When are ODE EF solutions orthogonal? PhL 1.28.10

The answer to this question is elusive, I have tried several times. Right now I have a particular problem with Q functions and I am wondering if they are orthogonal, but here let's look at "the general theory" as best I can find it.

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1. Regular BV Problem.

Start with the "regular BV problem" Stak p 268. Assume L is formally self adjoint, having his famous form shown there with p and q and interval (a,b). They key thing then is that "the parts" vanish from the throw-over of L from one side of the scalar product to the other. Stak shows you can write these parts as 4.29b p 269, [note complex scalar product]

$$(Lu,v) - (u,Lv) = \text{"the parts"} = p(b)W(u,v^*; b) - p(a)W(u,v^*; a)$$

If you assume that the ODE has a domain defined by two unmixed homo BC's $B_a u = 0$ and $B_b u = 0$, and if you require u, v to be in that domain of functions, then you can show that W(u,v*; b) = 0 and same for the other one. I now see I have discussed this buried deep in my Stak notes for Chap 4, so I will extract that stuff to right here:

"The BC's are standard unmixed ones as shown, one at each end. We now allow functions v and u to be complex (were real in Chap 1), so of course we use the full inner product. Simple algebra gives 4.29 and the RHS is 0 because each W is 0. This is forced by the form of the BC's! For example, assume that u and v are solutions to one of our equations at point a. Then we have:

$$W(u,\bar{v}; a) = \begin{vmatrix} u(a) & \bar{v}(a) \\ u'(a) & \bar{v}'(a) \end{vmatrix} \qquad \text{but } \alpha_1 u(a) + \alpha_2 u'(a) = 0 \text{ and same for } \bar{v} \quad (\alpha_i \text{ are real})$$

This means that the first column is a multiple of the second column, so W = 0. But we know from Abel p 60 that the W then vanishes everywhere. Of course our other BC says it vanishes at b as well.

Note 1 added 12.4.09. Lest this not be clear to stupid people (you rang), the BC's are these

$$\alpha_1 u(a) + \alpha_2 u'(a) = 0$$

$$\alpha_1 \bar{v}(a) + \alpha_2 \bar{v}'(a) = 0$$

[α_1 and α_2 must be real] which we write as

$$\alpha_1 \begin{pmatrix} u(a) \\ \bar{v}(a) \end{pmatrix} + \alpha_2 \begin{pmatrix} u'(a) \\ \bar{v}'(a) \end{pmatrix} = 0$$

And THIS says that the two ROWS of our matrix are DEPENDENT, therefore the det vanishes. Another way to say this is that we have

$$\begin{pmatrix} u(a) & u'(a) \\ \bar{v}(a) & \bar{v}'(a) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since we are claiming we have nonzero numbers α_1 and α_2, it must be true that the inverse of this matrix does not exist, and that means det M = 0 hence det M^T = 0 hence Wronksian = 0.

Note2added 12.4.09. On page 269 Stak says "For any u and v". But when it gets to the Wronskian issue, he says p 270 top " if u and v satisfy the homo BC's." Only then does the Wronskian vanish by the argument given above. The larger picture is this: you think of L and L* as operator and its adjoint. The operator is "formally" self-adjoint of the parts stuff is right. But it is "fully" self-adjoint only on a certain space of functions $D_A = D_{A^*}$ which consists of functions which are differentiable AND which meet the two BC's. "Vanishing at the BC's" is part of the *definition* of the domain of A = L. So our whole world of functions is this. So in all this proof stuff, we really mean "for any u and v which are in our space of functions" which includes the requirement that u and v separately respect both homo BC's. "

Stak then does his real EV's and orthogonal EF's proof on page 270 using only the fact that you can toss over the L operator. So this is the main idea. Notice that we are making no claims about "completeness" here. We don't have to convert to an integral equation and examine if Hilbert-Schmidt if we just care about orthogonality.

Example 1:

Now we pause to consider the usual Legendre P functions on (-1,1). *Although* the end points are singular in some sense (more below), we can still examine P at the two endpoints. Here is a quote from "questions about Legendre functions.doc"

"Conclusion: for any generic negative value of μ (make it non integral), if we select ν+μ = 1,2,3....) or the other string) then our function $P_\nu^\mu(z)$ vanishes both at z = 1 and z = -1, which is just what we need for our orange slice problem. " [ie, we get V=0 match along the + and - z axis]

Now we might just imagine that our two homo BC's are these, so that the $P_\nu^\mu(z)$ satisfy both:

$$u(-1) = 0 \qquad u(+1) = 0$$

These are pretty simple homo BC's, you must admit. So from this we can conclude that

$(P_\nu^\mu(z), P_{\nu'}^\mu(z)) = \delta_{\nu,\nu'} K_\nu$ where ν and ν' are discrete = $-\mu + N$ and $-\mu + N'$

Don't forget that in the Legendre equation, we regard $\nu(\nu+1) = \lambda$ as "the eigenvalue" of the EV problem, and so the P's are the eigenfunctions. So the λ are of course real. And $\nu = \mu-1, \mu-2....$ are the EV's. See referenced doc for much more detail on this example.

2. Singular BV Problem.

We turn now to Stak p 295. As usual, there is a weight function s(x) I have not yet mentioned that is part of regular and singular problems. If $\int dx\, s(x)\, |u|^2 < \infty$, the u is "finite s norm". We now assume that "b" is a singular point, perhaps $b = \infty$. Weyl's Theorem says that there are two choices for what happens at this singular endpoint. Either there are two finite s-norm solutions there (limit circle), or there is only one (limit point). In this latter case, he says you need $\mathrm{Im}(\lambda) \neq 0$, ie, $\lambda = \nu(\nu+1)$ must be real.

Now comes the question: what happens to our proofs of reality of EVs and orthogonality of EF's in this singular endpoint situation? Well, first of all, for any particular regular problem, we know the EV spectrum is discrete and we have orthogonality. For any particular singular BV problem, the spectrum might be discrete, continuous, or mixed (Stak p 304, last 2"). In this same place Stak says that if the spectrum is entirely discrete, then the eigenfunctions ARE orthogonal (with weight s). So this might explain our Legendre example 1 above where both endpoints are in fact singular. Stak does not prove this claim. He then says that if all endpoints are either regular or limit-circle singular, then the spectrum is entirely discrete (so from the earlier claim, then you will have orthogonal eigenfunctions). [On page 319 he shows that at least for u=0, both $z = \pm 1$ are limit circle for Legendre.]

Big Example 2: oblate positive imaginary axis Q

This is the one I am pondering today. It is Legendre on the positive imaginary axis which appears in oblate problems. I think the 0 endpoint is a regular endpoint, and the $i\infty$ endpoint is a singular endpoint. I think that endpoint is probably limit-point since only the Q function converges out there (is finite s norm). I think the integral of any Q^2 over this entire range is finite. We are staying away from $z = \pm 1$. I am not sure of the nature of the spectrum, nor of the nature of the BC at the z=0 endpoint, and of course I am not sure if there is any orthogonality. [But all three will be figured out shortly below!]

The first order of business then is to decide what to do at $z = 0+$. I know from Legendre properties that

$P_\nu^\mu(z=0\pm) = 2^\mu\sqrt{\pi}\; e^{\mp i\pi\mu/2} / [\, \Gamma(1/2 - \nu/2 - \mu/2)\, \Gamma(1 + \nu/2 - \mu/2)\,]$ // p 126 (22)

$Q_\nu^\mu(z=0\pm) = \sqrt{\pi}\, 2^{\mu-1}\, e^{\pm i\pi(-\nu-1)/2}\, [\, \Gamma(1/2 + \nu/2 + \mu/2) / \Gamma(1 + \nu/2 - \mu/2)]$ // p 134 (40)

In the case $\nu/2-\mu/2 = -1,-2$ etc, both these functions will in fact vanish since the bottom Γ has a pole and the top Γ is finite (assuming μ is not a half integer). This means:

$\nu = \mu - N$ N=2,4,6...

Suppose we had as our z=0 homo BC that u(0+) = 0. That will then force the above *spectrum* (for given μ), it is a discrete spectrum, and we *should* therefore have orthogonality of the EF's according to the Stak comments quoted above. But what then are the EF's? We know that $P_\nu^\mu(z)$ goes as $z^\nu + z^{-\nu-1}$, so it is certainly not finite s-norm at b = ∞ and cannot then be part of an EF. So the EF's are then precisely the Q functions. I would guess we are limit-point at the endpoint i∞.

The predicted orthogonality property will be this:

$$(u^\mu_\nu, u^\mu_{\nu'}) = \int_0^\infty d\zeta \; Q_\nu^\mu(i\zeta)^* \, Q_{\nu'}^\mu(i\zeta) = K_\nu^\mu \, \delta_{\nu,\nu'}$$

where $\nu = \mu - N$ $\nu' = \mu - N'$ N and N' are positive, even, and ≠ 0.

However, we know that for large argument

$$Q_\nu^\mu(z) = 2^\nu \, [\Gamma(1+\nu)\Gamma(1+\nu+\mu)/ \Gamma(2+2\nu)] \; z^{-\nu-1}$$ // Bateman p 132 (36)

Ignoring the gammas, we will have $z^{-2\nu-2}$ as the large ξ behavior of our integrand (ν=ν'), and this can only converge if 2ν+2 > 1 which means we need ν > -1/2. Eigenfunctions must be finite s-norm, so this *truncates our spectrum on the low end*. Our spectrum is then this:

ν = μ-2, μ-4 ..... such that ν > -1/2. SPECTRUM!!!

So our ODE EV "system" has a *finite* discrete spectrum. Is this possible? I am used to the particle in a square well with a finite spectrum. But not clear how eigenfunctions would be a complete set as Stak suggests, but maybe he meant if the spectrum is infinite. [but we will have lots of μ values...]

This seems a very strange result. In my usual expansions we have μ = m, an integer. So this seems to restrict the spectrum to m ≥ 2. If m = 2, we get only ν = 0 as the spectrum!

Can Maple Confirm Orthogonality of some Sample Eigenfunctions?

One is sorely tempted to see what Maple has to say about this. Just plotting the integrand gives us a good idea of whether things are orthogonal.

Maple Test: $g = \mathrm{Re}\,(\; Q_2^4(i\zeta)^* \, Q_0^4(i\zeta) \;)$ $\mu = 4$

Notice that with μ = 4, our spectrum would be ν = 2, 0 and that is it! So for μ = 4, my example is the only integral we can talk about that should be 0.

Here is what Maple has to say: (the imaginary part is < 10^{-12} over the same interval)

```
restart;
mu := 4:
nu1 := 2:
nu2 := 0:
f1 := LegendreQ(nu1,mu,I*z);
```

$$f1 := \text{LegendreQ}(2, 4, Iz)$$

```
f2 := LegendreQ(nu2,mu,I*z);
```

$$f2 := \text{LegendreQ}(0, 4, Iz)$$

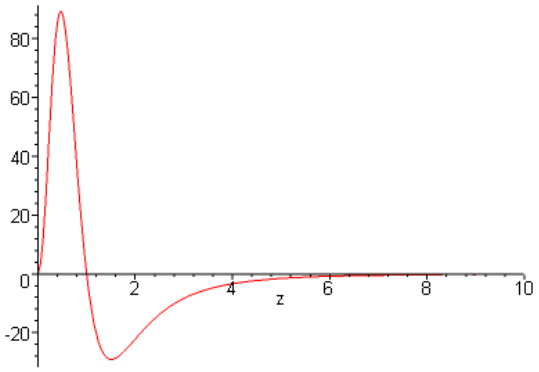
```
f := conjugate(f1)*f2;
```

$$f := \overline{\text{LegendreQ}(2, 4, Iz)}\ \text{LegendreQ}(0, 4, Iz)$$

```
g := Re(f);
```

$$g := \Re(\overline{\text{LegendreQ}(2, 4, Iz)}\ \text{LegendreQ}(0, 4, Iz))$$

```
plot(  g    , z=0..10);
```



Rather interesting! The integral *might* be 0, showing orthogonality! Well: (these are slow in Maple!)

```
evalf(Int(g, z=0..10));  evalf(Int(g, z=0..20));   evalf(Int(g, z=0..30));
```

.3727066168 .04764179252 .01417491996

I think I'm a **believer**. [this was the breakthrough moment] The last one took about 2 minutes or more.

If we set $\mu = 4$ and $\nu_1 = \nu_2 = 2$, we get:

```
restart;
mu := 4:
nu1 := 2:
nu2 := 2:
f1 := LegendreQ(nu1,mu,I*z);
```

$$f1 := \text{LegendreQ}(2, 4, Iz)$$

```
f2 := LegendreQ(nu2,mu,I*z);
```

$$f2 := \text{LegendreQ}(2, 4, Iz)$$

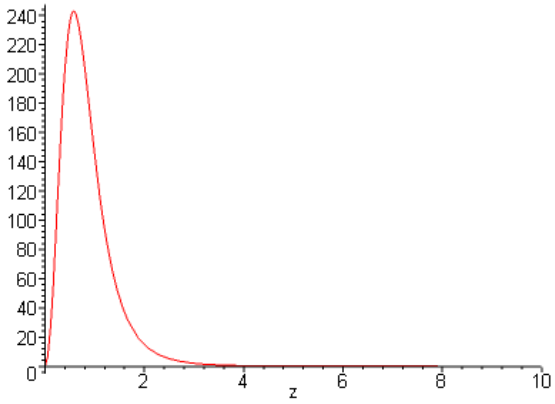
```
f := conjugate(f1)*f2;
```

$$f := \overline{\text{LegendreQ}(2, 4, Iz)}\ \text{LegendreQ}(2, 4, Iz)$$

```
g := Re(f);
```

$$g := \Re(\overline{\text{LegendreQ}(2, 4, Iz)}\ \text{LegendreQ}(2, 4, Iz))$$

```
plot(  g    , z=0..10);
```




```
evalf(Int(g, z=0..10));
```

$$226.1901922$$

Later we can use this as a check if we ever get a closed formula for the integral. Above is a plot in this case.

Example 2 Tentative Conclusion:

We consider the Legendre equation with interval (0, i∞). We impose the homo BC at the low end as u(0) = 0. We impose no BC at the other end since it is singular. At the high end we are (probably) limit point and we know that only $Q_\nu^\mu(z)$ will be finite s-norm at z = i∞. So this then is a complete description of our *ODE EV problem system*:

ODE = Legendre equation interval = (0+, i∞) BC: u(0+) = 0 EV = ν(ν+1)

In the ODE μ is "just a passive parameter", whereas ν(ν+1) is the eigenvalue. We find that the EF's of the above problem have a finite discrete spectrum and we conclude that the EF's are thus orthogonal, believing what Stak tells us. The quantization of the spectrum is caused by our imposed low end BC, and the finiteness by integrability at the high end. That spectrum is this: ν = μ – N N=2,4,6... . but ν > -1/2. Therefore, expect to have

$$(u^\mu_\nu, u^\mu_{\nu'}) = \int_0^\infty d\zeta\ Q_\nu^\mu(i\zeta)^*\ Q_{\nu'}^\mu(i\zeta) = K_\nu^\mu\ \delta_{\nu,\nu'}$$

We tested this in one simple Maple example. There must of course be some way to evaluate this integral to learn the K constant, but we will do that effort somewhere else. [have done if now for Q_2^4]

Example 2A Is there some other homo BC at z = 0 ? [no!]

Is there some OTHER homo BC we could choose at z = 0+ and get a different "system" perhaps with a larger or different spectrum? That homo BC would have to have this form:

$$f(\zeta) = Q_\nu^\mu(i\zeta) \qquad \partial_\zeta f(\zeta) = i\, Q_\nu^\mu{}'(i\zeta)$$

$$A\, Q_\nu^\mu(0+) + B\, \partial_\zeta\, Q_\nu^\mu(0+) = 0$$

where A and B depend only on μ. The BC's of an EV problem cannot depend on the EV λ ! Now in the case the μ = m = positive integer, I know that

$$Q_\nu^m{}'(0\pm) = \mp\, i\, Q_\nu^{m+1}(0\pm)$$

so our required homo BC would read:

$$A\, Q_\nu^m(0+) + B\, i\, Q_\nu^m{}'(0+) = 0$$

$$A_m\, Q_\nu^m(0+) + iB_m\, [\, -\, i\, Q_\nu^{m+1}(0\pm)\,] = 0$$

$$A_m\, Q_\nu^m(0+) + B_m Q_\nu^{m+1}(0+) = 0$$

I know that

$$Q_\nu^m(z=0\pm) = \sqrt{\pi}\, 2^{m-1}\, e^{\pm i\pi(-\nu-1)/2}\, [\, \Gamma(1/2 + \nu/2 + m/2) / \Gamma(1 + \nu/2 - m/2)] \quad \text{// p 134 (40)}$$

so our condition would be

$$A_m \sqrt{\pi}\, 2^{m-1}\, e^{\pm i\pi(-\nu-1)/2}\, [\, \Gamma(1/2 + \nu/2 + m/2) / \Gamma(1 + \nu/2 - m/2)]$$

$$+ B_m \sqrt{\pi}\, 2^m\, e^{\pm i\pi(-\nu-1)/2}\, [\, \Gamma(1/2 + \nu/2 + m/2 + 1/2) / \Gamma(1 + \nu/2 - m/2 - 1/2)] = 0$$

$$A_m\, 2^{-1}\, [\, \Gamma(1/2 + \nu/2 + m/2) / \Gamma(1 + \nu/2 - m/2)]$$

$$+ B_m\, [\, \Gamma(1/2 + \nu/2 + m/2 + 1/2) / \Gamma(1 + \nu/2 - m/2 - 1/2)] = 0$$

$$A_m\, 2^{-1}\, [\, \Gamma(1/2 + \nu/2 + m/2) / \Gamma(1 + \nu/2 - m/2)]$$

$$+ B_m\, [\, \Gamma(1 + \nu/2 + m/2\,) / \Gamma(1/2 + \nu/2 - m/2)] = 0$$

We have already considered the case $B_m = 0$ in our Example 2. So let's assume $B_m \neq 0$. If we solve our last equation above we get

$$A_m 2^{-1} [\, \Gamma(1/2 + \nu/2 + m/2) / \Gamma(1 + \nu/2 - m/2)] = -B_m [\, \Gamma(1 + \nu/2 + m/2\,) / \Gamma(1/2 + \nu/2 - m/2)]$$

$$A_m = -2\, B_m [\, \Gamma(1 + \nu/2 + m/2\,) / \Gamma(1/2 + \nu/2 - m/2)] * [\Gamma(1 + \nu/2 - m/2) / \Gamma(1/2 + \nu/2 + m/2)\,]$$

We assumed $B_m \neq 0$. Thus, we obtain A_m being a function of ν which is "illegal". But if we restrict our interest to a special spectrum of ν values, we might get $A_m = 0$ which would then be OK. Such ν values would cause a pole in one of the two denominators. So

$1/2 + \nu/2 - m/2 = 0,-1,-2... = -I \quad \Rightarrow \quad 2+\nu-m = -2I \quad \Rightarrow \quad \nu = m - 2 - 2I \quad I=0,1,2...$
$1/2 + \nu/2 + m/2 = 0,-1,-2... = -I \quad \Rightarrow \quad 2+\nu+m = -2I \quad \Rightarrow \quad \nu = -m - 2 - 2I \quad I=0,1,2...$

But we cannot have $\nu < -1/2$, so if $m > 0$ we through out the second spectrum, and for $m <$, the first. We end up with $\nu = |m| - N \qquad N = 2,4,6... \quad \nu > -1/2$ exactly as in Example 2. I am pretty sure this same conclusion would apply if m were not an integer.

My Conclusion: you cannot get anything "different" by assuming some different homo BC. So Example 2 is the whole ball of wax. Remember you must use Q functions to be finite s norm at the high end! The spectrum is

" $\Gamma(1 + \nu/2 - \mu/2)$ has a pole and $\nu > -1/2$. " $\qquad \nu = \mu - N \qquad N = 2,4,6...$

Listing of eigenfunctions in the case of integral m.

$m \leq 1$	none!
m=2	$Q_0^2(i\xi)$
m=3	$Q_1^3(i\xi)$
m=4	$Q_2^4(i\xi)$, $Q_0^4(i\xi)$
m=5	$Q_3^5(i\xi)$, $Q_1^5(i\xi)$
m=6	$Q_4^6(i\xi)$, $Q_2^6(i\xi)$, $Q_0^6(i\xi)$
m=7	$Q_5^7(i\xi)$, $Q_3^7(i\xi)$, $Q_1^7(i\xi)$
etc.	// in general, we have n = m-2, m-4.... (1 or 0)

Orthogonality says: [only meaningful for n in the spectrum]

$$(u^m_n, u^m_{n'}) = \int_0^\infty d\zeta\; Q_n^m(i\zeta)^*\, Q_{n'}^m(i\zeta) = K_n^m\, \delta_{n,n'}$$

We did one norm-constant evaluation and found that $K_2^4 = 72\pi$. I would have to write some kind of program to compute them all.