ODE Meta Notes PhL 12.16.09

These notes basically summarize Chapter 10 of Whittaker and Watson (W&W). But they also summarize most of what is in my horribly disorganized ODE1 notes doc.

WW Page 188. Here, they start with the form $[D^2 + p(z)\,D + q(z)\,]\,u(z) = 0$, they change functional form as shown from u(z) to v(z), which involves an integral of p, and convert this to $[\,D^2 + J(z)]\,v(z) = 0$. If z=b is an **ordinary** point (p and q analytic there), they show that you can directly compute some $v_n(z)$ functions (starting off with $v_0 = a_0 + a_1(z-b)$) and then $v(z) = \Sigma_{n=0} v_n(z)$ will solve the second ODE with the usual initial boundary conditions $v(b) = a_0$ and $v'(b) = a_1$. Somehow it is the fact that z = b is an ordinary point that lets us directly compute this solution near z = b. They then prove that this solution is unique. After this little introduction, they then move into the Frobenius method, which works even when p and q are not analytic at the point of interest, provided the point is regular. So Frobenius works for ordinary and regular points, but probably not for irregular points, all to be defined below.

Comment: Notice that no endpoints or intervals like (a,b) on the real axis are mentioned below, which play such a major role in Stakgold. We are talking here about ODE's without boundary conditions. The field is complex, and you really are talking about regions in the complex plane when you talk analyticity.

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1. Frobenius Method

I feel there is no substitute for actually "doing this live" and watching what happens. This is the starting point for everything. Assume the following "form" (form is key), where we are going to expand around z = 0 to make things simple to write out. We assume that P and Q are analytic at z=0 so can be expanded as shown. We also assume, as an ansatz, that the solution u(z) has the series form shown with $A_0 \neq 0$ (could set $A_0 = 1$)

$$[z^2D^2 + z\,P(z)\,D + Q(z)\,]\,u(z) = 0 \qquad \text{" Frobenius Form "}$$

$$P(z) = P_0 + zP_1 + z^2P_2 + \ldots = \Sigma_{i=0}\, P_i z^i$$
$$Q(z) = Q_0 + zQ_1 + z^2Q_2 + \ldots = \Sigma_{i=0}\, Q_i z^i$$

$$u(z) = \Sigma_{k=0} A_k\, z^{r+k} = A_0\, z^r + A_1 z^{r+1} + A_2\, z^{r+2} + \ldots = z^r\, \Sigma_{k=0} A_k\, z^k$$

We just jam everything in and watch what happens:

$$\Sigma_{k=0} A_k (r+k)(r+k-1)\, z^{r+k} + \Sigma_{k=0} A_k (r+k) \Sigma_{i=0} P_i\, z^{r+k+i} + \Sigma_{k=0} A_k \Sigma_{i=0} Q_i\, z^{r+k+i} = 0$$

$$z^r \Sigma_{k=0} A_k z^k \{ (r+k)(r+k-1) + \Sigma_{i=0} [(r+k) P_i + Q_i] z^i \} = 0$$

$$\Sigma_{k=0} A_k z^k \{ (r+k)(r+k-1) + \Sigma_{i=0} [(r+k) P_i + Q_i] z^i \} = 0$$

In order to have a solution, the coefficient of each power must vanish. We can then write out the first several coefficients and set them to zero.

0 z^0: $A_0[r^2 + (P_0-1)r + Q_0] = 0$
1 z^1: $A_1[(r+1)^2 + (P_0-1)(r+1) + Q_0] + A_0 [rP_1 + Q_1] = 0$
2 z^2: $A_2[(r+2)^2 + (P_0-1)(r+2) + Q_0] + A_1 [(r+1)P_1 + Q_1] + A_0 [rP_2 + Q_2] = 0$
....

The first equation is the super-famous and all-important **indicial equation**. It tells you that there are at most two possible solutions of the assumed form, each one corresponding to a root of the indicial equation, $u^{(r)}(z) = z^r \Sigma_{k=0} A_k^{(r)} z^k$. These two roots are called **the exponents** or the **indices** (at z = 0). For EACH exponent, the above ladder of equations lets you build out all the coefficients and in this way you get two "formal " solutions as powers times infinite series. They are formal in the sense that we have not shown anything about whether the series converge.

If we want to be completely general, and assume that P and Q have all coefficients being arbitrary (non-zero) values, then we are assuming that p includes a single pole, and that q has both a single pole and a double pole (p and q are defined below as P = zp and Q = z^2q). Our analysis here therefore includes z = 0 being an **ordinary** point ***and*** also being a **regular** singular point. We get the above indicial equation, and since P_0 and Q_0 are completely arbitrary, it seems that the roots r_i are also completely arbitrary. They could both be real fractions, for example, or in fact could both be complex. Thus, in general, both solutions might have a power-type branch point at z = 0.

<u>The ordinary case.</u> If we restrict our interest to the ordinary case only, then we are saying that p does not have a single pole, and q has neither a single nor a double pole, since ordinary means p and q are analytic. This means P has a zero and Q has a double zero, and that says $P_0 = 0$ and $Q_0 = Q_1 = 0$. In this case our three lowest ladder equations above become

0 z^0: $A_0[r^2 + (-1)r] = 0$ // r(r-1) = 0 "indicial"
1 z^1: $A_1[(r+1)^2 + (-1)(r+1)] + A_0 [rP_1] = 0$
2 z^2: $A_2[(r+2)^2 + (-1)(r+2)] + A_1 [(r+1)P_1] + A_0 [rP_2 + Q_2] = 0$

The indicial equation now says the roots are r=0 and r=1. The second ladder equation is this:

$$A_1[r(r+1)] + A_0 [rP_1] = 0$$

which we can write out in its two cases:

$$A_1^{(0)} [0] + A_0^{(0)} [0] = 0$$

$A_1^{(1)}\,[2\,] + A_0^{(1)}\,[P_1] = 0$

At this point, let's agree to scale things so both A_0 coefficients are just "1". The first equation above says that $A_1^{(0)}$ is *arbitrary*, so we just set it to 0, this being the simplest thing to do. The second equation says that $2\,A_1^{(1)} = -P_1$. So far, then, we have

$A_0^{(0)} = 1 \qquad A_0^{(1)} = 1$
$A_1^{(0)} = 0 \qquad A_1^{(1)} = -P_1/2$

The third ladder equation is this: (again we set $A_0 = 1$ for both solutions)

$A_2[\,(r+2)^2 + (-1)(r+2)\,] + A_1\,[\,(r+1)P_1] + [\,rP_2 + Q_2] = 0$

which we can write out in the two cases:

$A_2^{(0)}\,[2\,] + A_1^{(0)}\,[\,P_1] + [\,Q_2] = 0$
$A_2^{(1)}\,[6\,] + A_1^{(1)}\,[\,2P_1] + [\,P_2 + Q_2] = 0$

$A_2^{(0)}\,[2\,] + [\,Q_2] = 0$
$A_2^{(1)}\,[6\,] + (-P_1/2)\,[\,2P_1] + [\,P_2 + Q_2] = 0$

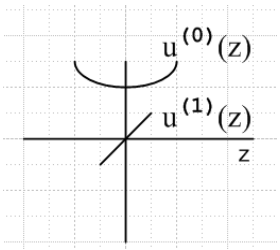
$A_2^{(0)} = -Q_2/2$
$A_2^{(1)} = [\,P_1^2 - P_2 - Q_2]/6$

So through this term, we have the following two solutions to our ODE at an ordinary point:

$u^{(0)}(z) = 1 - (1/2)Q_2\,z^2 + \ldots$

$u^{(1)}(z) = z - (1/2)P_1\,z^2 + (1/6)\,[\,P_1^2 - P_2 - Q_2]\,z^3 + \ldots$

These two solutions compare pretty well with M&F p531, and are called the **basic set**. (at z = 0). Here is what these solutions look like near the ordinary point z = 0 (which we could imagine as z = a):



MF point out that the general initial BC solution can then be written this way,

$u(z) = u(0)\,u^{(0)}(z) + u'(0)\,u^{(1)}(z)$

Our conclusion here is this: if we seek series solutions near an ordinary point , z = 0 in this case, we find that both solutions to the ODE are analytic at this point. MF claim that the convergence radius of the solution will be however large it can be until the circle hits a singular point of the ODE.

Irregular cases. Suppose Q(z) had a pole at z = 0. We then add a term $Q_{-1}z^{-1}$ into the Q expansion. Since this is then the only z^{-1} term in the equation, we cannot balance it without setting $Q_{-1} = 0$. The same if we try higher poles in Q. This would be an "indicial equation" of degree 0.

Suppose we have a pole in P(z)? Then we need $(r+0)P_{-1} = 0$ and then we are OK if r = 0. This would also work if we had a double pole and higher in P. This is fine, but then we have only one solution, not two solutions. I think one says in this case that the indicial equation is linear instead of quadratic. The indicial equation is what you get for the lowest power in the balance scheme.

Suppose we have a single pole in P and Q at the same time. Then we get $rP_{-1} + Q_{-1} = 0$ and again we have a linear indicial equation and there is one solution (that fits our assumed form).

These cases where P and Q are not analytic at z = 0 are called **irregular** and I think in these cases you either get 1 solution or maybe 0 solutions, put that off for now. The situation where P and Q are analytic at z = 0 are called regular. That is to say, z = 0 is a "**regular singular point**" if P and Q are analytic there. See below for implications for p and q.

Higher power balance. Go back to where we were:

$$\Sigma_{k=0} A_k z^k \{ (r+k)(r+k-1) + \Sigma_{i=0} [(r+k) P_i + Q_i] z^i \} = 0$$

The z^1 term says this:

$$A_1 z \{ (r+1)(r+1-1) + [(r+0) P_0 + Q_0] \} + A_0 1 [(r+0) P_1 + Q_1] z = 0$$

This says that, for each solution value of r, A_1 is a function of A_0 and of P_1, P_0, Q_1, Q_0 . As W&W show on page 192. if you look at the higher power balance, you find that, in general

A_n = function of all the lower A_i and of the matching and all lower P_i and Q_i

This then is the famous **recursion relation**. Here are the first few from my ODE1 notes, where the first is the indicial equation:

0 z^r: $A_0[r^2 + (P_0-1)r + Q_0] = 0$

1 z^{r+1}: $A_1[(r+1)^2 + (P_0-1)(r+1) + Q_0] + A_0 [rP_1 + Q_1] = 0$

2 z^{r+2}: $A_2[(r+2)^2 + (P_0-1)(r+2) + Q_0] + A_1 [(r+1)P_1 + Q_1] + A_0 [rP_2 + Q_2] = 0$

So, for each value of r_i (from solving the regular indicial equation), we get a full **"formal" power series** solution with an infinite number of terms. We hope that there is some "disk" around the point z = 0 where these two series converge and are thus actual solutions of the ODE. In general, ignoring special cases discussed below, if both these power series have disks of convergence around z = 0, we have by this method found ALL possible solutions to the ODE which have the form z^r *times* a function analytic around z = 0. Note that z^r might be $\sqrt{z}$ which is not analytic at z = 0.

Question: Could there be some third solution to the ODE that has some different form? Maybe that third solution could be ln(z) * power series. The answer here is that we know, from different sources, that there can be at most n independent solutions to our n^{th} order ODE. M&F show this for n = 2 using the Wronksian and a compact little argument that I think is OK. Forsyth has a different argument based on the number of integration constants. I suspect there is some basic operator argument that I have maybe forgotten from Stakgold Chapter 2. Let's not digress more right now. So the point here is that our ansatz form does not prove there are at most two solutions, that fact comes from elsewhere.

Comment: We showed above that both solutions near an ordinary point are analytic since p and q are analytic at that point. In the regular singular case, P and Q are still analytic even though p and/or q are not. In this case, the two roots r could take any values, so it could turn out still that both solutions are analytic at the regular singular point (both r_i are non-negative integers), or it could turn out that one or both solutions have a fractional power branch point there. This comment assumes we don't have one of the special cases mentioned below for the exponents. The point is that non-analytic p and q does not *force* one or both solutions to be non-analytic.

Ordinary vs Regular Points. Historically, one writes a 2nd order ODE in this **canonical form**

$$[D^2 + p(z)\, D + q(z)\,]\, u(z) = 0$$

which we can compare to the our **Frobenius form** above

$$[z^2 D^2 + z\, P(z)\, D + Q(z)\,]\, u(z) = 0$$

$$P(z) = z\, p(z) \qquad p_i = P_{i+1}$$

$$Q(z) = z^2\, q(z) \qquad q_i = Q_{i+2}$$

So saying P and Q are analytic at z=0, as we did, allows p to have a single pole, and q to have a single pole and/or a double pole, all at z = 0. Again, historically, if p and q are themselves analytic at z = 0, then z = 0 is called an **ordinary** point. If they are NOT analytic, *but* p has a worst a single pole, and q at worst a double pole, then P and Q are still analytic, and our work above still applies. Then z = 0 is called a **regular** (or regular singular) point. In this case, z = 0 is "sort of singular" but not really when we think of P and Q and Frobenius. However, if p and q are "worse than this" at z = 0, then z=0 is an **irregular** point. One might draw the classification this way:

ordinary point
singular point
  regular singular point
  irregular singular point

We can write all our recurrence formulas in terms of p_i and q_i instead of P_i and Q_i . In particular, the indicial equation looks like this:

$$r^2 + (P_0 - 1)\, r + Q_0 = 0 \qquad \Rightarrow \qquad r^2 + (p_{-1} - 1)\, r + q_{-2} = 0$$

Generalizing to z = c: The entire analysis above could be repeated replacing $z \to (z-c)$ as done by W&W. In that case, we would be assuming that P and Q were analytic at z = c and we would expand them around that point, and our solution would be around that point, etc. For example, we would then have

$$P(z) = P_0 + (z-c)P_1 + (z-c)^2P_2 + \ldots = \Sigma_{i=0}\, P_i(z-c)^i$$
$$Q(z) = Q_0 + (z-c)Q_1 + (z-c)^2Q_2 + \ldots = \Sigma_{i=0}\, Q_i(z-c)^i$$

$$u(z) = \Sigma_{k=0}A_k\,(z-c)^{r+k} = A_0\,(z-c)^r + A_1(z-c)^{r+1} + A_2\,(z-c)^{r+2} + \ldots = (z-c)^r\,\Sigma_{k=0}A_k\,(z-c)^k$$

Everywhere we had z^s we now have $(z-c)^s$. The recursion formulas are unchanged, including the indicial equation which is still $r^2 + (P_0-1)\,r + Q_0 = 0$. *However*, one must remember that P_i and Q_i are coefficients *relative to* the point z = c (as are the A_k), so different points will have different P_i and Q_i. Therefore, the values of r_1 and r_2 (the exponents) are different for different expansion points.

What about $z = \infty$? You can study this situation by changing variables to $z' = 1/z$ and then examine the $z' = 0$ situation. I think this is called a Mobius transformation. The conclusions are stated by W&W p 196:

$2z - z^2p(z)$ and $z^4q(z)$ both analytic at $z = \infty$ $\Leftrightarrow$ $z = \infty$ is an ordinary point

$z\,p(z)$ and $z^2q(z)$ both analytic at $z = \infty$ $\Leftrightarrow$ $z = \infty$ is an regular point

For example, if $p(z) = 1/z$ and $q(z) = 1/z^2$, then $z = \infty$ is a regular point. The same for higher inverse powers for these two. The first situation is harder to achieve. You could do it with $p(z) = (2z-1)/z^2$ and $q(z) = 1/z^4$.

Essential singularities. The claim is made in MF that at any **irregular singular point**, the solution of the ODE will have an **essential singularity**, the prototype case of this being $u(z) = e^{1/(z-a)}$ at z = a. An essential singularity is "worse than a pole of any finite order". I have not proved this claim, but MF talk about it and probably prove it. They also talk more about essential singularities. For example, such a singularity has a **species** and this a pecking order. For example, $e^{1/(z-a)^2}$ is worse than $e^{1/(z-a)}$. Finally, they point out that, like a pole of any finite order, an essential singularity is **isolated**, it is not a branch point of a cut, you can sail around it and get back to where you started.

Special Cases for Indicial equation solutions:

What happens if $r_1 = r_2 \equiv r$? The two series solutions are then identical, so there must be some other independent solution. This is given in equation p 195 A of W&W, where notice the h sum starts at n=1. In this case we have (they have $A_0 = 1$)

$$w_1(z) = z^r\,[\,1 + \Sigma_{n=1}^{\infty}\, A_n z^n]$$

$$w_2(z) = z^r\,\,[\Sigma_{n=1}^{\infty}\, h_n z^n\,] + w_1(z)\,\ln(z)$$

where we are not given a rule for computing the h_n, we just know they exist. You must include the series part of the second solution for w_2 to actually solve the ODE; the term $w_1(z)\ln(z)$ alone, although independent from $w_1(z)$, would not be a solution of the ODE.

What happens if $r_1 = r_2 + s$ with s = positive integer ? In this case the solutions are p 195 B of W&W:

$$w_1(z) = z^{r1}\,[\,1 + \Sigma_{n=1}^{\infty} A_n z^n]$$

$$w_2(z) = z^{r2}\;\; [\,-1/s + \Sigma_{n=1}^{\infty} h_n z^n\,] \; + g_s\, w_1(z)\ln(z)$$

where r_2 is the "smaller" exponent. The constant g_s depends in a complicated way on both P and Q and so is really a function of the ODE itself. It is the coefficient of the power z^s in a function g(z) shown on page 194 which you see is a messy combination of an exponentiated integral of P(z) and $w_1(z)$.

The main point about g_s is that for some ODE's it *might vanish*. If that happens, there is no ln(z) term, and the second solution is just the power series part. I would guess that in this case, the h_n would be obtained from the ladder of equations just as the A_n are, as shown above, just a different value of r. Probably in the $g_s \neq 0$ case, the ladder causes some $A_n^{(2)} = \infty$ and that is why we have the ln(z) thing.

Here is an "alternate form" for the w_2 solution where we just multiply through by constant -s,

$$w_2(z) = z^{r2}\;\; [\,1 + \Sigma_{n=1}^{\infty} (-sh_n) z^n\,] \; +(-sg_s)\, w_1(z)\ln(z)$$

The nice feature here is that the n=0 term in the series part is 1, just as in the $w_1(z)$ series. And of course the issue of $g_s = 0$ is unaltered.

Generalization to n^{th} order ODE. The indicial equation will be of degree n, and there will then in general be n solutions of the series form, each with its r_i.

Frobenius at ordinary points. If z = 0 is an "ordinary point", we know that P has a zero and Q has a double zero at z = 0. This means that $P_0 = 0$ and $Q_0 = Q_1 = 0$. In this case, our indicial equation becomes simply $r^2 - r = 0 = r(r-1)$, so the roots are r=0 and r=1. I suspect that in this case, $g_1 = 0$. The two solutions are then just power series

$$w_1(z) = z^1\,[\,1 + \Sigma_{n=1}^{\infty} A^{(1)}{}_n z^n]$$
$$w_2(z) = z^0\,[\,1 + \Sigma_{n=1}^{\infty} A^{(2)}{}_n z^n]$$

Each solution is analytic in a disc about z = 0. So that is the main idea of an "ordinary point". [This we treated in more detail above.]

2. Classification by Number of Singular Points

In the above Frobenius discussion, we considered "what if" when z = 0 was an ordinary or regular singular point, and we noted that this could just as well have been at some z = c. Consider:

$[D^2 + p(z)\,D + q(z)\,]\,u(z) = 0$ // canonical form

$[(z-c)^2D^2 + (z-c)\,P(z)\,D + Q(z)\,]\,u(z) = 0$ // Frobenius form

$P(z) = (z-c)\,p(z)$ $\quad$ $Q(z) = (z-c)^2\,q(z)$

where we are focusing on the point z = c. We could then write some sample p(z) forms:

$p(z) =$ analytic	$P(z) =$ analytic	// z=c is ordinary
$p(z) = (z-c)$ + analytic (= ana)	$P(z) =$ analytic	// z=c is ordinary
$p(z) = 1/(z-c)$ + analytic	$P(z) =$ analytic	// z=c is regular
$p(z) = \sqrt{z-c}$	$P(z) = (z-c)\sqrt{z-c}$ + analytic	// z=c is irregular
$p(z) = 1/\sqrt{z-c}$	$P(z) = \sqrt{z-c}$ + analytic	// z=c is irregular
$p(z) = 1/(z-c)^2$ + analytic	$P(z) = 1/(z-c)$ + analytic	// z=c is irregular

In general, an ODE will have some number of irregular points, and some number of regular points, and of course then all other points are ordinary. Consider now two separate points of interest c and d:

$[D^2 + p(z)\,D + q(z)\,]\,u(z) = 0$ // canonical form

$[(z-c)^2D^2 + (z-c)\,P_c(z)\,D + Q_c(z)\,]\,u(z) = 0$ // Frobenius form

$[(z-d)^2D^2 + (z-d)\,P_d(z)\,D + Q_d(z)\,]\,u(z) = 0$ // Frobenius form

$P_c(z) = (z-c)\,p(z)$ $\quad$ $Q_c(z) = (z-c)^2\,q(z)$

$P_d(z) = (z-d)\,p(z)$ $\quad$ $Q_d(z) = (z-d)^2\,q(z)$

Now there is a distinct pair of functions P and Q for each point of interest. You have to analyze each point separately. This shows the "advantage" of the canonical equation form where there is only one pair of functions p and q to worry about. Here are some examples to ponder:

$p(z) = 1/(z-c)$ + analytic everywhere	// regular at z = c, ordinary at z = d
$p(z) = \sqrt{z-c}$ + $1/(z-d)$ + analytic everywhere	// irregular at z = c, regular at z = d

You could make up an ODE for example with 11 regular points, and 3 irregular points. All you have to do is write down a p and q which explicitly do this, generalizing the c and d examples above.

You can therefore "**characterize**" an ODE by the number and type of singular points it has, and at what values of z these points occur. One needs to include $z = \infty$ in this analysis in the way noted above. We can *try* a series expansion about *any* point $z = z_1$ we like, be it ordinary, regular or irregular. Each (ordinary or regular) point z_1 will have its own pair of exponents from the indicial equation at z_1. But we are often going to be interested in expansions at the regular points (and maybe the irregular points as well). So in our characterization of an ODE, we might list off all the singular points, and the pair of exponents at each such point. [You can try a series at an irregular point. Since P/Q are not analytic at that point, it is likely that the solution is complicated at that point, so a simple series probably won't fly. This chapter of W&W really has nothing much to say about the nature of irregular point solutions.]

3. ODE with Five Regular Singular Points, and possible Confluences

The claim is made by W&W that page 197 that "the most general" ODE you can write, which has five regular points (including one at z = ∞) (and no irregular points) is given by p 197 E. You can see that the p(z) function has (in general) a single pole at each finite regular point, and q(z) has a double pole. The pole residues are exactly what they need to be so the indicial equation at each point gives the right exponents. The other term in q(z) has single poles at all the finite regular points. The exponents at the finite regular points are called α_r and β_r where r = 1,2,3,4 (has nothing to do with r = variable in indicial equation). The point z = ∞ obviously requires extra thinking which I don't want to do right now, but he claims that the exponents at z = ∞, which he calls $\mu_{1,2}$ are affected by the value of constant A (as indicated in equation F), while the constants B and C in equation E are completely free constants.

Why on earth would anyone be interested in an ODE with exactly 5 regular singular points? His reason is this: pretty much all ODE's which arise in "mathematical physics" (think "special functions") are particular forms of this general 5-regular-point ODE where (1) some of the singular points are the same, meaning they are "confluent"; and (2) all five exponent differences are 1/2 before confluence ! I see no mention of the five-ness in M&F Chap 5. [Ah, but now that I have read some there, I see that when you consider just the Helmholtz equation in various curvilinear coordinates, you get this confluence situation happening as you specialize from ellipsoidal to simpler systems. Although it is always "the Helmholtz equation", the post-separation ODE singularity structure depends on the coordinates you choose!]

By the way, an ODE whose singular points are all regular is called **Fuchsian**. The Google-sampled book of Wang Guo seems to have a lot of data on this subject. But let's continue in W&W.

So W&W go on to set $\beta_r = \alpha_r + 1/2$ (I write as Δ=1/2 below) and rewrite the general form as in p 197 G. I have no idea why they suddenly decide to write z as ζ, but OK. You can still of course see the various poles in the p and q functions. He calls this 5-point equation the **generalized Lamé equation**.

What happens if, say, the singular points a_1 and a_2 "conflow" (interestingly, there is no verb meaning to become confluent, OED mentions conflux). The second term in q(x) then develops a double pole at a_1 , and you have to add the residues in the p(z) function. The main double-pole part of q(z) also requires addition of residues. You do NOT generate a third order pole anywhere, but the exponent stuff gets shuffled around (sort of QM repulsion idea) and you find that this new equation has exponents α and β at this confluence point a_1 which are functions of just about everything as shown in page 198 G, and the difference is no longer 1/2 but can in fact be anything you want based on constants B and C. In the table below, this new regular point is called "other".

What happens if you conflow three regular points? Ah! Then the second q(z) term makes a triple pole, and your new confluence point is then an irregular point! Same for four or five.

I will now try to replicate the W&W table on page 198. Notice that if you only conflow two regular points, you will have left three Δ=1/2 points, and one new regular point which does not have Δ = 1/2, and that is the second line of the table. If you conflow three points, then only 2 Δ=1/2 left, and you have created an irregular point, as just discussed above -- the Mathieu line of the table. If you separately conflow two pairs, then only one Δ = 1/2 left, and you have created two "other" regulars, the Legendre row. You could consider instead conflowing one pair and one triplet, giving the Bessel line below. What if you conflow a quartet? Then one Δ=1/2 left and one irregular, the Weber/Hermite line. And finally you could flow all 5 together in a quintet to get the Stokes last line.

# Δ=1/2 regular points	# other regular	# irregular	NAME	tot reg

5	0	0	generalized Lamé	5
3	1	0	Lamé	4
2	0	1	Mathieu	2
1	2	0	Legendre	3
0	1	1	Bessel	1
1	0	1	Weber, Hermite	1
0	0	1	Stokes	0

4. ODE with Three Regular Singular Points

You could, I think, regard this as case 3 above (Legendre), but for some reason W&W just sort of start from scratch here. They make z = ∞ be an ordinary point, and then the three regular points are a,b,c with exponents α,α' at a, β,β' at b and γ,γ' at c. He then writes what I guess is the most general possible ODE with these three finite regular singular points (with these exponents) as equation E, crediting Papperitz 1885, but we are going to call this thing the Riemann P equation. [There are no free constants, so perhaps this is just a particular "prototype" equation with these properties, not the most general.] A solution of this equation is then written in the famous **Riemann P notation**, where the 9 parameters appear on three lines as shown. He then notes that the hypergeometric equation is a special case of Riemann P. In the hypergeometric, we have (see p 201 A,B).

a = 0	b = ∞	c = 1
α = 0	β = a	γ = 0
α' = 1-c	β' = b	γ' = c - a - b

Now there are two basic "transformations" you can prove are valid, which I copy from Wang & Guo,

First, we have this one, where the points (coordinate and singular) move, but the exponents don't move:

It is not difficult to show that a fractional linear transformation from z to z_1, namely

$$z_1 = \lambda \frac{z-\mu}{z-\nu}, \tag{9}$$

leaves the form of (1) unchanged, only the singular points a, b, c now becoming a_1, b_1, c_1 respectively, corresponding to z to z_1 according to (9). The whole transformation scheme may be expressed as

$$P\begin{Bmatrix} a & b & c & \\ \alpha_1 & \beta_1 & \gamma_1, & z \\ \alpha_2 & \beta_2 & \gamma_2 & \end{Bmatrix} = P\begin{Bmatrix} a_1 & b_1 & c_1 & \\ \alpha_1 & \beta_1 & \gamma_1, & z_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \end{Bmatrix}. \tag{10}$$

Note that the exponent pairs of the corresponding singularities remain unaltered under this sort of transformation.

Second, we have this one where the points don't move, but the exponents do:

$$\left(\frac{z-a}{z-c}\right)^k \left(\frac{z-b}{z-c}\right)^l P\left\{\begin{matrix} a & b & c & \\ \alpha_1 & \beta_1 & \gamma_1, & z \\ \alpha_2 & \beta_2 & \gamma_2 & \end{matrix}\right\}$$
$$= P\left\{\begin{matrix} a & b & c & \\ \alpha_1 + k & \beta_1 + l & \gamma_1 - k - l, & z \\ \alpha_2 + k & \beta_2 + l & \gamma_2 - k - l & \end{matrix}\right\}. \qquad (11)$$

Using these transformation rules, we can express an arbitrary P-equation solution in terms of the standard hypergeometric function as follows:

The solutions are denoted by the *Riemann P-symbol*

$$w = P\left\{\begin{matrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\}$$

The standard hypergeometric function may be expressed as

$${}_2F_1(a,b;c;z) = P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & a & 0 & z \\ 1-c & b & c-a-b & \end{matrix}\right\}$$

The P-functions obey a number of identities; one of them allows a general P-function to be expressed in terms of the hypergeometric function. It is

$$P\left\{\begin{matrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\} = \left(\frac{z-a}{z-b}\right)^\alpha \left(\frac{z-c}{z-b}\right)^\gamma P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & \alpha+\beta+\gamma & 0 & \frac{(z-a)(c-b)}{(z-b)(c-a)} \\ \alpha'-\alpha & \alpha+\beta'+\gamma & \gamma'-\gamma & \end{matrix}\right\}$$

In other words, one may write the solutions in terms of the hypergeometric function as

$$w = \left(\frac{z-a}{z-b}\right)^\alpha \left(\frac{z-c}{z-b}\right)^\gamma {}_2F_1\left(\alpha+\beta+\gamma, \alpha+\beta'+\gamma; 1+\alpha-\alpha'; \frac{(z-a)(c-b)}{(z-b)(c-a)}\right)$$

The full complement of Kummer's 24 solutions may be obtained in this way; see the article hypergeometric differential equation for a treatment of Kummer's solutions.

So this says that if you know all about the hypergeometric equation solutions, then you know all about the P equation solutions. My bateman1.doc has lots of better notes on the P equation and Kummer etc.

Note Added: Something is confusing me in the above discussion. Let's quote from WW 1990,

10·72. *The connexion of Riemann's P-equation with the hypergeometric equation.*

By means of the results of § 10·71 it follows that

$$P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = \left(\frac{z-a}{z-b}\right)^{\alpha}\left(\frac{z-c}{z-b}\right)^{\gamma} P\begin{Bmatrix} a & b & c & \\ 0 & \beta+\alpha+\gamma & 0 & z \\ \alpha'-\alpha & \beta'+\alpha+\gamma & \gamma'-\gamma & \end{Bmatrix}$$

$$= \left(\frac{z-a}{z-b}\right)^{\alpha}\left(\frac{z-c}{z-b}\right)^{\gamma} P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & \beta+\alpha+\gamma & 0 & x \\ \alpha'-\alpha & \beta'+\alpha+\gamma & \gamma'-\gamma & \end{Bmatrix},$$

where $$x = \frac{(z-a)(c-b)}{(z-b)(c-a)}.$$

The first equality is stated for *finite* a,b,c (arbitrary). *Then* the P function is *replaced* by another one where Rule II is used, which rule is this:

$$\text{(II)} \quad P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = P\begin{Bmatrix} a_1 & b_1 & c_1 & \\ \alpha & \beta & \gamma & z_1 \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} \qquad z = \frac{Az_1 + B}{Cz_1 + D}$$ etc

In the general Rule II, the constants A,B,C,D are arbitrary. The general rule II applies where *all four points* transform in the same way:

a = (Aa₁+B)/(Ca₁+D)
b = (Ab₁+B)/(Cb₁+D)
c = (Ac₁+B)/(Cc₁+D)
z = (Az₁+B)/(Cz₁+D)

This fact is not clearly stated in some sources I have. We can of course rewrite these "homographic" transformations (Ahlfors calls these things "linear fractional transformations", recall).

z = (Az₁+B)/(Cz₁+D) = A(z₁+B/A) / C(z₁+D/C) = (A/B) (z₁+B/A)/ (z₁+D/C)

= k' (z₁–α')/(z₁–β')

where now k', α', β' are our free constants of the linear transformation. The inverse transformations have exactly this same form, so we could say

z₁ = k (z–α)/(z–β)

where now the constants are all unprimed for the inverse transformations. This last form is really the most convenient for our purposes here, so let's write

$z_1 = k\,(z-\alpha)/(z-\beta)$
$a_1 = k\,(a-\alpha)/(a-\beta)$
$b_1 = k\,(b-\alpha)/(b-\beta)$
$c_1 = k\,(c-\alpha)/(c-\beta)$

Now, finally, we are "set up" to continue. Identify $z_1 = x$. So

$$x = k\,(z-\alpha)/(z-\beta) = [\,(c\text{-}b)/(c\text{-}a)\,]\,(z-a)/(z-b) \qquad k = (c\text{-}b)/(c\text{-}a) \quad \alpha = a \quad \beta = b$$

Then we can look out our transformed singular points

$$a_1 = [\,(c\text{-}b)/(c\text{-}a)\,]\,(a-a)/(a-b) = 0$$

$$b_1 = [\,(c\text{-}b)/(c\text{-}a)\,]\,(b-a)/(b-b) = \infty$$

$$c_1 = [\,(c\text{-}b)/(c\text{-}a)\,]\,(c-a)/(c-b) = 1$$

and there you have it.

Now in our HG connection deal above, we have a specific set of A,B,C,D in mind. Think of $x = z_1$. Then we have

5. ODE with Two Regular Singular Points

W&W are unclear here, but I think their conclusion is that, in the case of just two regular points, the general ODE of this type has simple power type solutions, see p 202.

This brings WW Chapter 10 to a close.

6. Confluent Hypergeometric Equation

Just some short comments here, my own topic. We say that if we started with the general-form 5-regular-point ODE (generalized Lamé), conflowing any *pair* of regular points did not generate an irregular singularity. What happens then if you start with our 3-regular-point ODE and do this? We can stare at the Riemann P equation on page 200 WW and we see that p(z) has only poles and q(z) has double poles. I suspect that this is "the most general form" which you consider that you have required $z = \infty$ to be an ordinary point. If we conflow a and b then q(z) has a triple pole, and this we have indeed generated an irregular singular point at a. But point c should remain regular.

WW confirm this notion on page 330 in their chapter on the confluent HG function.

In the Riemann P equation, we talk about a,b,c as the regular points. These are NOT the same a,b,c that appear in the HG function F(a,b;c;z).

Similarly, WW talk about $c,0,\infty \rightarrow \infty,0,\infty$ to get the CHG case. After confluence, they find constants m and k in the ODE and talk about functions $W_{k,m}$ and $M_{k,m}$ which we now call "Whittaker functions". Bateman mostly uses the Kummer $\Phi(a,c;x)$ and $\Psi(a,c;x)$ notation with constants a and c, and these constants have nothing to do with the location of the regular singularities!