(\varphi) &= \frac{3}{2} - \frac{3}{2} \cos 2\varphi, \\
 p_3^0(\varphi) &= \frac{3}{8} \cos \varphi + \frac{5}{8} \cos 3\varphi, & p_3^1(\varphi) &= \frac{3}{8} \sin \varphi + \frac{15}{8} \sin 3\varphi, \\
 p_3^2(\varphi) &= \frac{15}{4} \cos \varphi - \frac{15}{4} \cos 3\varphi, & p_3^3(\varphi) &= \frac{45}{4} \sin \varphi - \frac{15}{4} \sin 3\varphi, \\
 p_4^0(\varphi) &= \frac{9}{64} + \frac{5}{16} \cos 2\varphi + \frac{35}{64} \cos 4\varphi, & p_4^1(\varphi) &= \frac{5}{8} \sin 2\varphi + \frac{35}{16} \sin 4\varphi, \\
 p_4^2(\varphi) &= \frac{45}{16} + \frac{15}{4} \cos 2\varphi - \frac{105}{16} \cos 4\varphi, & p_4^3(\varphi) &= \frac{105}{4} \sin 2\varphi - \frac{105}{8} \sin 4\varphi, \\
 p_4^4(\varphi) &= \frac{315}{8} - \frac{105}{2} \cos 2\varphi + \frac{105}{8} \cos 4\varphi.
 \end{aligned} \tag{18.32}$$

It is also instructive to plot the eigenfunctions in terms of the zenith angle φ ; see Figure 18.2. As in Figure 18.1, the vertical scales are not the same.

At this stage, we have determined both angular components of our separable solutions

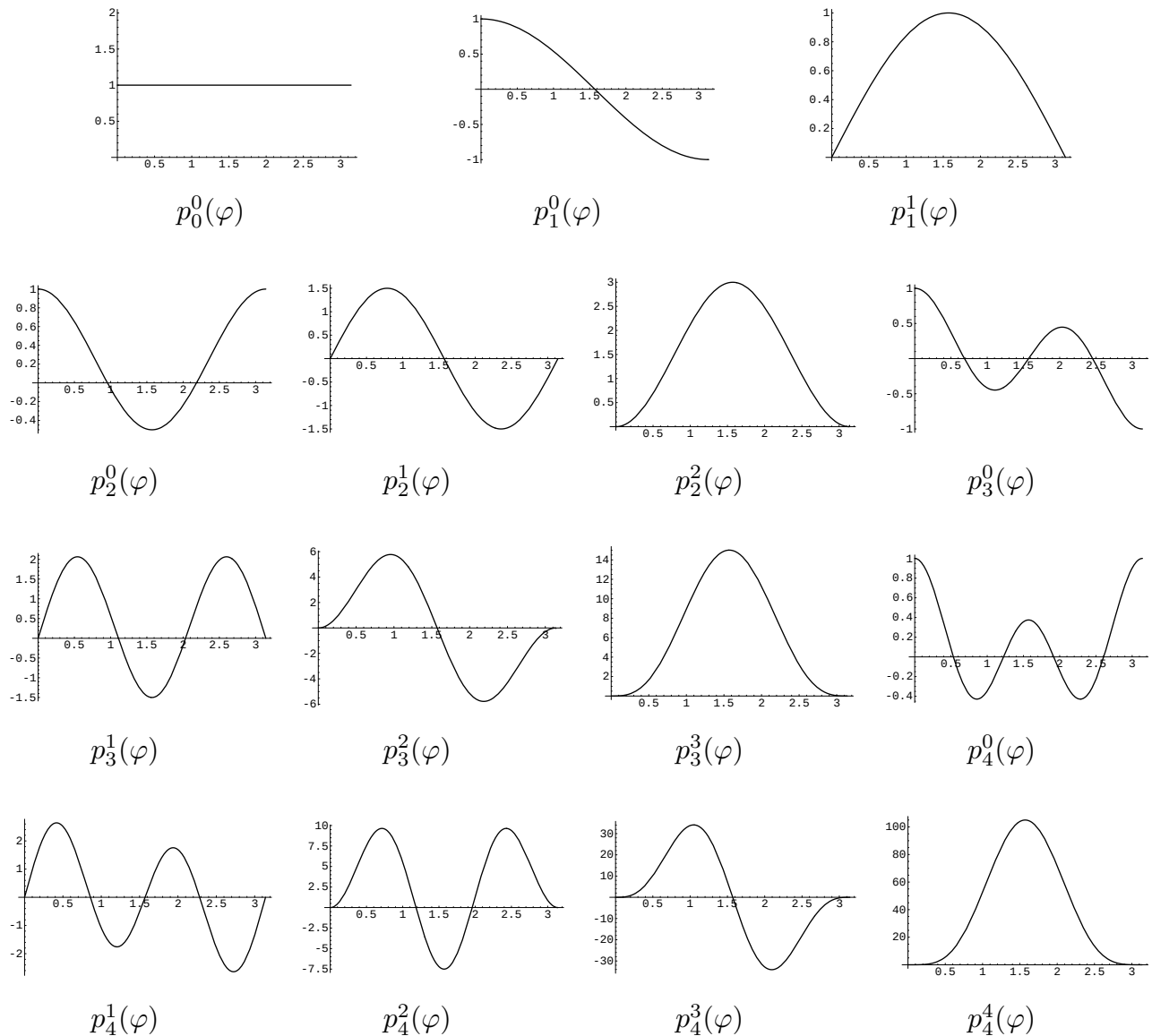


Figure 18.2. Trigonometric Legendre Functions.

(18.20). Multiplying the two parts together results in the spherical angle functions

$$\begin{aligned} Y_n^m(\varphi, \theta) &= P_n^m(\cos \varphi) \cos m\theta, & n &= 0, 1, 2, \dots, \\ \tilde{Y}_n^m(\varphi, \theta) &= P_n^m(\cos \varphi) \sin m\theta, & m &= 0, 1, \dots, n, \end{aligned} \quad (18.33)$$

known as *spherical harmonics*. They satisfy the spherical Helmholtz equation

$$\Delta_S Y_n^m + n(n+1) Y_n^m = 0 = \Delta_S \tilde{Y}_n^m + n(n+1) \tilde{Y}_n^m, \quad (18.34)$$

and so are eigenfunctions for the spherical Laplacian operator, (18.17), with associated

eigenvalues $\mu_n = n(n+1)$ for $n = 0, 1, 2, \dots$. The n^{th} eigenvalue μ_n admits a $(2n+1)$ -dimensional eigenspace, spanned by the spherical harmonics

$$Y_n^0(\varphi, \theta), \quad Y_n^1(\varphi, \theta), \quad \dots \quad Y_n^n(\varphi, \theta), \quad \tilde{Y}_n^1(\varphi, \theta), \quad \dots \quad \tilde{Y}_n^n(\varphi, \theta).$$

(The omitted function $\tilde{Y}_n^0(\varphi, \theta) \equiv 0$ is trivial, and so does not contribute.) In Figure 18.3 we plot the first few spherical harmonic surfaces $r = Y_n^m(\varphi, \theta)$. In these graphs, in view of the spherical coordinate formula (18.13), points with a negative value of r appear on the opposite side of the origin from the point on the unit sphere with angles φ, θ . Incidentally, the graphs of their counterparts $r = \tilde{Y}_n^m(\varphi, \theta)$, when $m \neq 0$, are obtained by rotation around the z axis by 90° . On the other hand, the graphs of Y_n^0 are cylindrically symmetric (why?), and hence unaffected by such a rotation.

Self-adjointness of the spherical Laplacian, cf. Exercise ■, implies that the spherical harmonics are orthogonal with respect to the L^2 inner product

$$\langle f; g \rangle = \iint_{S_1} f g dS = \int_0^\pi \int_0^{2\pi} f(\varphi, \theta) g(\varphi, \theta) \sin \varphi d\theta d\varphi \quad (18.35)$$

given by integrating the product of the functions with respect to surface area over the unit sphere $S_1 = \{\|\mathbf{x}\| = 1\}$, cf. (B.42). More correctly, self-adjointness only guarantees orthogonality for the harmonics corresponding to distinct eigenvalues. However, the orthogonality relations

$$\begin{aligned} \langle Y_n^m; Y_l^k \rangle &= \iint_{S_1} Y_n^m Y_l^k dS = 0, & (m, n) &\neq (k, l), \\ \langle Y_n^m; \tilde{Y}_l^k \rangle &= \iint_{S_1} Y_n^m \tilde{Y}_l^k dS = 0, & \text{for all } (m, n), (k, l), \\ \langle \tilde{Y}_n^m; \tilde{Y}_l^k \rangle &= \iint_{S_1} \tilde{Y}_n^m \tilde{Y}_l^k dS = 0 & (m, n) &\neq (k, l), \end{aligned} \quad (18.36)$$

do, in fact, hold in full generality; Exercise ■ contains the details. Their norms can be explicitly computed:

$$\|Y_n^0\|^2 = \frac{4\pi}{2n+1}, \quad \|Y_n^m\|^2 = \|\tilde{Y}_n^m\|^2 = \frac{2\pi(n+m)!}{(2n+1)(n-m)!}. \quad (18.37)$$

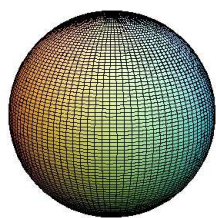
A proof of this formula appears in Exercise ■.

With some further work, it can be shown that the spherical harmonics form a complete orthogonal system of functions on the unit sphere. This means that any reasonable function $h: S_1 \rightarrow \mathbb{R}$, e.g., piecewise C^1 , can be expanded into a convergent *spherical Fourier series*

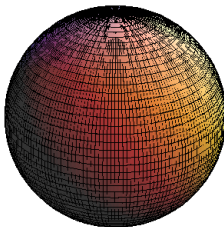
$$h(\varphi, \theta) = \frac{c_{0,0}}{2} + \sum_{n=1}^{\infty} \left(\frac{c_{0,n}}{2} Y_n^0(\varphi) + \sum_{m=1}^n \left[c_{m,n} Y_n^m(\varphi, \theta) + \tilde{c}_{m,n} \tilde{Y}_n^m(\varphi, \theta) \right] \right) \quad (18.38)$$

in the spherical harmonics. Applying the orthogonality relations (18.36), we find that the spherical Fourier coefficients are given by the inner products

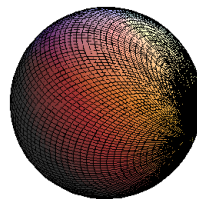
$$c_{0,n} = \frac{2\langle h; Y_n^0 \rangle}{\|Y_n^0\|^2}, \quad c_{m,n} = \frac{\langle h; Y_n^m \rangle}{\|Y_n^m\|^2}, \quad \tilde{c}_{m,n} = \frac{\langle h; \tilde{Y}_n^m \rangle}{\|\tilde{Y}_n^m\|^2}, \quad \begin{array}{l} 0 \leq n, \\ 1 \leq m \leq n, \end{array}$$



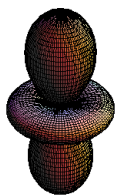
$$Y_0^0(\varphi, \theta)$$



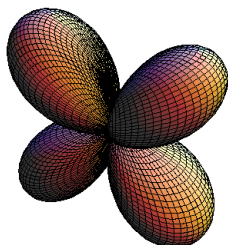
$$Y_1^0(\varphi, \theta)$$



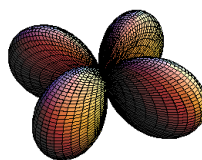
$$Y_1^1(\varphi, \theta)$$



$$Y_2^0(\varphi, \theta)$$



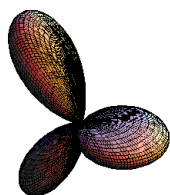
$$Y_2^1(\varphi, \theta)$$



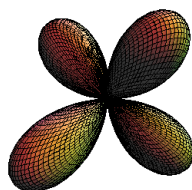
$$Y_2^2(\varphi, \theta)$$



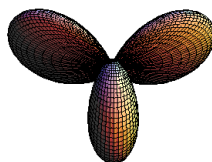
$$Y_3^0(\varphi, \theta)$$



$$Y_3^1(\varphi, \theta)$$



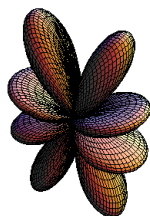
$$Y_3^2(\varphi, \theta)$$



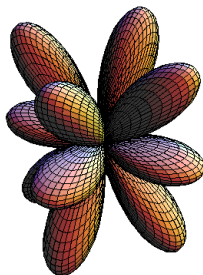
$$Y_3^3(\varphi, \theta)$$



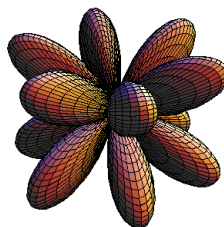
$$Y_4^0(\varphi, \theta)$$



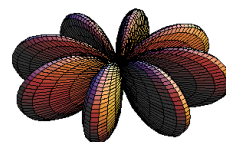
$$Y_4^1(\varphi, \theta)$$



$$Y_4^2(\varphi, \theta)$$



$$Y_4^3(\varphi, \theta)$$



$$Y_4^4(\varphi, \theta)$$

Figure 18.3. Spherical Harmonics.

or, explicitly, using (18.35) and the formulae (18.37) for the norms,

$$\begin{aligned} c_{m,n} &= \frac{(2n+1)(n-m)!}{2\pi(n+m)!} \int_0^{2\pi} \int_0^\pi h(\varphi, \theta) P_n^m(\cos \varphi) \cos m\theta \sin \varphi \, d\varphi \, d\theta, \\ \tilde{c}_{m,n} &= \frac{(2n+1)(n-m)!}{2\pi(n+m)!} \int_0^{2\pi} \int_0^\pi h(\varphi, \theta) P_n^m(\cos \varphi) \sin m\theta \sin \varphi \, d\varphi \, d\theta. \end{aligned} \quad (18.39)$$

As with an ordinary Fourier series, the extra $\frac{1}{2}$ was appended to the $c_{0,n}$ terms in the series (18.38) so that the formulae (18.39) are valid for all m, n . In particular, the constant term in the spherical harmonic series is the mean of the function h over the unit sphere:

$$\frac{c_{0,0}}{2} = \frac{1}{4\pi} \iint_{S_1} h \, dS = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi h(\varphi, \theta) \sin \varphi \, d\varphi \, d\theta \quad (18.40)$$

Remark: An alternative approach is to replace the real trigonometric functions by complex exponentials, and work with the *complex spherical harmonics*

$$\mathcal{Y}_n^m(\theta, \varphi) = Y_n^m(\theta, \varphi) + i\tilde{Y}_n^m(\theta, \varphi) = P_n^m(\cos \varphi) e^{im\theta}, \quad \begin{aligned} n &= 0, 1, 2, \dots, \\ m &= -n, -n+1, \dots, n. \end{aligned} \quad (18.41)$$

The complex orthogonality and expansion formulas are relegated to the exercises.

To complete our solution to the Laplace equation on the solid ball, we still need to analyze the ordinary differential equation (18.18) for the radial component $v(r)$. In view of our analysis of the spherical Helmholtz equation, the original separation constant is $\mu = n(n+1)$ for some non-negative integer $n \geq 0$, and so the radial equation takes the form

$$r^2 v'' + 2r v' - n(n+1)v = 0. \quad (18.42)$$

As noted earlier, to solve such a second order linear equation of Euler type (3.84), we substitute the power ansatz $v(r) = r^\alpha$. The exponent α must satisfy the quadratic equation

$$\alpha^2 + \alpha - n(n+1) = 0, \quad \text{and hence} \quad \alpha = n \quad \text{or} \quad \alpha = -(n+1).$$

Therefore, the two linearly independent solutions are

$$v_1(r) = r^n \quad \text{and} \quad v_2(r) = r^{-n-1}. \quad (18.43)$$

Since here we are only interested in solutions that remain bounded at $r = 0$ — the center of the ball — we should just retain the first solution $v(r) = r^n$ in our subsequent analysis.

At this stage, we have solved all three ordinary differential equations for the separable solutions. We combine the results (18.21, 33, 43) together to produce the following spherically separable solutions to the Laplace equation:

$$\begin{aligned} H_n^m &= r^n Y_n^m(\varphi, \theta) = r^n P_n^m(\cos \varphi) \cos m\theta & n &= 0, 1, 2, \dots, \\ \tilde{H}_n^m &= r^n \tilde{Y}_n^m(\varphi, \theta) = r^n P_n^m(\cos \varphi) \sin m\theta & m &= 0, 1, \dots, n. \end{aligned} \quad (18.44)$$

Although apparently complicated, these solutions are, surprisingly, elementary polynomial functions of the rectangular coordinates x, y, z , and hence *harmonic polynomials*. The first few are

$$\begin{aligned}
H_0^0 &= 1, & H_1^0 &= z, & H_2^0 &= z^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2, & H_3^0 &= z^3 - \frac{3}{2}x^2z - \frac{3}{2}y^2z \\
H_1^1 &= x, & H_2^1 &= 3xz, & & & H_3^1 &= 6xz^2 - \frac{3}{2}x^3 - \frac{3}{2}xy^2 \\
\tilde{H}_1^1 &= y, & \tilde{H}_2^1 &= 3yz, & & & \tilde{H}_3^1 &= 6yz^2 - \frac{3}{2}x^2y - \frac{3}{2}y^3 \\
H_2^2 &= 3x^2 - 3y^2, & & & & & H_3^2 &= 15x^2z - 15y^2z \\
\tilde{H}_2^2 &= 6xy, & & & & & \tilde{H}_3^2 &= 30xyz \\
H_3^3 &= 15x^3 - 45xy^2, & & & & & & \\
\tilde{H}_3^3 &= 45x^2y - 15y^3. & & & & & &
\end{aligned} \tag{18.45}$$

The polynomials

$$H_n^0, H_n^1, \dots, H_n^n, \tilde{H}_n^1, \dots, \tilde{H}_n^n$$

form a basis for the vector space $\mathcal{H}^{(n)}$ of all homogeneous harmonic polynomials of degree n , which therefore has dimension $2n + 1$.

The harmonic polynomials form a complete orthogonal system, and therefore the general solution to the Laplace equation inside the unit ball can be written as a harmonic polynomial series:

$$u(x, y, z) = \frac{c_{0,0}}{2} + \sum_{n=1}^{\infty} \left(\frac{c_{0,n}}{2} H_n^0(x, y, z) + \sum_{m=1}^n \left[c_{m,n} H_n^m(x, y, z) + \tilde{c}_{m,n} \tilde{H}_n^m(x, y, z) \right] \right), \tag{18.46}$$

or, equivalently in spherical coordinates,

$$u(r, \varphi, \theta) = \frac{c_{0,0}}{2} + \sum_{n=1}^{\infty} \left(\frac{c_{0,n}}{2} r^n Y_n^0(\varphi) + \sum_{m=1}^n \left[c_{m,n} r^n Y_n^m(\varphi, \theta) + \tilde{c}_{m,n} r^n \tilde{Y}_n^m(\varphi, \theta) \right] \right). \tag{18.47}$$

The coefficients $c_{m,n}, \tilde{c}_{m,n}$ are uniquely prescribed by the boundary conditions. Indeed, substituting (18.47) into the Dirichlet boundary conditions on the unit sphere $r = 1$ yields

$$u(1, \varphi, \theta) = \frac{c_{0,0}}{2} + \sum_{n=1}^{\infty} \left(\frac{c_{0,n}}{2} Y_n^0(\varphi) + \sum_{m=1}^n \left[c_{m,n} Y_n^m(\varphi, \theta) + \tilde{c}_{m,n} \tilde{Y}_n^m(\varphi, \theta) \right] \right) = h(\varphi, \theta). \tag{18.48}$$

Thus, the coefficients $c_{m,n}, \tilde{c}_{m,n}$ are given by the orthogonality formulae (18.39). If the terms in the resulting series are uniformly bounded — which occurs for all integrable functions h , as well as also certain generalized functions including the delta function — then the harmonic polynomial series (18.47) converges everywhere, and, in fact, uniformly on any smaller ball $\|\mathbf{x}\| = r \leq r_0 < 1$.

In rectangular coordinates, the n^{th} summand of the series (18.46) is a homogeneous polynomial of degree n . Therefore, repeating the argument used on the two-dimensional polar coordinate solution (15.39), we conclude that the harmonic polynomial series is,

in fact, a power series, and hence provides the *Taylor expansion for the harmonic function* $u(x, y, z)$ *at the origin!* In particular, this implies that the harmonic function $u(x, y, z)$ is analytic at $\mathbf{0}$.

The constant term in such a Taylor series can be identified with the value of the function at the origin: $u(0, 0, 0) = \frac{1}{2} c_{0,0}$. On the other hand, since $u = h$ on $S_1 = \partial\Omega$, the coefficient formula (18.40) tells us that

$$u(0, 0, 0) = \frac{c_{0,0}}{2} = \frac{1}{4\pi} \iint_{S_1} u \, dS. \quad (18.49)$$

Therefore, we have established the three-dimensional counterpart of Theorem 15.8: the value of the harmonic function at the center of the sphere is equal to the average of its values on the sphere's surface. In addition, the higher order coefficients $c_{m,n}, \tilde{c}_{m,n}$ serve to prescribe the partial derivatives $\frac{\partial^{i+j+k} u}{\partial x^i \partial y^j \partial z^k}(0, 0, 0)$. In this way, the orthogonality formulae (18.39) can be re-interpreted as three-dimensional counterparts of the Cauchy formulae (16.139) for the derivatives of the real and imaginary parts of a complex analytic function; see Exercise ■ for details.

So far, we have restricted our attention to the sphere of unit radius. A simple scaling argument serves to establish the general result.

Theorem 18.3. *If $u(\mathbf{x})$ is a harmonic function defined on a domain $\Omega \subset \mathbb{R}^3$, then u is analytic inside Ω . Moreover, its value at any point $\mathbf{x}_0 \in \Omega$ is obtained by averaging its values on any sphere centered at $\mathbf{x}_0$, so*

$$u(\mathbf{x}_0) = \frac{1}{4\pi a^2} \iint_{\|\mathbf{x}-\mathbf{x}_0\|=a} u \, dS, \quad (18.50)$$

provided the enclosed ball $\{\|\mathbf{x} - \mathbf{x}_0\| \leq a\} \subset \Omega$ lies entirely within the domain of definition.

Proof: It is easily checked that, under the hypothesis of the theorem, the rescaled and translated function

$$U(\mathbf{y}) = u(a\mathbf{y} + \mathbf{x}_0) = u(\mathbf{x}), \quad \text{where} \quad \mathbf{y} = \frac{\mathbf{x} - \mathbf{x}_0}{a}, \quad (18.51)$$

is harmonic on the unit ball $\|\mathbf{y}\| \leq 1$, and hence solves the boundary value problem (18.12) with boundary values $h(\mathbf{y}) = U(\mathbf{y}) = u(a\mathbf{y} + \mathbf{x}_0)$ on $\|\mathbf{y}\| = 1$. By the preceding remarks, $U(\mathbf{y})$ is analytic at $\mathbf{y} = \mathbf{0}$, and so $u(\mathbf{x}) = U\left(\frac{\mathbf{x} - \mathbf{x}_0}{a}\right)$ is analytic at $\mathbf{x} = \mathbf{x}_0$. Since $\mathbf{x}_0$ can be any point inside Ω , this proves analyticity of u everywhere in Ω . Moreover, according to the integral formula (18.49),

$$u(\mathbf{x}_0) = U(\mathbf{0}) = \frac{1}{4\pi} \iint_{\|\mathbf{y}\|=1} U \, dS = \frac{1}{4\pi a^2} \iint_{\|\mathbf{x}-\mathbf{x}_0\|=a} u \, dS,$$

since the change of variables (18.51) has the effect of rescaling the spherical surface integral. *Q.E.D.*

Arguing as in the planar case of Theorem 15.9, we readily establish the corresponding Maximum Principle for harmonic functions of three variables.

Theorem 18.4. *A non-constant harmonic function cannot have a local maximum or minimum at any interior point of its domain of definition. Moreover, its global maximum or minimum (if any) can only occur on the boundary of the domain.*

For instance, the Maximum Principle implies that the maximum and minimum temperatures in a solid body in thermal equilibrium are to be found only on its boundary. In physical terms, since heat energy must flow away from any internal maximum and towards any internal minimum, any local temperature extremum inside the body would preclude it from being in thermal equilibrium.

Example 18.5. In this example, we shall determine the electrostatic potential inside a hollow sphere when the upper and lower hemispheres are held at different constant potentials. This device is called a *spherical capacitor* and is realized experimentally by separating the two charged conducting hemispherical shells by a thin insulating ring at the equator. A straightforward scaling argument allows us to choose our units so that the sphere has radius 1, while the potential is set equal to 1 on the upper hemisphere and equal to 0, i.e., grounded, on the lower hemisphere. The resulting electrostatic potential satisfies the Laplace equation

$$\Delta u = 0 \quad \text{inside a solid ball} \quad \|\mathbf{x}\| < 1,$$

and is subject to Dirichlet boundary conditions

$$u(x, y, z) = \begin{cases} 1, & z > 0, \\ 0, & z < 0, \end{cases} \quad \text{on the unit sphere} \quad \|\mathbf{x}\| = 1. \quad (18.52)$$

The solution will be prescribed by a harmonic polynomial series (18.46) whose coefficients are fixed by the boundary values (18.52). Before taking on the required computation, let us first note that since the boundary data does not depend upon the azimuthal angle θ , the solution $u = u(r, \varphi)$ will also be independent of θ . Therefore, we need only consider the θ -independent spherical harmonic polynomials (18.33), which are those with $m = 0$, and hence

$$u(r, \varphi) = \frac{1}{2} \sum_{n=0}^{\infty} c_n H_n^0(x, y, z) = \frac{1}{2} \sum_{n=0}^{\infty} c_n r^n P_n(\cos \varphi),$$

where we abbreviate $c_n = c_{0,n}$. The boundary conditions (18.52) require

$$u(1, \varphi) = \frac{1}{2} \sum_{n=0}^{\infty} c_n P_n(\cos \varphi) = h(\varphi) = \begin{cases} 1, & 0 \leq \varphi < \frac{1}{2}\pi, \\ 0, & \frac{1}{2}\pi < \varphi \leq \pi. \end{cases}$$

The coefficients are given by (18.39), which, in the case $m = 0$, reduce to

$$c_n = \frac{2n+1}{2\pi} \iint_{S_1} f Y_n^0 dS = (2n+1) \int_0^{\pi/2} P_n(\cos \varphi) \sin \varphi d\varphi = (2n+1) \int_0^1 P_n(t) dt, \quad (18.53)$$

since $f = 0$ when $\frac{1}{2}\pi < \varphi \leq \pi$. The first few are

$$c_0 = 1, \quad c_1 = \frac{3}{2}, \quad c_2 = 0, \quad c_3 = -\frac{7}{8}, \quad c_4 = 0, \quad \dots$$

Therefore, the solution has the explicit Taylor expansion

$$\begin{aligned} u(x, y, z) &= \frac{1}{2} + \frac{3}{4} r \cos \varphi - \frac{21}{128} r^3 \cos \varphi - \frac{35}{128} r^3 \cos 3\varphi + \dots \\ &= \frac{1}{2} + \frac{3}{4} z + \frac{21}{32} (x^2 + y^2) z - \frac{7}{16} z^3 + \dots \end{aligned} \quad (18.54)$$

Note in particular that the value $u(0, 0, 0) = \frac{1}{2}$ at the center of the sphere is the average of its boundary values, in accordance with Theorem 18.3. Observe that the solution only depends upon the cylindrical coordinates r, z . This follows from the invariance of the Laplace equation under general rotations, coupled with the invariance of the boundary data under rotations around the z axis.

Remark: The same solution $u(x, y, z)$ describes the thermal equilibrium in a solid sphere whose upper hemisphere is held at temperature 1° and lower hemisphere at 0° .

Example 18.6. A closely related problem is to determine the electrostatic potential *outside* a spherical capacitor. As in the preceding example, we take our capacitor of radius 1, with electrostatic charge of 1 on the upper hemisphere and 0 on the lower hemisphere. Here, we need to solve the Laplace equation $\Delta u = 0$ in the unbounded domain $\Omega = \{\|\mathbf{x}\| > 1\}$ — the exterior of the unit sphere — subject to same Dirichlet boundary conditions (18.52). We anticipate that the potential will be vanishingly small at large distances away from the capacitor: $r = \|\mathbf{x}\| \gg 1$. Therefore, the harmonic polynomial solutions (18.44) will not help us solve this problem, since (except for the constant case) they become unboundedly large far away from the origin.

However, reconsideration of our original separation of variables argument will produce a different class of solutions having the desired decay properties. When we solved the radial equation (18.42), we discarded the solution $v_2(r) = r^{-n-1}$ because it had a singularity at the origin. In the present situation, the behavior of the function at $r = 0$ is irrelevant; our requirement is that the solution decays as $r \rightarrow \infty$, and $v_2(r)$ has this property. Therefore, we will utilize the complementary harmonic functions

$$\begin{aligned} K_n^m(x, y, z) &= r^{-2n-1} H_n^m(x, y, z) = r^{-n-1} Y_n^m(\varphi, \theta) = r^{-n-1} P_n^m(\cos \varphi) \cos m\theta, \\ \tilde{K}_n^m(x, y, z) &= r^{-2n-1} \tilde{H}_n^m(x, y, z) = r^{-n-1} \tilde{Y}_n^m(\varphi, \theta) = r^{-n-1} P_n^m(\cos \varphi) \sin m\theta, \end{aligned} \quad (18.55)$$

for solving such exterior problems. For the capacitor problem, we need only those that are independent of θ , which have $m = 0$. We write the resulting solution as a series

$$u(r, \varphi) = \frac{1}{2} \sum_{n=0}^{\infty} c_n K_n^0(x, y, z) = \frac{1}{2} \sum_{n=0}^{\infty} c_n r^{-n-1} P_n(\cos \varphi).$$

The boundary conditions

$$u(1, \varphi) = \frac{1}{2} \sum_{n=0}^{\infty} c_n P_n(\cos \varphi) = f(\varphi) = \begin{cases} 1, & 0 \leq \varphi < \frac{1}{2}\pi, \\ 0, & \frac{1}{2}\pi < \varphi \leq \pi, \end{cases}$$

are identical with those in the previous example. Therefore, the coefficients are given by (18.53), leading to the series expansion

$$u = \frac{1}{2r} + \frac{3 \cos \varphi}{4r^2} - \frac{21 \cos \varphi + 35 \cos 3\varphi}{128r^4} + \cdots = \frac{1}{2r} + \frac{3z}{4r^3} + \frac{21(x^2 + y^2)z - 14z^3}{32r^7} + \cdots, \quad (18.56)$$

where $r = \sqrt{x^2 + y^2 + z^2}$. Interestingly, at large distances, the higher order terms become negligible, and the potential looks like that associated with a point charge of magnitude $\frac{1}{2}$ — the average of the potential over the sphere — that is concentrated at the origin. This is indicative of a general fact, to be explored in Exercise ■.

18.3. The Green's Function.

We now turn to the inhomogeneous form of the three-dimensional Laplace equation: the *Poisson equation*

$$-\Delta u = f \quad \text{for all } \mathbf{x} \in \Omega \quad (18.57)$$

on a solid domain $\Omega \subset \mathbb{R}^3$. In order to uniquely specify the solution, we must impose appropriate boundary conditions: Dirichlet or mixed. We only need to discuss the case of homogeneous boundary conditions, since, by linearity, an inhomogeneous boundary value problem can be split up into a homogeneous boundary value problem for the inhomogeneous Poisson equation and an inhomogeneous boundary value problem for the homogeneous Laplace equation.

As in Chapters 11 and 15, we begin by analyzing the case of a delta function inhomogeneity that is concentrated at a single point in the domain. Thus, for each $\boldsymbol{\xi} = (\xi, \eta, \zeta) \in \Omega$, the *Green's function* $G(\mathbf{x}; \boldsymbol{\xi}) = G(x, y, z; \xi, \eta, \zeta)$ is the unique solution to the Poisson equation

$$-\Delta u = \delta(\mathbf{x} - \boldsymbol{\xi}) = \delta(x - \xi) \delta(y - \eta) \delta(z - \zeta) \quad \text{for all } \mathbf{x} \in \Omega, \quad (18.58)$$

subject to the chosen homogeneous boundary conditions. The solution to the general Poisson equation (18.57) is then obtained by superposition: We write the forcing function

$$f(x, y, z) = \iiint_{\Omega} f(\xi, \eta, \zeta) \delta(x - \xi) \delta(y - \eta) \delta(z - \zeta) d\xi d\eta d\zeta$$

as a linear superposition of delta functions. By linearity, the solution

$$u(x, y, z) = \iiint_{\Omega} f(\xi, \eta, \zeta) G(x, y, z; \xi, \eta, \zeta) d\xi d\eta d\zeta \quad (18.59)$$

to the homogeneous boundary value problem for the Poisson equation (18.57) is then given as the corresponding superposition of the Green's function solutions.

The Green's Function in Space

Only in a few specific instances is the explicit formula for the Green's function known. Nevertheless, certain general guiding features can be readily established. The starting point is to investigate the Poisson equation (18.58) when the domain $\Omega = \mathbb{R}^3$ is all of

three-dimensional space. We impose boundary constraints by seeking a solution that goes to zero, $u(\mathbf{x}) \rightarrow 0$, at large distances $\|\mathbf{x}\| \rightarrow \infty$. Since the Laplacian is invariant under translations we can, without loss of generality, place our delta impulse at the origin, and concentrate on solving the particular case

$$-\Delta u = \delta(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3.$$

Since $\delta(\mathbf{x}) = 0$ for all $\mathbf{x} \neq \mathbf{0}$, the desired solution will, in fact, be a solution to the homogeneous Laplace equation

$$\Delta u = 0, \quad \mathbf{x} \neq \mathbf{0},$$

save, possibly, for a singularity at the origin.

The Laplace equation models the equilibria of a homogeneous, isotropic medium, and so is also invariant under three-dimensional rotations; details can be found in Exercise ■. This suggests that, in any radially symmetric configuration, the solution should only depend upon the distance $r = \|\mathbf{x}\|$ from the origin. Referring to the spherical coordinate form (18.14) of the Laplacian operator, if u only depends upon r , its derivatives with respect to the angular coordinates φ, θ are zero, and so $u(r)$ solves the ordinary differential equation

$$\frac{d^2 u}{dr^2} + \frac{2}{r} \frac{du}{dr} = 0. \quad (18.60)$$

This equation is, in effect, a first order linear ordinary differential equation for $v = du/dr$ and hence is particularly easy to solve:

$$\frac{du}{dr} = v(r) = -\frac{b}{r^2}, \quad \text{and hence} \quad u(r) = a + \frac{b}{r},$$

where a, b are arbitrary constants. The constant solution $u(r) = a$ does not die away at large distances, nor does it have a singularity at the origin. Therefore, if our intuition is valid, the desired solution should be of the form

$$u = \frac{b}{r} = \frac{b}{\|\mathbf{x}\|} = \frac{b}{\sqrt{x^2 + y^2 + z^2}}. \quad (18.61)$$

Indeed, this function is harmonic — solves Laplace's equation — everywhere away from the origin, and has a singularity at $\mathbf{x} = \mathbf{0}$.

The solution (18.61) is, up to constant multiple, the three-dimensional Newtonian gravitational potential due to a point mass at the origin. Its gradient

$$\mathbf{g}(\mathbf{x}) = \nabla \left(\frac{b}{\|\mathbf{x}\|} \right) = -\frac{b\mathbf{x}}{\|\mathbf{x}\|^3}. \quad (18.62)$$

defines the gravitational force vector at the point $\mathbf{x}$. When $b > 0$, the force $\mathbf{g}(\mathbf{x})$ points towards the mass at the origin. Its magnitude

$$\|\mathbf{g}\| = \frac{b}{\|\mathbf{x}\|^2} = \frac{b}{r^2}$$

is proportional to one over the squared distance, which is the well-known inverse square law of three-dimensional Newtonian gravity. Thus, (18.61) can also be interpreted as the electrostatic Coulomb potential on a charged mass at position $\mathbf{x}$ due to a concentrated electric charge at the origin, with (18.62) the corresponding electrostatic force. The constant b is positive when the charges are of opposite signs, leading to an attractive force, and negative in the repulsive case of like charges.

Returning to our problem, the remaining task is to fix the multiple b such that the Laplacian of our candidate solution (18.61) has a delta function singularity at the origin; equivalently, we must determine $c = 1/b$ such that

$$-\Delta r^{-1} = c \delta(\mathbf{x}). \quad (18.63)$$

This equation is certainly valid away from the origin, since $\delta(\mathbf{x}) = 0$ when $\mathbf{x} \neq \mathbf{0}$. To investigate near the singularity, we integrate both sides of (18.63) over a small solid ball $B_\varepsilon = \{\|\mathbf{x}\| \leq \varepsilon\}$ of radius ε :

$$-\iint\int_{B_\varepsilon} \Delta r^{-1} dx dy dz = \iint\int_{B_\varepsilon} c \delta(\mathbf{x}) dx dy dz = c, \quad (18.64)$$

where we used the definition of the delta function to evaluate the right hand side. On the other hand, since $\Delta r^{-1} = \nabla \cdot \nabla r^{-1}$, we can use the divergence theorem (B.85) to evaluate the left hand integral, whence

$$\iint\int_{B_\varepsilon} \Delta r^{-1} dx dy dz = \iint\int_{B_\varepsilon} \nabla \cdot \nabla r^{-1} dx dy dz = \iint_{S_\varepsilon} \frac{\partial}{\partial \mathbf{n}} \left(\frac{1}{r} \right) dS,$$

where the surface integral is over the bounding sphere $S_\varepsilon = \partial B_\varepsilon = \{\|\mathbf{x}\| = \varepsilon\}$. The sphere's unit normal $\mathbf{n}$ points in the radial direction, and hence the normal derivative coincides with differentiation with respect to r ; in particular,

$$\frac{\partial}{\partial \mathbf{n}} \left(\frac{1}{r} \right) = \frac{\partial}{\partial r} \left(\frac{1}{r} \right) = -\frac{1}{r^2}.$$

The surface integral can now be explicitly evaluated:

$$\iint_{S_\varepsilon} \frac{\partial}{\partial \mathbf{n}} \left(\frac{1}{r} \right) dS = - \iint_{S_\varepsilon} \frac{1}{r^2} dS = - \iint_{S_\varepsilon} \frac{1}{\varepsilon^2} dS = -4\pi,$$

since S_ε has surface area $4\pi\varepsilon^2$. Substituting this result back into (18.64), we conclude that

$$c = 4\pi, \quad \text{and hence} \quad -\Delta r^{-1} = 4\pi \delta(\mathbf{x}). \quad (18.65)$$

This is our desired formula! We conclude that the solution to Poisson's equation for a delta function impulse at the origin is

$$G(x, y, z) = \frac{1}{4\pi r} = \frac{1}{4\pi \|\mathbf{x}\|} = \frac{1}{4\pi \sqrt{x^2 + y^2 + z^2}}, \quad (18.66)$$

which is the three-dimensional Newtonian potential due to a unit point mass situated at the origin.

If the singularity is concentrated at some other point $\boldsymbol{\xi} = (\xi, \eta, \zeta)$, then we merely translate the preceding solution. This leads immediately to the Green's function

$$G(\mathbf{x}; \boldsymbol{\xi}) = G(\mathbf{x} - \boldsymbol{\xi}) = \frac{1}{4\pi \|\mathbf{x} - \boldsymbol{\xi}\|} = \frac{1}{4\pi \sqrt{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2}}. \quad (18.67)$$

The superposition principle (18.59) implies the following integral formula for the solutions to the Poisson equation on all of three-dimensional space.

Theorem 18.7. *A particular solution to the Poisson equation*

$$-\Delta u = f \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^3 \quad (18.68)$$

is given by

$$u_*(\mathbf{x}) = \frac{1}{4\pi} \iiint_{\mathbb{R}^3} \frac{f(\boldsymbol{\xi})}{\|\mathbf{x} - \boldsymbol{\xi}\|} d\boldsymbol{\xi} = \frac{1}{4\pi} \iiint_{\mathbb{R}^3} \frac{f(\xi, \eta, \zeta) d\xi d\eta d\zeta}{\sqrt{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2}}. \quad (18.69)$$

The general solution is $u(x, y, z) = u_*(x, y, z) + w(x, y, z)$, where $w(x, y, z)$ is an arbitrary harmonic function.

Example 18.8. In this example, we compute the gravitational (or electrostatic) potential in three-dimensional space due to a uniform solid ball, e.g., a spherical planet such as the earth. By rescaling, it suffices to consider the case when the forcing function

$$f(\mathbf{x}) = \begin{cases} 1, & \|\mathbf{x}\| < 1, \\ 0, & \|\mathbf{x}\| > 1, \end{cases}$$

is equal to 1 inside a solid ball of radius 1 and zero outside. The particular solution to the resulting Poisson equation (18.68) is given by the integral

$$u(\mathbf{x}) = \frac{1}{4\pi} \iiint_{\|\boldsymbol{\xi}\| < 1} \frac{1}{\|\mathbf{x} - \boldsymbol{\xi}\|} d\xi d\eta d\zeta. \quad (18.70)$$

Clearly, since the forcing function is radially symmetric, the solution $u = u(r)$ is also radially symmetric. To evaluate the integral, then, we can take $\mathbf{x} = (0, 0, z)$ to lie on the z axis, so that $r = \|\mathbf{x}\| = |z|$. We use cylindrical coordinates $\boldsymbol{\xi} = (\rho \cos \theta, \rho \sin \theta, \zeta)$, so that

$$\|\mathbf{x} - \boldsymbol{\xi}\| = \sqrt{\rho^2 + (z - \zeta)^2}.$$

The integral in (18.70) can then be explicitly computed:

$$\begin{aligned} \frac{1}{4\pi} \int_{-1}^1 \int_0^{\sqrt{1-\zeta^2}} \int_0^{2\pi} \frac{\rho d\theta d\rho d\zeta}{\sqrt{\rho^2 + (z - \zeta)^2}} &= \\ &= \frac{1}{2} \int_{-1}^1 \left(\sqrt{1 + z^2 - 2z\zeta} - |z - \zeta| \right) d\zeta = \begin{cases} \frac{1}{3|z|}, & |z| \geq 1, \\ \frac{1}{2} - \frac{z^2}{6}, & |z| \leq 1. \end{cases} \end{aligned}$$

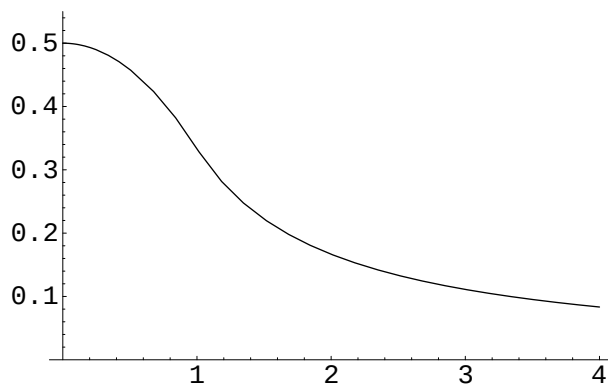


Figure 18.4. Solution to Poisson's Equation in a Solid Ball.

Therefore, by radial symmetry, the solution is

$$u(\mathbf{x}) = \begin{cases} \frac{1}{3r}, & r = \|\mathbf{x}\| \geq 1, \\ \frac{1}{2} - \frac{r^2}{6}, & r = \|\mathbf{x}\| \leq 1, \end{cases} \quad (18.71)$$

plotted, as a function of $r = \|\mathbf{x}\|$ in Figure 18.4. Note that, outside the solid ball, the solution is a Newtonian potential corresponding to a concentrated point mass of magnitude $\frac{4}{3}\pi$ — the total mass of the planet. We have thus demonstrated a well-known result in gravitation and electrostatics: the exterior potential due to a spherically symmetric mass (or electric charge) is the same as if all the mass (charge) were concentrated at its center. In outer space if you can't see a spherical planet, you can only determine its mass, not its size, by measuring its external gravitational force.

Bounded Domains and the Method of Images

Suppose we now wish to solve the inhomogeneous Poisson equation (18.57) on a bounded domain $\Omega \subset \mathbb{R}^3$. To construct the desired Green's function, we proceed as follows. The Newtonian potential (18.67) is a particular solution to the underlying inhomogeneous equation

$$-\Delta u = \delta(\mathbf{x} - \boldsymbol{\xi}), \quad \mathbf{x} \in \Omega, \quad (18.72)$$

but it almost surely does not have the proper boundary values on $\partial\Omega$. By linearity, the general solution to such an inhomogeneous linear equation is of the form

$$u(\mathbf{x}) = \frac{1}{4\pi\|\mathbf{x} - \boldsymbol{\xi}\|} - v(\mathbf{x}), \quad (18.73)$$

where the first summand is a particular solution, which we now know, while[†] $v(\mathbf{x})$ is an arbitrary solution to the homogeneous equation $\Delta v = 0$, i.e., an arbitrary harmonic function. The solution (18.73) satisfies the homogeneous boundary conditions provided

[†] The minus sign is for later convenience.

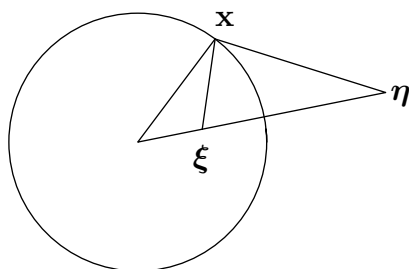


Figure 18.5. Method of Images for the Unit Ball.

the boundary values of $v(\mathbf{x})$ match those of the Green's function. Let us explicitly state the result in the Dirichlet case.

Theorem 18.9. *The Green's function for the homogeneous Dirichlet boundary value problem*

$$-\Delta u = f, \quad \mathbf{x} \in \Omega, \quad u = 0, \quad \mathbf{x} \in \partial\Omega,$$

for the Poisson equation in a domain $\Omega \subset \mathbb{R}^3$ has the form

$$G(\mathbf{x}; \boldsymbol{\xi}) = \frac{1}{4\pi \|\mathbf{x} - \boldsymbol{\xi}\|} - v(\mathbf{x}; \boldsymbol{\xi}), \quad \mathbf{x}, \boldsymbol{\xi} \in \Omega, \quad (18.74)$$

where $v(\mathbf{x}; \boldsymbol{\xi})$ is the harmonic function of $\mathbf{x}$ that satisfies

$$v(\mathbf{x}; \boldsymbol{\xi}) = \frac{1}{4\pi \|\mathbf{x} - \boldsymbol{\xi}\|} \quad \text{for all} \quad \mathbf{x} \in \partial\Omega.$$

In this manner, we have reduced the determination of the Green's function to the solution to a particular family of Laplace boundary value problems, which are parametrized by the point $\boldsymbol{\xi} \in \Omega$.

In certain domains with simple geometry, the Method of Images can be used to produce an explicit formula for the Green's function. As in Section 15.3, the idea is to match the boundary values of the free space Green's function due to a delta impulse at a point inside the domain with one or more additional Green's functions corresponding to impulses at points outside the domain — the “image points”.

The case of a solid ball of radius 1 with Dirichlet boundary conditions is the easiest to handle. Indeed, the *same* geometrical construction that we used for a planar disk, redrawn in Figure 18.5, applies here. Although the same as Figure 15.8, we are re-interpreting it as a three-dimensional diagram, with the circle representing the unit sphere, while the lines remain lines. The required image point is given by *inversion*:

$$\boldsymbol{\eta} = \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|^2}, \quad \text{whereby} \quad \|\boldsymbol{\xi}\| = \frac{1}{\|\boldsymbol{\eta}\|}.$$

By the similar triangles argument used before, we find

$$\frac{\|\boldsymbol{\xi}\|}{\|\mathbf{x}\|} = \frac{\|\mathbf{x}\|}{\|\boldsymbol{\eta}\|} = \frac{\|\mathbf{x} - \boldsymbol{\xi}\|}{\|\mathbf{x} - \boldsymbol{\eta}\|}, \quad \text{and therefore} \quad \|\mathbf{x}\| = 1.$$

As a result, the function

$$v(\mathbf{x}, \boldsymbol{\xi}) = \frac{1}{4\pi} \frac{\|\boldsymbol{\eta}\|}{\|\mathbf{x} - \boldsymbol{\eta}\|} = \frac{1}{4\pi} \frac{\|\boldsymbol{\xi}\|}{\|\boldsymbol{\xi} - \|\boldsymbol{\xi}\|^2 \mathbf{x}\|}$$

has the same boundary values on the unit sphere as the Newtonian potential:

$$\frac{1}{4\pi} \frac{\|\boldsymbol{\eta}\|}{\|\mathbf{x} - \boldsymbol{\eta}\|} = \frac{1}{4\pi \|\mathbf{x} - \boldsymbol{\xi}\|} \quad \text{whenever} \quad \|\mathbf{x}\| = 1.$$

We conclude that their difference

$$G(\mathbf{x}; \boldsymbol{\xi}) = \frac{1}{4\pi} \left(\frac{1}{\|\mathbf{x} - \boldsymbol{\xi}\|} - \frac{\|\boldsymbol{\xi}\|}{\|\boldsymbol{\xi} - \|\boldsymbol{\xi}\|^2 \mathbf{x}\|} \right) \quad (18.75)$$

has the required properties of the Green's function: it satisfies the Laplace equation inside the unit ball except at the delta function singularity $\mathbf{x} = \boldsymbol{\xi}$, and, moreover, $G(\mathbf{x}; \boldsymbol{\xi}) = 0$ has homogeneous Dirichlet conditions on the spherical boundary $\|\mathbf{x}\| = 1$.

With the Green's function in hand, we can apply the general superposition formula (18.59) to arrive at a solution to the Dirichlet boundary value problem for the Poisson equation in the unit ball.

Theorem 18.10. *The solution to the homogeneous Dirichlet boundary value problem*

$$-\Delta u = f, \quad \text{for} \quad \|\mathbf{x}\| < 1, \quad u = 0, \quad \text{for} \quad \|\mathbf{x}\| = 1$$

is

$$u(\mathbf{x}) = \frac{1}{4\pi} \iiint_{\|\boldsymbol{\xi}\| \leq 1} \left(\frac{1}{\|\mathbf{x} - \boldsymbol{\xi}\|} - \frac{\|\boldsymbol{\xi}\|}{\|\boldsymbol{\xi} - \|\boldsymbol{\xi}\|^2 \mathbf{x}\|} \right) f(\boldsymbol{\xi}) d\xi d\eta d\zeta. \quad (18.76)$$

The Green's function can also be used to solve the inhomogeneous boundary value problem

$$-\Delta u = 0, \quad \mathbf{x} \in \Omega, \quad u = h, \quad \mathbf{x} \in \partial\Omega. \quad (18.77)$$

The same argument used in the two-dimensional situation produces the solution

$$u(\mathbf{x}) = - \iint_{\partial\Omega} \frac{\partial G(\mathbf{x}; \boldsymbol{\xi})}{\partial \mathbf{n}} h(\boldsymbol{\xi}) dS. \quad (18.78)$$

In the case when Ω is a solid ball, this integral formula effectively sums the spherical harmonic series (18.46).

18.4. The Heat Equation in Three-Dimensional Media.

Thermal diffusion in a homogeneous, isotropic solid body $\Omega \subset \mathbb{R}^3$ is governed by the three-dimensional *heat equation*

$$\frac{\partial u}{\partial t} = \gamma \Delta u = \gamma \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \quad (x, y, z) \in \Omega. \quad (18.79)$$

The positivity of the body's thermal diffusivity $\gamma > 0$ is required on both physical and mathematical grounds. The physical derivation is exactly the same as the two-dimensional version (17.1), and does not need to be repeated in detail. Briefly, the heat flux vector is proportional to the temperature gradient, $\mathbf{w} = -\kappa \nabla u$, while its divergence is proportional to the rate of change of temperature: $\nabla \cdot \mathbf{w} = -\sigma u_t$. Combining these two physical laws and assuming homogeneity, whereby κ and σ are constant, produces (18.79) with $\gamma = \kappa/\sigma$.

As always, we must impose suitable boundary conditions: Dirichlet conditions $u = h$ that specify the boundary temperature; (homogeneous) Neumann conditions $\partial u / \partial \mathbf{n} = 0$ corresponding to an insulated boundary; or a mixture of the two. Given the initial temperature of the body

$$u(t_0, x, y, z) = f(x, y, z) \quad (18.80)$$

at the initial time t_0 , it can be proved, [48], that the resulting initial-boundary value problem is well-posed, and so there is a unique solution $u(t, x, y, z)$ that is defined for all subsequent times $t \geq t_0$ and depends continuously on the initial data.

As in the one- and two-dimensional versions, we do not lose generality by restricting our attention to homogeneous boundary conditions. Separation of variables method works as usual, and we quickly review the basic ideas. One begins by imposing an exponential ansatz $u(t, \mathbf{x}) = e^{-\lambda t} v(\mathbf{x})$. Substituting into the differential equation and canceling the exponentials, it follows that v satisfies the Helmholtz eigenvalue problem

$$\gamma \Delta v + \lambda v = 0,$$

subject to the relevant boundary conditions. For Dirichlet and mixed boundary conditions, the Laplacian is a positive definite operator, and hence the eigenvalues are all strictly positive,

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots, \quad \text{with} \quad \lambda_n \longrightarrow \infty, \quad \text{as} \quad n \rightarrow \infty.$$

Moreover, on a bounded domain, the Helmholtz eigenfunctions are complete, and so linear superposition implies that the solution can be written as an eigenfunction series

$$u(t, \mathbf{x}) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} v_n(\mathbf{x}). \quad (18.81)$$

The coefficients c_n are uniquely prescribed by the initial condition (18.80):

$$u(t_0, \mathbf{x}) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t_0} v_n(\mathbf{x}) = f(\mathbf{x}). \quad (18.82)$$

Self-adjointness of the boundary value problem implies orthogonality of the eigenfunctions, and hence the coefficients are given by the usual orthogonality formulae

$$c_n = e^{\lambda_n t_0} \frac{\langle f; v_n \rangle}{\|v_n\|^2} = e^{-\lambda_n t_0} \frac{\iiint_{\Omega} f(\mathbf{x}) v_n(\mathbf{x}) dx dy dz}{\iiint_{\Omega} v_n(\mathbf{x})^2 dx dy dz}. \quad (18.83)$$

The resulting solution $u(t, \mathbf{x}) \rightarrow 0$ decays exponentially fast to thermal equilibrium, at a rate equal to the smallest positive eigenvalue $\lambda_1 > 0$. Since the higher modes — the terms with $n \gg 0$ — go to zero extremely rapidly with increasing t , the solution can be well approximated by the first few terms in its Fourier expansion. As a consequence, the heat equation rapidly smoothes out discontinuities and eliminates high frequency noise in the initial data, and so can be used to process three-dimensional images and video — although better nonlinear techniques are now available, [162].

Unfortunately, the explicit formulae for the eigenfunctions and eigenvalues few and far between, [134]. Most explicit eigensolutions of the Helmholtz boundary value problem require a further separation of variables. In a rectangular box, one separates into a product of functions depending upon the individual Cartesian coordinates, and the eigenfunctions are written as products of trigonometric and hyperbolic functions; see Exercise ■ for details. In a cylindrical domain, the separation is effected in cylindrical coordinates, and leads to eigensolutions involving trigonometric and Bessel functions, as outlined in Exercise ■. The most interesting and enlightening case is a spherical domain, and we treat this particular problem in complete detail.

Heating of a Ball

Our goal is to study heat propagation in a solid spherical body, e.g., the earth[†]. For simplicity, we take the diffusivity $\gamma = 1$, and consider the heat equation on a solid spherical ball $B_1 = \{\|\mathbf{x}\| < 1\}$ of radius 1, subject to homogeneous Dirichlet boundary conditions. Once we know how to solve this particular case, an easy scaling argument, as outlined in Exercise ■, will allow us to find the solution for a ball of arbitrary radius and with a general diffusion coefficient.

As usual, when dealing with a spherical geometry, we adopt spherical coordinates r, φ, θ as in (18.13), in terms of which the heat equation takes the form

$$\frac{\partial u}{\partial t} = \Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} + \frac{\cos \varphi}{r^2 \sin \varphi} \frac{\partial u}{\partial \varphi} + \frac{1}{r^2 \sin^2 \varphi} \frac{\partial^2 u}{\partial \theta^2}, \quad (18.84)$$

where we have used our handy formula (18.14) for the Laplacian in spherical coordinates. The standard diffusive separation of variables ansatz

$$u(t, r, \varphi, \theta) = e^{-\lambda t} v(r, \varphi, \theta)$$

[†] In this simplified model, we are assuming that the earth is composed of a completely homogeneous and isotropic solid material.

requires us to analyze the spherical coordinate form of the Helmholtz equation

$$\Delta v + \lambda v = \frac{\partial^2 v}{\partial r^2} + \frac{2}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \varphi^2} + \frac{\cos \varphi}{r^2 \sin \varphi} \frac{\partial v}{\partial \varphi} + \frac{1}{r^2 \sin^2 \varphi} \frac{\partial^2 v}{\partial \theta^2} + \lambda v = 0 \quad (18.85)$$

on the unit ball $\Omega = \{r < 1\}$ with homogeneous Dirichlet boundary conditions. To make further progress, we invoke a second variable separation, splitting off the radial coordinate by setting

$$v(r, \varphi, \theta) = p(r) w(\varphi, \theta).$$

The function w must be 2π periodic in θ and well-defined at the poles $\varphi = 0, \pi$. Substituting our ansatz into (18.85), and separating all the r -dependent terms from those terms depending upon the angular variables φ, θ leads to a pair of differential equations: The first is an ordinary differential equation

$$r^2 \frac{d^2 p}{dr^2} + 2r \frac{dp}{dr} + (\lambda r^2 - \mu)p = 0, \quad (18.86)$$

for the radial component $p(r)$, while the second is a familiar partial differential equation

$$\Delta_S w + \mu w = \frac{\partial^2 w}{\partial \varphi^2} + \frac{\cos \varphi}{\sin \varphi} \frac{\partial w}{\partial \varphi} + \frac{1}{\sin^2 \varphi} \frac{\partial^2 w}{\partial \theta^2} + \mu w = 0, \quad (18.87)$$

for its angular counterpart $w(\varphi, \theta)$. The operator Δ_S is the *spherical Laplacian* from (18.17). In Section 18.2, we showed that its eigenvalues are

$$\mu_m = m(m+1) \quad \text{for} \quad m = 0, 1, 2, 3, \dots$$

The m^{th} eigenvalue admits $2m+1$ linearly independent eigenfunctions — the spherical harmonics $Y_m^0, \dots, Y_m^m, \tilde{Y}_m^1, \dots, \tilde{Y}_m^m$ defined in (18.33).

The radial ordinary differential equation (18.86) can be solved by setting

$$p(r) = \sqrt{r} q(r).$$

We use the product rule to relate their derivatives

$$p = \frac{1}{\sqrt{r}} q, \quad \frac{dp}{dr} = \frac{1}{\sqrt{r}} \frac{dq}{dr} - \frac{1}{2r^{3/2}} q, \quad \frac{d^2 p}{dr^2} = \frac{1}{\sqrt{r}} \frac{d^2 q}{dr^2} - \frac{1}{r^{3/2}} \frac{dq}{dr} + \frac{3}{4r^{5/2}} q.$$

Substituting these expressions back into (18.86) with $\mu = \mu_m = m(m+1)$, and multiplying the resulting equation by $\sqrt{r}$, we discover that $q(r)$ must solve the differential equation

$$r^2 \frac{d^2 q}{dr^2} + r \frac{dq}{dr} + \left[\lambda r^2 - \left(m + \frac{1}{2}\right)^2 \right] q = 0, \quad (18.88)$$

which turns out to be the rescaled Bessel equation (17.52) of half integer order $m + \frac{1}{2}$. As a result, the solution to (18.88) that remains bounded at $r = 0$ is (up to scalar multiple) the rescaled Bessel function

$$q(r) = J_{m+1/2}(\sqrt{\lambda} r).$$

The corresponding solution

$$p(r) = r^{-1/2} J_{m+1/2}(\sqrt{\lambda} r) \quad (18.89)$$

to (18.86) is important enough to warrant a special name.

Definition 18.11. The *spherical Bessel function* of order $m \geq 0$ is defined by the formula

$$S_m(x) = \sqrt{\frac{\pi}{2x}} J_{m+1/2}(x). \quad (18.90)$$

Remark: The multiplicative factor $\sqrt{\pi/2}$ is included in the definition so as to avoid annoying factors of $\sqrt{\pi}$ and $\sqrt{2}$ in subsequent formulae.

Surprisingly, unlike the Bessel functions of integer order, the spherical Bessel functions are elementary functions! According to formula (C.67), the spherical Bessel function of order 0 is

$$S_0(x) = \frac{\sin x}{x}. \quad (18.91)$$

The higher order spherical Bessel functions can be obtained by use of the general recurrence relation

$$S_{m+1}(x) = -\frac{dS_m}{dx} + \frac{m}{x} S_m(x), \quad (18.92)$$

which is a consequence of Proposition C.13. The next few are, therefore,

$$\begin{aligned} S_1(x) &= -\frac{dS_0}{dx} = -\frac{\cos x}{x} + \frac{\sin x}{x^2}, \\ S_2(x) &= -\frac{dS_1}{dx} + \frac{S_1}{x} = -\frac{\sin x}{x} - \frac{3 \cos x}{x^2} + \frac{3 \sin x}{x^3}, \\ S_3(x) &= -\frac{dS_2}{dx} + \frac{2 S_2}{x} = \frac{\cos x}{x} - \frac{6 \sin x}{x^2} - \frac{15 \cos x}{x^3} + \frac{15 \sin x}{x^4}, \end{aligned} \quad (18.93)$$

and so on. Graphs can be found in Figure 18.6. Our radial solution (18.89) is, apart from an inessential constant multiple, a rescaled spherical Bessel function of order m :

$$v_m(r) = S_m(\sqrt{\lambda} r).$$

So far, we have not taken into account the (homogeneous) Dirichlet boundary condition at $r = 1$. This requires

$$p(1) = 0, \quad \text{and hence} \quad S_m(\sqrt{\lambda}) = 0.$$

Therefore, $\sqrt{\lambda}$ must be a root of the m^{th} order spherical Bessel function. We introduce the notation

$$0 < \sigma_{m,1} < \sigma_{m,2} < \sigma_{m,e} < \dots$$

to denote the successive (positive) *spherical Bessel roots*, satisfying

$$S_m(\sigma_{m,n}) = 0 \quad \text{for} \quad n = 1, 2, \dots \quad (18.94)$$

In particular the roots of the zeroth order spherical Bessel function $S_0(x) = x^{-1} \sin x$ are just the integer multiples of π :

$$\sigma_{0,n} = n\pi \quad \text{for} \quad n = 1, 2, \dots$$

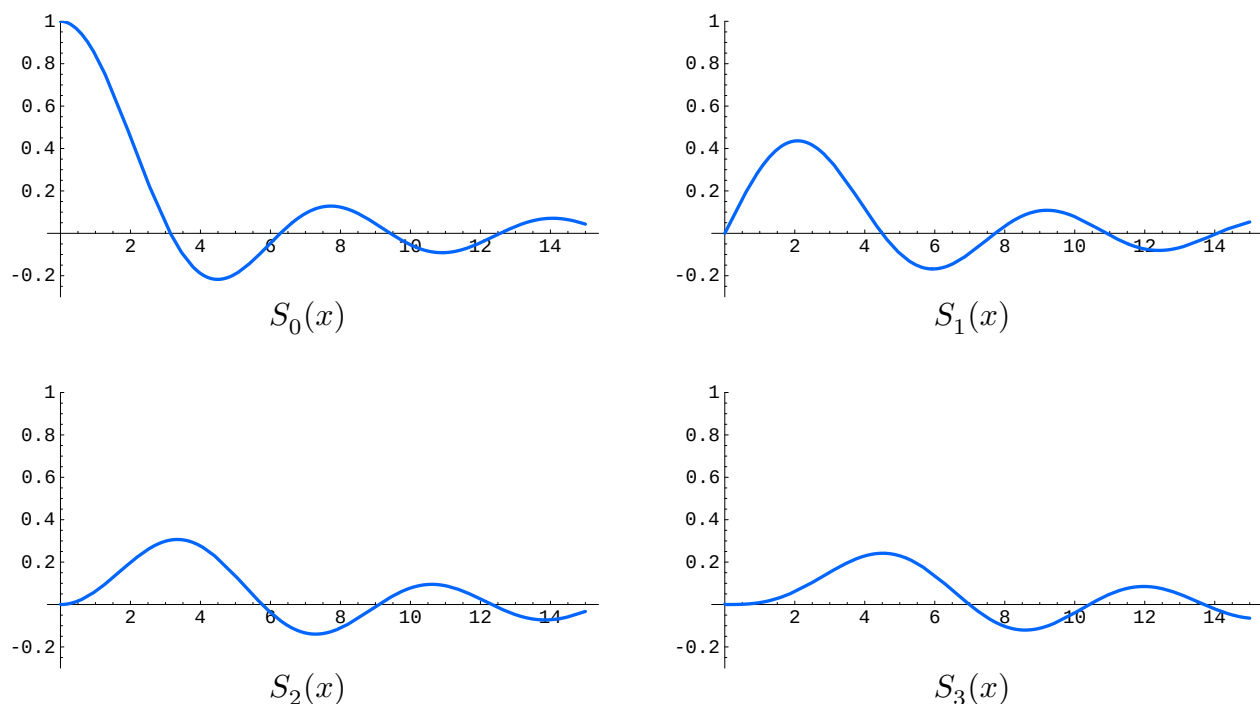


Figure 18.6. Spherical Bessel Function.

Spherical Bessel Roots $\sigma_{m,n}$

$n \backslash m$	0	1	2	3	4	5	6	7	
1	3.1416	4.4934	5.7635	8.1826	9.3558	10.5128	11.6570	12.7908	...
2	6.2832	7.7253	9.0950	11.7049	12.9665	$\vdots$	$\vdots$	$\vdots$	
3	9.4248	10.9041	12.3229	$\vdots$	$\vdots$				
4	12.5664	$\vdots$	$\vdots$						
$\vdots$	$\vdots$								

A table of all spherical Bessel roots that are < 13 appears above. The columns of the table are indexed by m , the order, while the rows are indexed by n , the root number.

Re-assembling the individual constituents, we have now demonstrated that the separable eigenfunctions of the Helmholtz equation on a solid ball of radius 1, when subject to homogeneous Dirichlet boundary conditions, are products of spherical Bessel functions and spherical harmonics,

$$v_{k,m,n}(r, \varphi, \theta) = S_m(\sigma_{m,n} r) Y_m^k(\varphi, \theta), \quad \tilde{v}_{k,m,n}(r, \varphi, \theta) = S_m(\sigma_{m,n} r) \tilde{Y}_m^k(\varphi, \theta). \quad (18.95)$$

The corresponding eigenvalues

$$\lambda_{m,n} = \sigma_{m,n}^2, \quad m = 0, 1, 2, \dots, \quad n = 1, 2, 3, \dots, \quad (18.96)$$

are given by the squared spherical Bessel roots. Since there are $2m + 1$ independent spherical harmonics of order m , the eigenvalue $\lambda_{m,n}$ admits $2m + 1$ linearly independent eigenfunctions, namely $v_{0,m,n}, \dots, v_{m,m,n}, \tilde{v}_{1,m,n}, \dots, \tilde{v}_{m,m,n}$. In particular, the radially symmetric solutions are the eigenfunctions with $k = m = 0$, namely

$$v_n(r) = v_{0,0,n}(r) = S_0(\sigma_{0,n} r) = \frac{\sin n \pi r}{n \pi r}, \quad n = 1, 2, \dots. \quad (18.97)$$

Further analysis demonstrates that the separable solutions (18.95) form a complete system of eigenfunctions for the Helmholtz equation on the unit ball subject to homogeneous Dirichlet boundary conditions, cf. [47].

We have thus completely determined the basic separable solutions to the heat equation on a solid unit ball subject to homogeneous Dirichlet boundary conditions. They are products of exponential functions of time, spherical Bessel functions of the radius and spherical harmonics:

$$\begin{aligned} u_{k,m,n}(t, r, \varphi, \theta) &= e^{-\sigma_{m,n}^2 t} S_m(\sigma_{m,n} r) Y_m^k(\varphi, \theta), \\ \tilde{u}_{k,m,n}(t, r, \varphi, \theta) &= e^{-\sigma_{m,n}^2 t} S_m(\sigma_{m,n} r) \tilde{Y}_m^k(\varphi, \theta). \end{aligned} \quad (18.98)$$

The general solution can be written as an infinite “Fourier–Bessel–spherical harmonic” series in these fundamental modes

$$\begin{aligned} u(t, r, \theta, \varphi) &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} e^{-\sigma_{m,n}^2 t} S_m(\sigma_{m,n} r) \left(\frac{c_{0,m,n}}{2} Y_m^0(\varphi, \theta) + \right. \\ &\quad \left. + \sum_{k=1}^m \left[c_{k,m,n} Y_m^k(\varphi, \theta) + \tilde{c}_{k,m,n} \tilde{Y}_m^k(\varphi, \theta) \right] \right). \end{aligned} \quad (18.99)$$

The series’ coefficients $c_{k,m,n}, \tilde{c}_{k,m,n}$ are uniquely prescribed by the initial data; explicit formulae follow from the usual orthogonality relations among the eigenfunctions. Detailed formulae are relegated to the exercises. In particular, the slowest decaying mode is the spherically symmetric function

$$u_{0,0,1}(t, r) = \frac{e^{-\pi^2 t} \sin \pi r}{r} \quad (18.100)$$

corresponding to the smallest eigenvalue $\lambda_{0,1} = \sigma_{0,1}^2 = \pi^2$. Therefore, typically, the decay to thermal equilibrium of a unit sphere is at an exponential rate of $\pi^2 \approx 9.8696$, or, to a very rough approximation, 10.

The Fundamental Solution to the Heat Equation

For the heat equation (as well as more general diffusion equations), the fundamental solution measures the response of the body to a concentrated unit heat source. Thus, given a point $\boldsymbol{\xi} = (\xi, \eta, \zeta) \in \Omega$ within the body, the fundamental solution

$$u(t, \mathbf{x}) = F(t, \mathbf{x}; \boldsymbol{\xi}) = F(t, x, y, z; \xi, \eta, \zeta)$$

solves the initial-boundary value problem

$$u_t = \Delta u, \quad u(0, \mathbf{x}) = \delta(\mathbf{x} - \boldsymbol{\xi}), \quad \text{for } \mathbf{x} \in \Omega, \quad t > 0, \quad (18.101)$$

subject to the selected homogeneous boundary conditions — Dirichlet, Neumann or mixed.

In general, the fundamental solution has no explicit formula, although in certain domains it is possible to construct it as an eigenfunction series. The one case amenable to a complete analysis is when the heat is distributed over all of three-dimensional space, so $\Omega = \mathbb{R}^3$. We recall that Lemma 17.1 showed how to construct solutions of the two-dimensional heat equation as products of one-dimensional solutions. In a similar manner, if $v(t, x)$, $w(t, x)$ and $q(t, x)$ are any three solutions to the one-dimensional heat equation $u_t = \gamma u_{xx}$, then their product

$$u(t, x, y, z) = p(t, x) q(t, y) r(t, z) \quad (18.102)$$

is a solution to the three-dimensional heat equation

$$u_t = \gamma (u_{xx} + u_{yy} + u_{zz}).$$

In particular, choosing

$$p(t, x) = \frac{e^{-(x-\xi)^2/4\gamma t}}{2\sqrt{\pi\gamma t}}, \quad q(t, y) = \frac{e^{-(y-\eta)^2/4\gamma t}}{2\sqrt{\pi\gamma t}}, \quad r(t, z) = \frac{e^{-(z-\zeta)^2/4\gamma t}}{2\sqrt{\pi\gamma t}},$$

to all be one-dimensional fundamental solutions, we are immediately led to the fundamental solution in the form of a three-dimensional Gaussian kernel.

Theorem 18.12. *The fundamental solution*

$$F(t, \mathbf{x}; \boldsymbol{\xi}) = F(t, \mathbf{x} - \boldsymbol{\xi}) = \frac{e^{-\|\mathbf{x}-\boldsymbol{\xi}\|^2/4\gamma t}}{8(\pi\gamma t)^{3/2}} \quad (18.103)$$

solves the three-dimensional heat equation $u_t = \gamma \Delta u$ on $\mathbb{R}^3$ with an initial temperature equal to a delta function concentrated at the point $\mathbf{x} = \boldsymbol{\xi}$.

Thus, the initially concentrated heat energy immediately begins to spread out in a radially symmetric manner, with a minuscule, but nonzero effect felt at arbitrarily large distances away from the initial concentration. At each individual point $\mathbf{x} \in \mathbb{R}^3$, after an initial warm-up, the temperature decays back to zero at a rate proportional to $t^{-3/2}$ — even more rapidly than in two dimensions because, intuitively, there are more directions for the heat energy to disperse.

To solve a more general initial value problem with the initial temperature $u(0, x, y, z) = f(x, y, z)$ distributed over all of space, we first write

$$f(x, y, z) = \iiint f(\boldsymbol{\xi}) \delta(\mathbf{x} - \boldsymbol{\xi}) d\xi d\eta d\zeta$$

as a linear superposition of delta functions. By linearity, the solution to the initial value problem is given by the corresponding superposition

$$u(t, \mathbf{x}) = \frac{1}{8(\pi\gamma t)^{3/2}} \iiint f(\boldsymbol{\xi}) e^{-\|\mathbf{x}-\boldsymbol{\xi}\|^2/4\gamma t} d\xi d\eta d\zeta. \quad (18.104)$$

of the fundamental solutions. Since the fundamental solution has exponential decay as $\|\mathbf{x}\| \rightarrow \infty$, the superposition formula is valid even for initial temperature distributions which are moderately increasing at large distances. We remark that the integral (18.104) has the form of a three-dimensional convolution

$$u(t, \mathbf{x}) = F(t, \mathbf{x}) * f(\mathbf{x}) = \iiint f(\boldsymbol{\xi}) F(t, \mathbf{x} - \boldsymbol{\xi}) d\xi d\eta d\zeta \quad (18.105)$$

of the initial data with a one-parameter family of increasingly spread out Gaussian filters. Consequently, convolution with the Gaussian kernel has a smoothing effect on the initial temperature distribution.

18.5. The Wave Equation in Three-Dimensional Media.

The three-dimensional wave equation

$$u_{tt} = c^2 \Delta u = c^2(u_{xx} + u_{yy} + u_{zz}), \quad (18.106)$$

in which c denotes the velocity of light, governs the propagation of electromagnetic waves (light, radio, X-rays, etc.) in a homogeneous medium, including (in the absence of gravitational effects) empty space. While the electric and magnetic vector fields $\mathbf{E}, \mathbf{B}$ are intrinsically coupled by the more complicated system of Maxwell's equations, each individual component satisfies the wave equation; see Exercise ■ for details.

The wave equation also models certain classes[†] of vibrations of a uniform solid body. The solution $u(t, \mathbf{x}) = u(t, x, y, z)$ represents a scalar-valued displacement of the body at time t and position $\mathbf{x} = (x, y, z) \in \Omega \subset \mathbb{R}^3$. For example, $u(t, \mathbf{x})$ might represent the radial displacement of the body. One imposes suitable boundary conditions, e.g., Dirichlet, Neumann or mixed, on $\partial\Omega$, along with a pair of initial conditions

$$u(0, \mathbf{x}) = f(\mathbf{x}), \quad \frac{\partial u}{\partial t}(0, \mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in \Omega, \quad (18.107)$$

that specify the body's initial displacement and initial velocity. As long as the initial and boundary data are reasonably nice, there exists a unique solution to the initial-boundary value problem for all $-\infty < t < \infty$, cf. [47]. Thus, in contrast to the heat equation, one can follow solutions to the wave equation both forwards and backwards in time; see also Exercise ■.

Let us fix our attention on the homogeneous boundary value problem. The fundamental vibrational modes are found by imposing our usual trigonometric ansatz

$$u(t, x, y, z) = \cos(\omega t) v(x, y, z).$$

[†] Since the solution $u(t, \mathbf{x})$ to the wave equation is scalar-valued, it cannot measure the full range of possible three-dimensional motions of a solid body. The more complicated dynamical systems governing the elastic motions of solids are discussed in Exercise ■.

Substituting into the wave equation (18.106), we discover (yet again) that $v(x, y, z)$ must be an eigenfunction solving the associated Helmholtz eigenvalue problem

$$\Delta v + \lambda v = 0, \quad \text{where} \quad \lambda = \frac{\omega^2}{c^2}, \quad (18.108)$$

coupled to the relevant boundary conditions. In the positive definite cases, i.e., Dirichlet and mixed boundary conditions, the eigenvalues $\lambda_k = \omega_k^2/c^2 > 0$ are all positive; each eigenfunction $v_k(x, y, z)$ yields two normal vibrational modes

$$u_k(t, x, y, z) = \cos(\omega_k t) v_k(x, y, z), \quad \tilde{u}_k(t, x, y, z) = \sin(\omega_k t) v_k(x, y, z),$$

of frequency $\omega_k = c \sqrt{\lambda_k}$ equal to the square root of the corresponding eigenvalue multiplied by the wave speed. The general solution is a quasi-periodic linear combination

$$u(t, x, y, z) = \sum_{k=1}^{\infty} (a_k \cos \omega_k t + b_k \sin \omega_k t) v_k(x, y, z) \quad (18.109)$$

of these fundamental vibrational modes. The coefficients a_k, b_k are uniquely prescribed by the initial conditions (18.107). Thus,

$$\begin{aligned} u(0, x, y, z) &= \sum_{k=1}^{\infty} a_k v_k(x, y, z) = f(x, y, z), \\ \frac{\partial u}{\partial t}(0, x, y, z) &= \sum_{k=1}^{\infty} \omega_k b_k v_k(x, y, z) = g(x, y, z). \end{aligned}$$

The explicit formulas follow immediately from the orthogonality of the eigenfunctions:

$$a_k = \frac{\langle f; v_k \rangle}{\|v_k\|^2} = \frac{\iiint_{\Omega} f v_k \, dx \, dy \, dz}{\iiint_{\Omega} v_k^2 \, dx \, dy \, dz}, \quad b_k = \frac{1}{\omega_k} \frac{\langle g; v_k \rangle}{\|v_k\|^2} = \frac{\iiint_{\Omega} g v_k \, dx \, dy \, dz}{\omega_k \iiint_{\Omega} v_k^2 \, dx \, dy \, dz}. \quad (18.110)$$

In the positive semi-definite Neumann boundary value problem, there is an additional zero eigenvalue $\lambda_0 = 0$ corresponding to the constant null eigenfunction $v_0(x, y, z) \equiv 1$. This results in two additional terms in the eigenfunction expansion — a constant term

$$a_0 = \frac{1}{\text{vol } \Omega} \iiint_{\Omega} f(x, y, z) \, dx \, dy \, dz$$

that equals the average initial displacement, and an unstable mode $b_0 t$ that grows linearly in time, whose speed

$$b_0 = \frac{1}{\text{vol } \Omega} \iiint_{\Omega} g(x, y, z) \, dx \, dy \, dz$$

is the average of the initial velocity over the entire body. The unstable mode will be excited if and only if there is a non-zero net initial velocity: $b_0 \neq 0$.

Most of the basic solution techniques we learned in the two-dimensional case apply here, and we will not dwell on the details. The case of a rectangular box is a particularly straightforward application of the method of separation of variables, and is outlined in the exercises. A similar analysis, now in cylindrical coordinates, can be applied to the case of a vibrating cylinder. The most interesting case is that of a solid spherical ball, which is the subject of the next subsection.

Vibrations of a Ball

Let us focus on the radial vibrations of a solid ball, as modeled by the three-dimensional wave equation (18.106). The solution $u(t, x, y, z)$ represents the radial displacement of the “atom” that is situated at position (x, y, z) when the ball is at rest.

For simplicity, we look at the Dirichlet boundary value problem on the unit ball $B_1 = \{\|\mathbf{x}\| < 1\}$. The normal modes of vibration are governed by the Helmholtz equation (18.108) subject to homogeneous Dirichlet boundary conditions. According to (18.95), the eigenfunctions are

$$\begin{aligned} v_{k,m,n}(r, \varphi, \theta) &= S_n(\sigma_{n,m} r) Y_m^k(\varphi, \theta), \\ \tilde{v}_{k,m,n}(r, \varphi, \theta) &= S_m(\sigma_{m,n} r) \tilde{Y}_m^k(\varphi, \theta), \end{aligned} \quad \begin{array}{ll} n = 1, 2, 3, \dots, \\ \text{for } m = 0, 1, 2, \dots, \\ k = 0, 1, \dots, m. \end{array} \quad (18.111)$$

Here S_m denotes the m^{th} order spherical Bessel function (18.90), $\sigma_{m,n}$ is its n^{th} root, while $Y_n^m, \tilde{Y}_n^m$ are the spherical harmonics (18.33). Each eigenvalue

$$\lambda_{m,n} = \sigma_{m,n}^2, \quad m = 0, 1, 2, \dots, \quad n = 1, 2, 3, \dots,$$

corresponds to $2m + 1$ independent eigenfunctions, namely

$$v_{k,m,0}(r, \varphi, \theta), \quad v_{k,m,1}(r, \varphi, \theta), \quad \dots \quad v_{k,m,m}(r, \varphi, \theta), \quad \tilde{v}_{k,m,1}(r, \varphi, \theta), \quad \dots \quad \tilde{v}_{k,m,m}(r, \varphi, \theta).$$

Consequently, the fundamental vibrational frequencies of a solid ball

$$\omega_{m,n} = c \sqrt{\lambda_{m,n}} = c \sigma_{m,n}, \quad m = 0, 1, 2, \dots, \quad n = 1, 2, 3, \dots, \quad (18.112)$$

are equal to the spherical Bessel roots $\sigma_{m,n}$ multiplied by the wave speed. There are a total of $2(2m + 1)$ independent vibrational modes associated with each distinct frequency (18.112), namely

$$\begin{aligned} u_{k,m,n}(t, r, \varphi, \theta) &= \cos(c \sigma_{m,n} t) S_m(\sigma_{m,n} r) Y_m^k(\varphi, \theta), \\ \hat{u}_{k,m,n}(t, r, \varphi, \theta) &= \sin(c \sigma_{m,n} t) S_m(\sigma_{m,n} r) Y_m^k(\varphi, \theta), \\ \tilde{u}_{k,m,n}(t, r, \varphi, \theta) &= \cos(c \sigma_{m,n} t) S_m(\sigma_{m,n} r) \tilde{Y}_m^k(\varphi, \theta), \\ \hat{\tilde{u}}_{k,m,n}(t, r, \varphi, \theta) &= \sin(c \sigma_{m,n} t) S_m(\sigma_{m,n} r) \tilde{Y}_m^k(\varphi, \theta). \end{aligned} \quad \begin{array}{l} n = 1, 2, 3, \dots, \\ m = 0, 1, 2, \dots, \\ k = 0, 1, \dots, m. \end{array} \quad (18.113)$$

Relative Spherical Bessel Roots $\sigma_{k,m}/\sigma_{0,1}$

$n \backslash m$	0	1	2	3	4	6	7	8	...
1	1.0000	1.4303	1.8346	2.2243	2.6046	2.9780	3.3463	3.7105	...
2	2.0000	2.4590	2.8950	3.3159	3.7258	$\vdots$	$\vdots$	$\vdots$	
3	3.0000	3.4709	3.9225	$\vdots$	$\vdots$				
4	4.0000	$\vdots$	$\vdots$						
$\vdots$	$\vdots$								

In particular, the radially symmetric modes of vibration have, according to (18.91), the elementary form

$$u_{0,0,n}(r, \varphi, \theta) = \cos(cn\pi t) S_0(n\pi r) = \frac{\cos cn\pi t \sin n\pi r}{r}, \quad k = 1, 2, 3, \dots \quad (18.114)$$

$$\hat{u}_{0,0,n}(r, \varphi, \theta) = \sin(cn\pi t) S_0(n\pi r) = \frac{\sin cn\pi t \sin n\pi r}{r},$$

Their vibrational frequencies, $\omega_{0,n} = cn\pi$, are integral multiples of the lowest frequency $\omega_{0,1} = \pi$. Therefore, interestingly, if you only excite the radially symmetric modes, the resulting motion of the ball is periodic.

More generally, adopting the same scaling argument as in (17.111), we conclude that the fundamental frequencies for a solid ball of radius R and wave speed c are given by $\omega_{m,n} = c\sigma_{m,n}/R$. The relative vibrational frequencies

$$\frac{\omega_{m,n}}{\omega_{0,1}} = \frac{\sigma_{m,n}}{\sigma_{0,1}} = \frac{\sigma_{m,n}}{\pi} \quad (18.115)$$

are independent of the size of the ball R or the wave speed c . In the accompanying table, we display all relative vibrational frequencies that are less than 4 in magnitude.

The purely radial modes of vibration (18.114) have individual frequencies

$$\omega_{0,n} = \frac{n\pi c}{R}, \quad \text{so} \quad \frac{\omega_{0,n}}{\omega_{0,1}} = n,$$

and appear in the first column of the table. The lowest frequency is $\omega_{0,1} = \pi c/R$, corresponding to a vibration with period $2\pi/\omega_{0,1} = 2R/c$. In particular, for the earth, the radius $R \approx 6,000$ km and the wave speed in rock is, on average, $c \approx 5$ km/sec, so that the fundamental mode of vibration has period $2R/c \approx 2400$ seconds, or 40 minutes. Vibrations of the earth are also known as *seismic waves* and, of course, earthquakes are their most severe manifestation. Understanding the modes of vibration is an issue of critical importance in geophysics and civil engineering, including the design of structures, buildings and bridges and the avoidance of resonant frequencies.

Of course, we have suppressed almost all interesting terrestrial geology in this very crude approximation, which has been based on the assumption that the earth is a uniform body, vibrating only in its radial direction. A more realistic modeling of the vibrations of the earth requires an understanding of the basic partial differential equations of linear and nonlinear elasticity, [87]. Nonuniformities in the earth lead to scattering of the vibrational waves, which are then used to locate subterranean geological structures, e.g., oil and gas deposits. We refer the interested reader to [6] for a comprehensive introduction to mathematical seismology.

Example 18.13. The radial vibrations of a hollow spherical shell (e.g., an elastic balloon) are governed by the differential equation

$$u_{tt} = c^2 \Delta_S[u] = c^2 \left(\frac{\partial^2 u}{\partial \varphi^2} + \frac{\cos \varphi}{\sin \varphi} \frac{\partial u}{\partial \varphi} + \frac{1}{\sin^2 \varphi} \frac{\partial^2 u}{\partial \theta^2} \right), \quad (18.116)$$

where Δ_S denotes the spherical Laplacian (18.17). The radial displacement $u(t, \varphi, \theta)$ of a point on the sphere only depends on time t and the angular coordinates φ, θ . The solution $u(t, \varphi, \theta)$ is required to be 2π periodic in the azimuthal angle θ and bounded at the poles $\varphi = 0, \pi$.

According to (18.33), the n^{th} eigenvalue $\lambda_n = n(n+1)$ of the spherical Laplacian possesses $2n+1$ linearly independent eigenfunctions, namely, the spherical harmonics

$$Y_n^0(\varphi, \theta), Y_n^1(\varphi, \theta), \dots, Y_n^n(\varphi, \theta), \tilde{Y}_n^1(\varphi, \theta), \dots, \tilde{Y}_n^n(\varphi, \theta).$$

As a consequence, the fundamental frequencies of vibration for a spherical shell are

$$\omega_n = c \sqrt{\lambda_n} = c \sqrt{n(n+1)}, \quad n = 1, 2, \dots \quad (18.117)$$

The vibrational solutions are quasi-periodic combinations of the fundamental spherical harmonic modes

$$\begin{aligned} \cos(\sqrt{n(n+1)} t) Y_n^m(\varphi, \theta), & \quad \sin(\sqrt{n(n+1)} t) Y_n^m(\varphi, \theta), \\ \cos(\sqrt{n(n+1)} t) \tilde{Y}_n^m(\varphi, \theta), & \quad \sin(\sqrt{n(n+1)} t) \tilde{Y}_n^m(\varphi, \theta). \end{aligned} \quad (18.118)$$

Representative graphs can be seen in Figure 18.3. The smallest positive eigenvalue is $\lambda_1 = 2$, yielding a lowest tone of frequency $\omega_1 = c\sqrt{2}$. The higher order frequencies are irrational multiples of the fundamental frequency, implying that a vibrating spherical bell sounds percussive to our ears.

One further remark is in order. The spherical Laplacian operator is only positive semi-definite, since the lowest mode has eigenvalue $\lambda_0 = 0$, which corresponds to the constant null eigenfunction $v_0(\varphi, \theta) = Y_0^0(\varphi, \theta) \equiv 1$. Therefore, the wave equation (18.116) admits an unstable mode $b_{0,0} t$, corresponding to a uniform radial inflation; its coefficient

$$b_{0,0} = \frac{3}{4\pi} \iint_{S_1} \frac{\partial u}{\partial t}(0, \varphi, \theta) dS$$

represents the sphere's average initial velocity. The existence of such an unstable mode is an artifact of the simplified linear model we are using, that fails to account for nonlinearly elastic effects that serve to constrain the inflation of a spherical balloon.

18.6. Spherical Waves and Huygens' Principle.

For any dynamical (time-varying) partial differential equation, the fundamental solution measures the effect of applying an instantaneous concentrated unit impulse at a single point. Two representative physical effects to keep in mind are the light waves emanating from a sudden concentrated blast, e.g., a lightning bolt or a stellar supernova, and the sound waves due to an explosion or thunderclap, propagating in air at a much slower speed. Linear superposition utilizes the fundamental solution to build up more general solutions to initial value problems. For the wave and other second order vibrational equations, the impulse can be applied either to the initial displacement or to the initial velocity, resulting in two types of fundamental solution.

In a uniform isotropic medium, an initial concentrated blast results in a spherically expanding wave, moving away at the speed of light (or sound) in all directions. Invoking translation invariance, we will assume that the source of the disturbance is at the origin, and so the solution $u(t, \mathbf{x})$ should only depend on the distance $r = \|\mathbf{x}\|$ from the source. We adopt spherical coordinates and look for a solution $u = u(t, r)$ to the three-dimensional wave equation with no angular dependence. Substituting the formula (18.14) for the spherical Laplacian and setting both angular derivatives to 0, we are led to the following partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} \right), \quad (18.119)$$

that governs the propagation of spherically symmetric waves in three-dimensional space. Surprisingly, we can explicitly solve this partial differential equation. The secret is to multiply both sides of the equation by r :

$$\frac{\partial^2 (ru)}{\partial t^2} = r \frac{\partial^2 u}{\partial t^2} = c^2 \left(r \frac{\partial^2 u}{\partial r^2} + 2 \frac{\partial u}{\partial r} \right) = c^2 \frac{\partial^2}{\partial r^2} (ru),$$

Thus, the function

$$w(t, r) = r u(t, r)$$

solves the one-dimensional wave equation

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial r^2}. \quad (18.120)$$

According to Theorem 14.9, the general solution to (18.120) can be written in d'Alembert form

$$w(t, r) = p(r - ct) + q(r + ct),$$

where $p(\xi)$ and $q(\eta)$ are arbitrary functions of a single characteristic variable. Therefore, spherically symmetric solutions to the three-dimensional wave equation assume the form

$$u(t, r) = \frac{p(r - ct)}{r} + \frac{q(r + ct)}{r}. \quad (18.121)$$

The first term

$$u(t, r) = \frac{p(r - ct)}{r} \quad (18.122)$$

represents a wave moving at speed c in the direction of increasing r , and so describes the effect of a variable light source that is concentrated at the origin, e.g., a pulsating quasar in interstellar space. To highlight this interpretation, let us concentrate on the case when $p(\xi) = \delta(\xi - a)$ is a delta function, keeping in mind that more general solutions can then be assembled by linear superposition. The induced solution

$$u(t, r) = \frac{\delta(r - ct - a)}{r} = \frac{\delta(r - c(t - t_0))}{r}, \quad \text{where} \quad t_0 = -\frac{a}{c}. \quad (18.123)$$

represents a spherical wave propagating through space. At the instant $t = t_0$, the light is entirely concentrated at the origin $r = 0$. The signal then moves away from the origin in all directions at speed c . At each later time $t > t_0$, the wave is concentrated on the surface of a sphere of radius $r = c(t - t_0)$. Its intensity at each point on the sphere, however, has decreased by a factor $1/r$, and so, the farther from the source, the dimmer the light. A stationary observer sitting at a fixed point in space will only see an instantaneous flash of light of intensity $1/r$ as the spherical wave passes by at time $t = t_0 + r/c$, where r is the observer's distance from the light source. A similar statement holds for sound waves — the sound of the explosion will only last momentarily. Thunder and lightning are the most familiar examples of this everyday phenomenon.

On the other hand, for $t < t_0$, the impulse is concentrated at a negative radius $r = c(t - t_0) < 0$. To interpret this, note that, for a given value of the spherical angles φ, θ , the point

$$x = r \sin \varphi \cos \theta, \quad y = r \sin \varphi \sin \theta, \quad z = r \cos \varphi,$$

for $r < 0$ lies on the antipodal point of the sphere of radius $|r|$, so that replacing r by $-r$ has the same effect as changing $\mathbf{x}$ to $-\mathbf{x}$. Thus, the solution (18.123) represents a concentrated spherically symmetric light wave arriving from the edges of the universe at speed c , that strengthens in intensity as it collapses into the origin at $t = t_0$. After collapse, it immediately reappears and expands back out into the universe.

The second solution in the d'Alembert formula (18.121) has, in fact, exactly the same physical form. Indeed, if we set

$$\tilde{r} = -r, \quad \tilde{p}(\xi) = -q(-\xi), \quad \text{then} \quad \frac{q(r + ct)}{r} = \frac{\tilde{p}(\tilde{r} - ct)}{\tilde{r}}.$$

Thus, the second d'Alembert solution is redundant, and we only need to consider solutions of the form (18.122) from now on.

To effectively utilize such spherical wave solutions, we need to understand the nature of their originating singularity. For simplicity, we set $a = 0$ in (18.123) and concentrate on the particular solution

$$u(t, r) = \frac{\delta(r - ct)}{r}, \quad (18.124)$$

which has a singularity at the origin $r = 0$ when $t = 0$. We need to pin down precisely which sort of distribution this solution represents. Invoking the limiting definition is tricky,

and it will be easier to work with the dual characterization of a distribution as a linear functional. Thus, at a fixed time $t \geq 0$, we must evaluate the inner product

$$\langle u; f \rangle = \iiint u(t, x, y, z) f(x, y, z) dx dy dz.$$

of the solution with a smooth test function $f(\mathbf{x}) = f(x, y, z)$. We rewrite the triple integral in spherical coordinates using the change of variables formula (B.68), whereby

$$\langle u; f \rangle = \int_0^{2\pi} \int_0^\pi \int_0^\infty \frac{\delta(r - ct)}{r} f(r, \varphi, \theta) r^2 \sin \varphi dr d\varphi d\theta$$

When $t \neq 0$, the r integration can be immediately computed, and so

$$\langle u; f \rangle = ct \int_0^{2\pi} \int_0^\pi f(ct, \varphi, \theta) \sin \varphi d\varphi d\theta = 4\pi ct M_{ct}^{\mathbf{0}}[f], \quad (18.125)$$

where, according to (B.43),

$$M_{ct}^{\mathbf{0}}[f] = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi f(ct, \varphi, \theta) \sin \varphi d\varphi d\theta = \frac{1}{4\pi c^2 t^2} \iint_{S_{ct}} f dS \quad (18.126)$$

is the mean or average value of the function f on the sphere $S_{ct} = \{ \|\mathbf{x}\| = ct \}$ of radius $r = ct$. In particular, the mean over the limiting sphere of radius $r = 0$ reduces to the value of the function at the origin:

$$M_0^{\mathbf{0}}[f] = f(\mathbf{0}). \quad (18.127)$$

Thus, in the limit as $t \rightarrow 0$, (18.125) implies that

$$\langle u; f \rangle = 0 \quad \text{for all functions } f,$$

and hence $u(0, r) \equiv 0$ represents a zero initial displacement.

In the absense of any intial displacement, how, then, can the solution (18.124) be non-zero? Clearly, this must be the result of a nonzero initial velocity. To find $u_t(0, r)$, we differentiate (18.125) with respect to t , whereby

$$\begin{aligned} \left\langle \frac{\partial u}{\partial t}; f \right\rangle &= \frac{\partial}{\partial t} \langle u; f \rangle = \frac{\partial}{\partial t} \left(ct \int_0^{2\pi} \int_0^\pi f(ct, \varphi, \theta) \sin \varphi d\varphi d\theta \right) \\ &= c \int_0^{2\pi} \int_0^\pi f(ct, \varphi, \theta) \sin \varphi d\varphi d\theta + c^2 t \int_0^{2\pi} \int_0^\pi \frac{\partial f}{\partial r}(ct, \varphi, \theta) \sin \varphi d\varphi d\theta \\ &= 4\pi c M_{ct}^{\mathbf{0}}[f] + 4\pi c^2 t M_{ct}^{\mathbf{0}} \left[\frac{\partial f}{\partial r} \right]. \end{aligned} \quad (18.128)$$

The result is a linear combination of the mean of f and of its radial derivative f_r over the sphere of radius ct . In particular,

$$\lim_{t \rightarrow 0} \langle u_t; f \rangle = 4\pi c M_0^{\mathbf{0}}[f] = 4\pi c f(\mathbf{0}).$$

Since this holds for all test functions, we conclude that the initial velocity

$$u_t(0, r) = 4\pi c \delta(\mathbf{x})$$

is a multiple of a delta function at the origin! Dividing through by $4\pi c$, we conclude that the spherical expanding wave

$$u(t, r) = \frac{\delta(r - ct)}{4\pi c r} \quad (18.129)$$

solves the initial value problem

$$u(0, \mathbf{x}) \equiv 0, \quad \frac{\partial u}{\partial t}(0, \mathbf{x}) = \delta(\mathbf{x}),$$

corresponding to an initial unit velocity impulse concentrated at the origin. This solution can be viewed as the three-dimensional version of the hammer-blow solution (14.128) to the one-dimensional wave equation. A significant difference is that, in three dimensions, there is no residual effect after the wave passes by.

More generally, if the unit impulse is concentrated at the point $\boldsymbol{\xi}$, we invoke translational symmetry to conclude that the function

$$G(t, \mathbf{x}; \boldsymbol{\xi}) = \frac{\delta(\|\mathbf{x} - \boldsymbol{\xi}\| - ct)}{4\pi c \|\mathbf{x} - \boldsymbol{\xi}\|}, \quad t \geq 0, \quad (18.130)$$

is the *fundamental solution* to the wave equation resulting from a concentrated unit velocity at the initial time $t = 0$:

$$G(0, \mathbf{x}; \boldsymbol{\xi}) = 0, \quad \frac{\partial G}{\partial t}(0, \mathbf{x}; \boldsymbol{\xi}) = \delta(\mathbf{x} - \boldsymbol{\xi}). \quad (18.131)$$

We can then apply linear superposition to solve the initial value problem

$$u(0, x, y, z) = 0, \quad \frac{\partial u}{\partial t}(0, x, y, z) = g(x, y, z), \quad (18.132)$$

with zero initial displacement. Namely, we write the initial velocity

$$g(\mathbf{x}) = \iiint g(\boldsymbol{\xi}) \delta(\mathbf{x} - \boldsymbol{\xi}) d\xi d\eta d\zeta$$

as a superposition of impulses, and immediately conclude that the relevant solution is the self-same superposition of spherical waves:

$$u(t, \mathbf{x}) = \frac{1}{4\pi c} \iiint g(\boldsymbol{\xi}) \frac{\delta(\|\mathbf{x} - \boldsymbol{\xi}\| - ct)}{\|\mathbf{x} - \boldsymbol{\xi}\|} d\xi d\eta d\zeta = \frac{1}{4\pi c^2 t} \iint_{\|\boldsymbol{\xi} - \mathbf{x}\| = ct} g(\boldsymbol{\xi}) dS. \quad (18.133)$$

Its value

$$u(t, \mathbf{x}) = t M_{ct}^{\mathbf{x}}[g], \quad (18.134)$$

at a point $\mathbf{x}$ and time $t \geq 0$, is t times the average of the initial velocity function g on a sphere of radius $r = ct$ centered at the point $\mathbf{x}$.

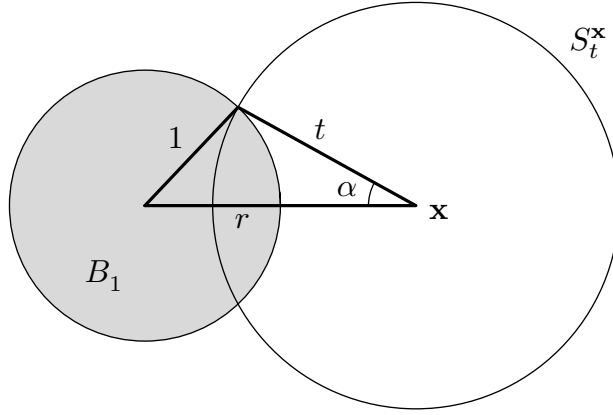


Figure 18.7. A Sphere Intersecting a Ball.

Example 18.14. Let us set the wave speed $c = 1$ for simplicity. Suppose that the initial velocity

$$g(\mathbf{x}) = \begin{cases} 1, & \|\mathbf{x}\| < 1, \\ 0, & \|\mathbf{x}\| > 1 \end{cases}$$

is 1 inside the unit ball B_1 centered at the origin, and 0 outside. To solve the initial value problem, we must compute the average value of g over a sphere $S_t^{\mathbf{x}}$ of radius $t > 0$ centered at a point $\mathbf{x} \in \mathbb{R}^3$. Since $g = 0$ outside the unit ball, its average will be equal to the surface area of that part of the sphere that is contained inside the unit ball, $S_t^{\mathbf{x}} \cap B_1$, divided by the total surface area of $S_t^{\mathbf{x}}$, namely $4\pi t^2$.

To compute this quantity, let $r = \|\mathbf{x}\|$. If $t > r + 1$ or $0 < t < r - 1$, then the sphere of radius t lies entirely outside the unit ball, and so the average is 0; if $0 < t < 1 - r$, then the sphere lies entirely within the unit ball and so the average is 1. Otherwise, referring to Figure 18.7, and referring to Exercise B.5.3, we see that the area of the spherical cap $S_t^{\mathbf{x}} \cap B_1$ is, by the Law of Cosines,

$$2\pi t^2(1 - \cos \alpha) = 2\pi t^2 \left(1 - \frac{r^2 + t^2 - 1}{2rt} \right) = \frac{\pi t}{r} [1 - (t - r)^2],$$

where α denotes the angle between the line joining the centers of the two spheres and the circle formed by their intersection. Assembling the different subcases, we conclude that

$$M_{ct}^{\mathbf{x}}[g] = \begin{cases} 1, & 0 \leq t \leq 1 - r, \\ \frac{1 - (t - r)^2}{4rt}, & |r - 1| \leq t \leq r + 1, \\ 0, & 0 \leq t \leq r - 1 \quad \text{or} \quad t \geq r + 1. \end{cases} \quad (18.135)$$

The solution (18.134) is obtained by multiplying by t , and hence for $t \geq 0$,

$$u(t, \mathbf{x}) = \begin{cases} t, & 0 \leq t \leq 1 - \|\mathbf{x}\|, \\ \frac{1 - (t - \|\mathbf{x}\|)^2}{4\|\mathbf{x}\|}, & |\|\mathbf{x}\| - 1| \leq t \leq \|\mathbf{x}\| + 1, \\ 0, & 0 \leq t \leq \|\mathbf{x}\| - 1 \quad \text{or} \quad t \geq \|\mathbf{x}\| + 1. \end{cases} \quad (18.136)$$

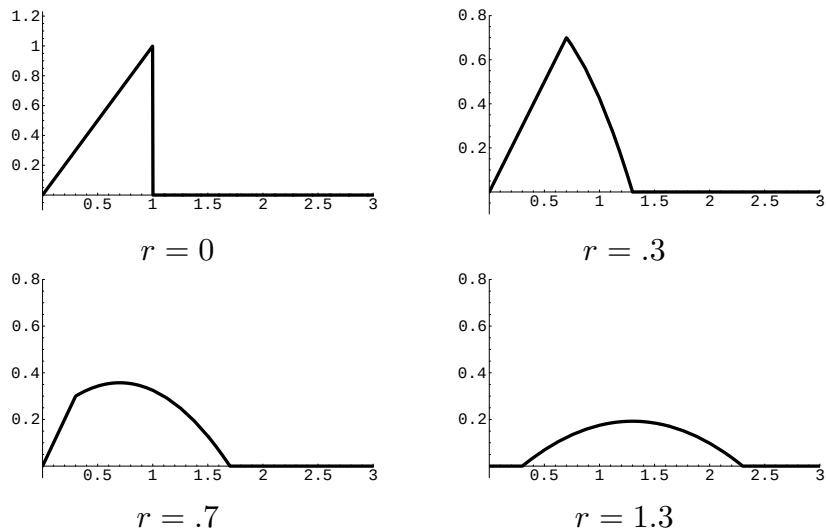


Figure 18.8. Solution to the Wave Equation Solution due to an Initial Concentrated Velocity.

Figure 18.8 plots the solution as a function of time for several fixed values of $r = \|\mathbf{x}\|$. An observer sitting at the origin will see a linearly increasing light intensity followed by a sudden decrease to 0. At other points inside the sphere, the decrease follows a parabolic arc; if the observer is closer to the edge than the center, the parabolic portion will continue to increase for a while before eventually tapering off. On the other hand, an observer sitting outside the sphere will experience, after an initially dark period, a symmetrical, parabolic increase to a maximal intensity and then decrease back to dark after a total time lapse of 2. We also show a plot of u as a function of r for various fixed times in Figure 18.9. Note that, up until time $t = 1$, the light spreads out while increasing in intensity near the origin, after which the solution is of gradually decreasing magnitude, supported within the domain lying between two concentric spheres of respective radii $t - 1$ and $t + 1$.

The solution described by formula (18.133) only handles initial velocities. What about solutions resulting from a nonzero initial displacement? Surprisingly, the answer is differentiation! The key observation is that if $u(t, \mathbf{x})$ is any (sufficiently smooth) solution to the wave equation, so is its time derivative

$$v(t, \mathbf{x}) = \frac{\partial u}{\partial t}(t, \mathbf{x}).$$

This follows at once from differentiating both sides of the wave equation with respect to t and using the equality of mixed partial derivatives. Physically, this implies that the velocity of a wave obeys the same evolutionary principle as the wave itself, which is a manifestation of the linearity and time-independence (autonomy) of the equation. Suppose u has initial conditions

$$u(0, \mathbf{x}) = f(\mathbf{x}), \quad u_t(0, \mathbf{x}) = g(\mathbf{x}).$$

What are the initial conditions for its derivative $v = u_t$? Clearly, its initial displacement

$$v(0, \mathbf{x}) = u_t(0, \mathbf{x}) = g(\mathbf{x})$$

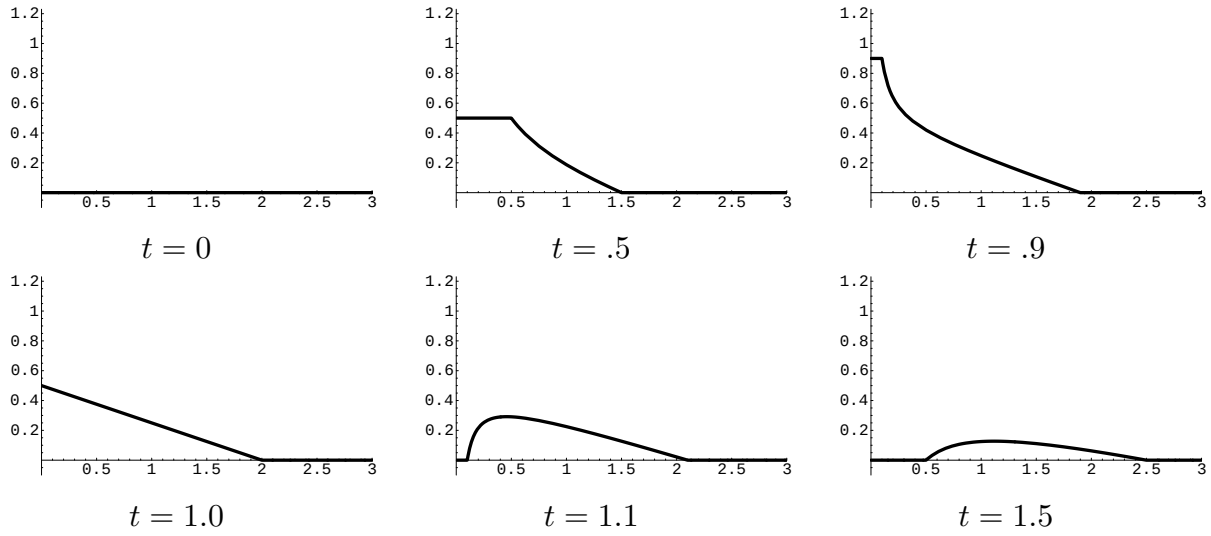


Figure 18.9. Solution to the Wave Equation Solution due to an Initial Concentrated Velocity.

equals the initial velocity of u . As for its initial velocity, we have

$$\frac{\partial v}{\partial t} = \frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$

because we are assuming that u solves the wave equation. Thus, at the initial time

$$\frac{\partial v}{\partial t}(0, \mathbf{x}) = c^2 \Delta u(0, \mathbf{x}) = c^2 \Delta f(\mathbf{x})$$

equals c^2 times the Laplacian of the initial displacement[†]. In particular, if u satisfies the initial conditions

$$u(0, \mathbf{x}) = 0, \quad u_t(0, \mathbf{x}) = g(\mathbf{x}), \quad (18.137)$$

then $v = u_t$ satisfies the initial conditions

$$v(0, \mathbf{x}) = g(\mathbf{x}), \quad v_t(0, \mathbf{x}) = 0. \quad (18.138)$$

Thus, paradoxically, to solve the initial displacement problem we differentiate the initial velocity solution (18.133) with respect to t , and hence

$$v(t, \mathbf{x}) = \frac{\partial u}{\partial t}(t, \mathbf{x}) = \frac{\partial}{\partial t} (t M_{ct}^{\mathbf{x}}[g]) = M_{ct}^{\mathbf{x}}[g] + ct M_{ct}^{\mathbf{x}} \left[\frac{\partial g}{\partial \mathbf{n}} \right], \quad (18.139)$$

using our computation in (18.128). Therefore, $v(t, \mathbf{x})$ is a linear combination of the mean of the function g and the mean of its normal or radial derivative $\partial g / \partial \mathbf{n} = \partial g / \partial r$, taken

[†] In Section 14.6, a similar device was used to initiate the numerical solutions to the wave equation.

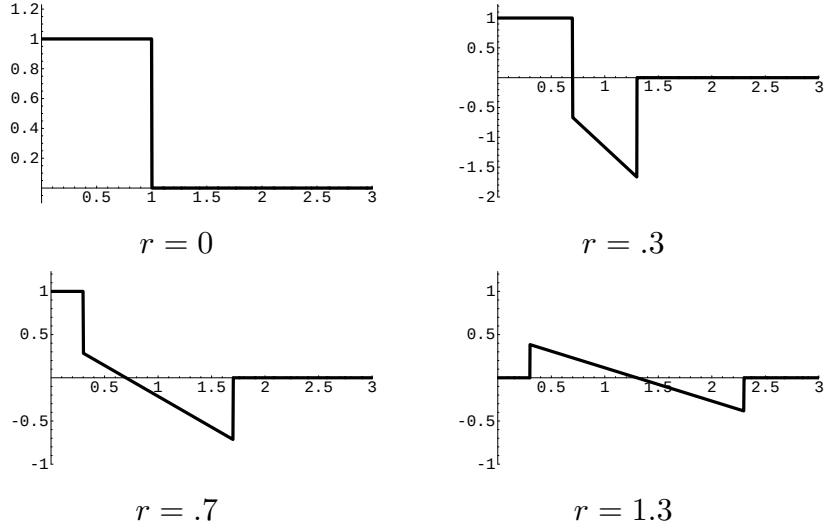


Figure 18.10. Solution to the Wave Equation due to an Initial Concentrated Displacement.

over a sphere of radius ct centered at the point $\mathbf{x}$. In particular, to obtain the solution corresponding to a concentrated initial displacement,

$$F(0, \mathbf{x}; \boldsymbol{\xi}) = \delta(\mathbf{x} - \boldsymbol{\xi}), \quad \frac{\partial F}{\partial t}(0, \mathbf{x}; \boldsymbol{\xi}) = 0, \quad (18.140)$$

we differentiate the solution (18.130), resulting in

$$F(t, \mathbf{x}; \boldsymbol{\xi}) = \frac{\partial G}{\partial t}(t, \mathbf{x}; \boldsymbol{\xi}) = -\frac{\delta'(\|\mathbf{x} - \boldsymbol{\xi}\| - ct)}{4\pi\|\boldsymbol{\xi} - \mathbf{x}\|}, \quad (18.141)$$

which represents a spherically expanding doublet, cf. Figure 11.10. Thus, interestingly, a concentrated initial displacement spawns an expanding spherical doublet wave, whereas a concentrated initial velocity spawns a spherical singlet or delta wave.

Example 18.15. Let $c = 1$ for simplicity. Consider the initial displacement

$$u(0, \mathbf{x}) = f(\mathbf{x}) = \begin{cases} 1, & \|\mathbf{x}\| < 1, \\ 0, & \|\mathbf{x}\| > 1 \end{cases}$$

along with zero initial velocity, modeling the effect of an instantaneously illuminated solid ball. To obtain the solution, we differentiate (18.136) with respect to t , leading to

$$u(t, \mathbf{x}) = \begin{cases} 1, & 0 \leq t < 1 - \|\mathbf{x}\|, \\ \frac{\|\mathbf{x}\| - t}{2\|\mathbf{x}\|}, & |\|\mathbf{x}\| - 1| \leq t \leq \|\mathbf{x}\| + 1, \\ 0, & 0 \leq t < \|\mathbf{x}\| - 1 \quad \text{or} \quad t > 1 + \|\mathbf{x}\|. \end{cases} \quad (18.142)$$

As illustrated in Figure 18.10, an observer sitting at the center of the ball will see a constant light intensity until $t = 1$, at which time the solution suddenly goes dark. At

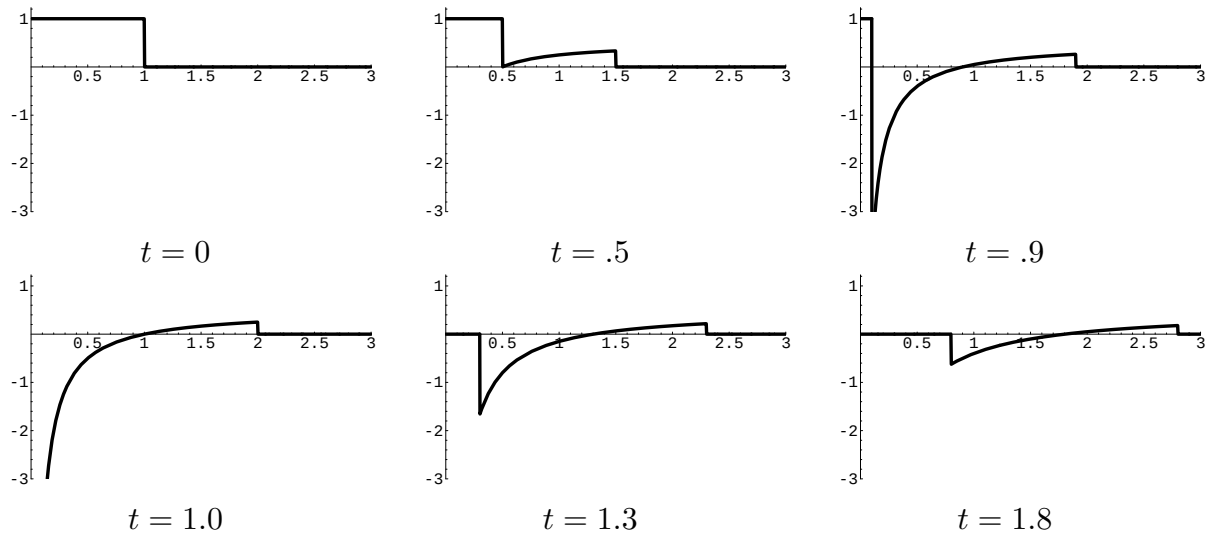


Figure 18.11. Solution to the Wave Equation due to an Initial Concentrated Displacement.

other points inside the ball, $0 < r < 1$, the downwards jump in intensity arrives sooner, and even goes below 0, followed by a further linear decrease, and finally a jump back to quiescent. An observer placed outside the ball will experience, after an initially dark period, a sudden increase in the light intensity, followed by a linear decrease to negative, followed by a jump back up to darkness. The farther away from the source, the fainter the light. In Figure 18.11 we plot the same solution as a function of r for different values of t . Note the sudden appearance of a $1/r$ singularity at the origin at time $t = 1$, due to the focussing of the initial discontinuities in u over the entire unit sphere. Afterwards, the residual disturbance moves off to ∞ while gradually decreasing in intensity.

Linearly combining the two solutions (18.134, 139) establishes *Kirchhoff's formula* — although it was first discovered by Poisson — which is the three-dimensional counterpart to the d'Alembert's solution formula for the wave equation.

Theorem 18.16. *The solution to the initial value problem*

$$u_{tt} = c^2 \Delta u, \quad u(0, \mathbf{x}) = f(\mathbf{x}), \quad \frac{\partial u}{\partial t}(0, \mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3, \quad (18.143)$$

for the wave equation in three-dimensional space is given by

$$u(t, \mathbf{x}) = \frac{\partial}{\partial t} (t M_{ct}^{\mathbf{x}}[f]) + t M_{ct}^{\mathbf{x}}[g] = M_{ct}^{\mathbf{x}}[f] + ct M_{ct}^{\mathbf{x}} \left[\frac{\partial f}{\partial \mathbf{n}} \right] + t M_{ct}^{\mathbf{x}}[g]. \quad (18.144)$$

Here, $M_{ct}^{\mathbf{x}}[f]$ denotes the average of the function f over a sphere of radius ct centered at position $\mathbf{x}$.

A crucially important consequence of the Kirchhoff solution formula is the celebrated *Huygens' Principle*, which was first highlighted the pioneering seventeenth century Dutch scientist Christiaan Huygens. Roughly, Huygens' Principle states that, in

three-dimensional space, localized solutions to the wave equation remain localized. More concretely, (18.144) implies that the value of the solution at a point $\mathbf{x}$ and time t only depends upon the values of the initial displacements and velocities at a distance ct away. Thus, all signals propagate along the light cone

$$c^2 t^2 = x^2 + y^2 + z^2$$

in four-dimensional Minkowski space-time. For electromagnetic waves, this fact lies at the foundation of special relativity. Physically, Huygens' Principle means that the light that we see at a given time t arrived from points at a distance exactly $d = ct$ away at time $t = 0$. In particular, a sharp, localized initial signal — whether initial displacement or initial velocity — that is concentrated near a point produces a sharp, localized response that remains concentrated on an ever expanding sphere surrounding the point. In our three-dimensional universe, we only witness the light from an explosion for a brief moment, after which if there is no subsequent light source, the view returns to darkness. Similarly, a sharp sound remains sharply concentrated, with diminishing magnitude, as it propagates through space. Remarkably, as we will show next, Huygens' Principle does not hold in a two dimensional universe! In the plane, concentrated impulses will be spread out as time progresses.

Descent to Two Dimensions

So far, we have explicitly determined the solution to the wave equation in one- and three-dimensional space. The two-dimensional case

$$u_{tt} = c^2 \Delta u = c^2 (u_{xx} + u_{yy}). \quad (18.145)$$

is, counter-intuitively, more complicated! For instance, seeking a radially symmetric solution $u(t, r)$ requires solving the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) \quad (18.146)$$

which, unlike its three-dimensional cousin (18.119), is not so easily integrated.

However, our solution to the three-dimensional problem can be easily adapted to construct a solution using the so-called *Method of Descent*. Any solution $u(t, x, y)$ to the two-dimensional wave equation (18.145) can be viewed as a solution to the three-dimensional wave equation (18.106) that does not depend upon the vertical z coordinate, whence $\partial u / \partial z = 0$. Clearly, if the three-dimensional initial data does not depend on z , then the resulting solution $u(t, x, y)$ will also be independent of z .

Consider first the zero initial displacement initial conditions

$$u(0, x, y) = 0, \quad \frac{\partial u}{\partial t}(0, x, y) = g(x, y). \quad (18.147)$$

We rewrite the solution formula (18.133) in the form of a surface integral over the sphere $S_{ct} = \{ \|\boldsymbol{\xi}\| = ct \}$ centered at the origin:

$$u(t, \mathbf{x}) = \frac{1}{4\pi c^2 t} \iint_{S_{ct}} g(\boldsymbol{\xi}) dS = \frac{1}{4\pi c^2 t} \iint_{\|\boldsymbol{\xi}\|=ct} g(\mathbf{x} + \boldsymbol{\xi}) dS. \quad (18.148)$$

Imposing the condition that $g(x, y)$ does not depend upon the z coordinate, we see that the integrals over the upper and lower hemispheres

$$S_{ct}^+ = \{ \|\boldsymbol{\xi}\| = ct, \quad \zeta \geq 0 \}, \quad S_{ct}^- = \{ \|\boldsymbol{\xi}\| = ct, \quad \zeta \leq 0 \},$$

are identical. As in (B.47), to evaluate the upper hemispherical integral, we parametrize the upper hemisphere as the graph of

$$\zeta = \sqrt{c^2 t^2 - \xi^2 - \eta^2} \quad \text{over the disk} \quad D_{ct} = \{ \xi^2 + \eta^2 \leq c^2 t^2 \},$$

We conclude that

$$u(t, x, y) = \frac{1}{2\pi c^2 t} \iint_{S_{ct}^+} g(\mathbf{x} + \boldsymbol{\xi}) dS = \frac{1}{2\pi c} \iint_{D_{ct}} \frac{g(x + \xi, y + \eta)}{\sqrt{c^2 t^2 - \xi^2 - \eta^2}} d\xi d\eta \quad (18.149)$$

solves the initial value problem (18.147). In particular, if we take the initial velocity

$$g(x, y) = \delta(x - \xi) \delta(y - \eta)$$

to be a concentrated impulse, then the resulting solution is

$$G(t, x, y; \xi, \eta) = \begin{cases} \frac{1}{2\pi c \sqrt{c^2 t^2 - (x - \xi)^2 - (y - \eta)^2}}, & (x - \xi)^2 + (y - \eta)^2 < c^2 t^2, \\ 0, & (x - \xi)^2 + (y - \eta)^2 > c^2 t^2. \end{cases} \quad (18.150)$$

An observer placed at position $\mathbf{x}$ will first experience a concentrated displacement singularity at time $t = \|\mathbf{x} - \boldsymbol{\xi}\|/c$. However, in contrast to the three-dimensional solution, even after the impulse passes by, the observer will continue to experience a decreasing, but non-zero signal of magnitude roughly proportional to $1/t$. In Figure 18.12, we plot the solution corresponding to a concentrated impulse at the origin, with unit wave speed $c = 1$. The first line shows the displacement at three different times as a function of $r = \|\mathbf{x}\|$; note the initial singularity, indicated by a spike in the graph, is followed by a progressively smaller residual displacement. The second line plots intensity as a function of t at three different radii; the further away from the initial impulse, the faster the residual displacement decays back to 0 — although it never entirely disappears.

Similarly, the solution to the initial displacement conditions

$$u(0, x, y) = f(x, y), \quad \frac{\partial u}{\partial t}(0, x, y) = 0, \quad (18.151)$$

can be obtained by differentiation with respect to t , and so

$$u(t, x, y) = \frac{\partial}{\partial t} \left(\frac{1}{2\pi c} \iint_{D_{ct}} \frac{f(x + \xi, y + \eta)}{\sqrt{c^2 t^2 - \xi^2 - \eta^2}} d\xi d\eta \right). \quad (18.152)$$

Again, for a concentrated impulse in the initial displacement, an observer will witness, after a certain time lapse, an abrupt impulse passing by that is followed by a progressively decaying residual effect. The general solution to the two-dimensional wave equation on all of $\mathbb{R}^2$ is a linear combination of these two types of solutions (18.149, 152).

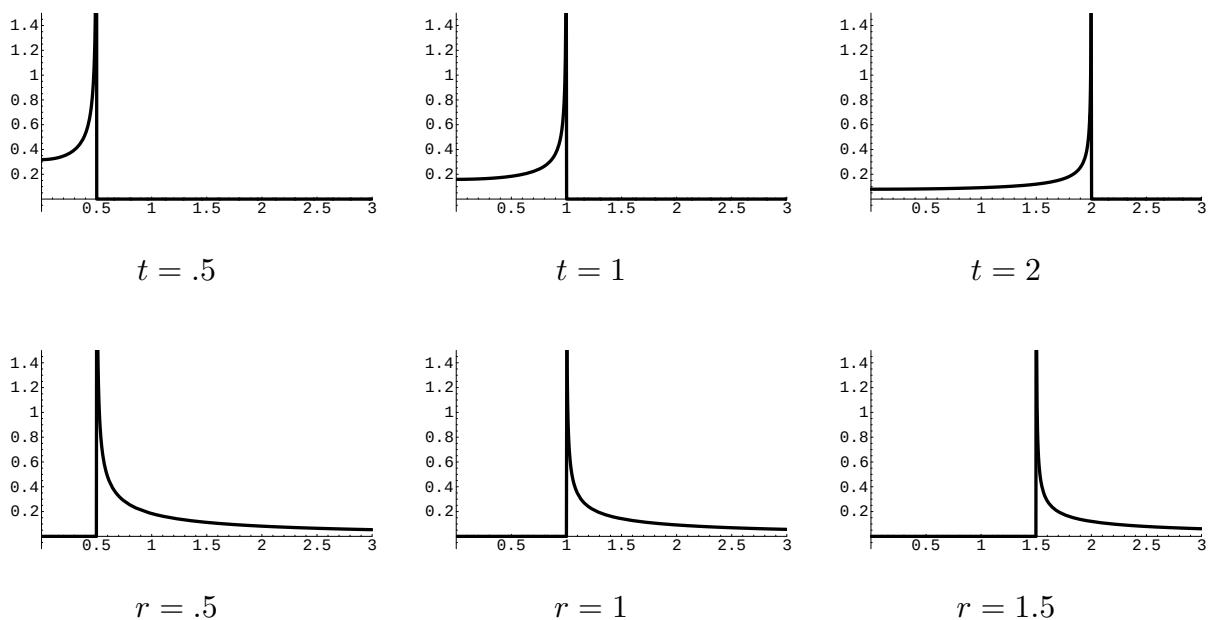


Figure 18.12. Solution to the Two-Dimensional Wave Equation for a Concentrated Impulse.

Thus, Huygens' Principle is *not* valid in a two-dimensional universe. The solution to the two-dimensional wave equation at a point $\mathbf{x}$ at time t depends upon the initial displacement and velocity on the entire disk of radius rt centered at the point, and not just on the points a distance ct away. So a two-dimensional creature would experience not only a initial effect of any sound or light wave but also an “afterglow” with slowly diminishing magnitude. It would be like living in a permanent echo chamber, and so understanding and acting upon sensory phenomena would more challenging. In general, Huygens' principle is only valid in odd-dimensional spaces; see also [16] for recent advances in the classification of partial differential equations that admit a Huygens' principle.

Remark: Since the solutions to the two-dimensional wave equation can be interpreted as three-dimensional solutions with no z dependence, a concentrated delta impulse in the two-dimensional wave equation would correspond to a concentrated impulse along an entire vertical line in three dimensions. If light starts propagating from the line at $t = 0$, after the initial signal reaches us, we will continue to receive light from points that lie progressively farther away along the line, and this accounts for the two-dimensional afterglow.

18.7. The Schrödinger Equation and the Hydrogen Atom.

A *hydrogen atom* consists of a single electron, of mass m and charge e , circling an atomic nucleus containing a single proton in three-dimensional space. As a result of quantization, the Schrödinger equation governing the dynamical behavior of the electron around

the nucleus takes the explicit form

$$i \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi - \frac{\alpha}{r} \psi = -\frac{\hbar^2}{2m} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right) - \frac{\alpha}{\sqrt{x^2 + y^2 + z^2}} \psi. \quad (18.153)$$

Here $\psi(t, x, y, z)$ denotes the electron's time-dependent wave function giving its quantum probability density as it moves around the nucleus, which, owing to its relatively small size, is assumed to be concentrated at the origin. The coefficient of the Laplacian depends on Planck's constant $\hbar$ and the electron mass m . The final term represents the electromagnetic (or Newtonian) potential function $V = \alpha/r$ attracting the electron to the nucleus, where $\alpha = e^2$ is the square of the electron's (and proton's) charge. More generally, if the nucleus contains Z protons, then one needs to multiply the potential accordingly: $\alpha = Ze^2$. Incidentally, the Schrödinger equation for multi-electron atoms or even molecules is not hard to write down, but its solution, even for, say, the helium atom, is *much* more difficult, and is still a major challenge for numerical approximations on today's supercomputers. Thus, we will only consider a single electron atom in this section.

According to the analysis in Section 14.7, the normal mode solutions to the Schrödinger equation are of the form

$$\psi(t, x, y, z) = e^{i\lambda t/\hbar} u(x, y, z),$$

where u is an eigenfunction of the Hamiltonian operator with eigenvalue λ , and hence

$$\frac{\hbar^2}{2m} \Delta u + \left(\lambda + \frac{\alpha}{r} \right) u = 0. \quad (18.154)$$

The *bound states* of the atom, in which the electron remains trapped by the nucleus, are represented the non-zero solutions to the eigenvalue problem with bounded L^2 norm:

$$\|u\|^2 = \iiint |u(x, y, z)|^2 dx dy dz < \infty.$$

The eigenvalue λ specifies the state's energy, which is necessarily negative: $\lambda < 0$. Unlike the Laplace equation, the bound states do *not* form a complete system of eigenfunctions, and so not every wave function $\varphi \in L^2(\mathbb{R}^3)$ can be approximated by an eigenfunction series. The missing data are the so-called *scattering states* arising from the *continuous spectrum* of the Schrödinger operator; these represent electrons that scatter off of the nucleus, and so do not remain bounded or trapped. We will leave the discussion of the scattering states and continuous spectrum to a more advanced treatment, [130, 154].

To understand the bound states, we begin by rewriting the eigenvalue problem (18.154) in spherical coordinates:

$$\frac{\hbar^2}{2m} \left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} + \frac{\cos \varphi}{r^2 \sin \varphi} \frac{\partial u}{\partial \varphi} + \frac{1}{r^2 \sin^2 \varphi} \frac{\partial^2 u}{\partial \theta^2} \right) + \left(\lambda + \frac{\alpha}{r} \right) u = 0. \quad (18.155)$$

We then separate off the radial coordinate, setting

$$u(r, \varphi, \theta) = v(r) w(\varphi, \theta).$$

The angular component satisfies the spherical Helmholtz equation

$$\Delta_S w + \mu w = \frac{\partial^2 w}{\partial \varphi^2} + \frac{\cos \varphi}{\sin \varphi} \frac{\partial w}{\partial \varphi} + \frac{1}{\sin^2 \varphi} \frac{\partial^2 w}{\partial \theta^2} + \mu w = 0,$$

that we have already solved. The eigensolutions are spherical harmonics which, because the quantum mechanical solutions are intrinsically complex-valued, we take in their complex form (18.41). The associated eigenvalue

$$\mu = l(l+1), \quad \text{where the integer} \quad l = 0, 1, 2, \dots, \quad (18.156)$$

known as the *angular quantum number*, admits a total of $2l+1$ linearly independent eigenfunctions

$$\mathcal{Y}_l^m(\theta, \varphi) = P_l^m(\cos \varphi) e^{im\theta}, \quad m = -l, -l+1, \dots, l-1, l. \quad (18.157)$$

The radial equation associated with the separation constant (18.156) is

$$\frac{\hbar^2}{2m} \left(\frac{d^2 v}{dr^2} + \frac{2}{r} \frac{dv}{dr} \right) + \left(\lambda + \frac{\alpha}{r} - \frac{l(l+1)}{r^2} \right) v = 0. \quad (18.158)$$

To eliminate the physical parameters, let's rescale the radial coordinate by setting

$$s = \sigma r, \quad \text{where} \quad \sigma = \sqrt{-\frac{8m\lambda}{\hbar^2}}, \quad (18.159)$$

where we use the fact that $\lambda < 0$. The resulting ordinary differential equation for the rescaled function

$$P(s) = v\left(\frac{s}{\sigma}\right)$$

is

$$\frac{d^2 P}{ds^2} + \frac{2}{s} \frac{dP}{ds} - \left(\frac{1}{4} - \frac{n}{s} + \frac{l(l+1)}{s^2} \right) P = 0, \quad (18.160)$$

where

$$n = \frac{2m\alpha}{\sigma \hbar^2} = \frac{2m\alpha}{\sigma \hbar^2} \sqrt{-\frac{m}{2\lambda}}. \quad (18.161)$$

Since we are searching for bound states, the relevant solution(s) of this second order ordinary differential equation should be defined on $0 \leq s < \infty$, remain bounded at $s = 0$, and go to zero as $s \rightarrow \infty$. The proof of the key result is outlined in the exercises in Chapter C.

Theorem 18.17. *The bound state solutions of (18.160) subject to the boundary conditions $P(0^+) < \infty$, $\lim_{s \rightarrow \infty} P(s) = 0$, only occur when $n \geq l+1$ is an integer, and are given by*

$$P(s) = s^l e^{-s/2} L_{n-l-1}^{2l+1}(s), \quad (18.162)$$

where

$$L_j^k(s) = \sum_{i=0}^k \frac{(-1)^i}{i!} \binom{j+k}{j-i} s^i, \quad j, k = 0, 1, 2, \dots, \quad (18.163)$$

is a certain polynomial function, known as an associated Laguerre polynomial.

The resulting integer n , whose physical value was noted in (18.161), is known as the *principle quantum number*. We further note that the scaling factor in (18.159) can be written as

$$\sigma = \frac{2m\alpha}{n\hbar^2} = \frac{2}{na} \quad \text{where} \quad a = \frac{\hbar^2}{m\alpha} \approx .529 \times 10^{-10} \text{meter}$$

is called the *Bohr radius*, in honor of the pioneering Danish quantum physicist Niels Bohr, and approximates the radius of the first atomic energy level. Reverting to physical coordinates, the bound states (18.162) become, up to an inessential constant multiple, the *radial wave functions*

$$v_l^n(r) = \left(\frac{2r}{na}\right)^l e^{-r/(na)} L_{n-l-1}^{2l+1} \left(\frac{2r}{na}\right). \quad (18.164)$$

Comining them with the spherical harmonics (18.157) yields the atomic eigenfunctions

$$U_{lmn}(r, \varphi, \theta) = v_l^n(r) \mathcal{Y}_l^m(\theta, \varphi). \quad (18.165)$$

These eigenstates depend upon three integers:

- $l = 0, 1, 2, 3, \dots$: the angular quantum number;
- $n = l + 1, l + 2, l + 3, \dots$: the principle quantum number;
- $m = -l, -l + 1, \dots, l - 1, l$: the magnetic quantum number.

The energy of the eigenstate is the associated eigenvalue

$$\lambda_n = -\frac{\alpha^2 m}{2\hbar^2} \frac{1}{n^2}, \quad n = 1, 2, 3, \dots$$

The fact that the ratios $\lambda_n/\lambda_1 = 1/n^2$ between the higher and lowest energy levels are inverse squares of integers was first established by Bohr. The n^{th} energy level has a total of

$$\sum_{l=0}^{n-1} (2l+1) = n^2$$

bound states with that energy, indicating the number of orbital shells in the atom associated with that energy level.

In chemistry, the electron levels are indexed by the angular quantum number, i.e., the order l of the spherical harmonic, and traditionally labeled by a letter in the sequence $p, s, d, f, \dots$. Thus, the two order $l = 0$ spherical harmonics correspond to the p shells; the six harmonics of order $l = 1$ are the s shells, and so on. Since electrons are allowed to have one of two possible spins, the Pauli exclusion principle tells us that each energy shell can be occupied by at most two electrons. Thus, the number of electrons that can reside in the atomic shell with angular quantum number l is, in fact, $2(2l+1)$. The configuration of energy shells and electrons in atoms are responsible for the periodic table. Thus, hydrogen has a single electron in the p shell. Helium has two electrons in the p shell. Lithium has 3 electrons, with two of them filling the first p shell and the third in the second p shell. Neon has 10 electrons filling the two p and first three s shells. And so on. The chemical properties of the elements are, to a very large extent, determined by the placement of the electrons within the different shells. See [Chem] for further details.