Essay on separable coordinates for Laplace PhL 1.25.10

At this point I have some **familiarity** with 6 of the 11 separable systems:

Cartesian,
spherical, spheroidal (oblate, some prolate), ellipsoidal
cylindrical

When I look at other systems, perhaps I will update this essay.

The Laplace equation is elliptical, the **signs of the curvatures** $\partial_i^2 u$ are the same and must add up to zero. All three directions cannot "curve up" at the same time, for example. At least two dimensions have to show opposite curvature at any point. This is why Laplace surfaces in 2D look the way they do.

When the Laplace equation is "separated" in the above coordinate systems, there are always two **separation constants**. One normally chooses the nature of these constants (real, imaginary, positive, negative) so the separated solutions match the nature of the physical problems one is trying to solve.

The **Cartesian separation** serves as the model which helps one understand the other cases. If we choose two dimensions to be sinusoidal, the other dimension must be expo and "blows up or blows down". The reason is that $k_x^2 + k_y^2 + k_z^2 = 0$ (this is Laplace), so if two of the k's are real, the third has to be imaginary. In the various coordinate systems, you can choose which two coordinates you want to be oscillatory, although there is a certain standard choice usually encountered.

This third expo dimension, perhaps it is "z", can be regarded as the "radial coordinate" in the following sense: it is by moving in this dimension that you can put yourself far away from the physical system of interest. Since this dimension has the possibility of exponential decay (while the other two dimensions do not, for a given choice of separation constant natures), this allows for a solution to a problem which has u = 0 at infinity at least in some direction(s). Of course not all problems require this particular condition, it is problem dependent.

Once the multi-dimensional Laplace equation Lu=0 is separated, each separated equation is a 1D **eigenvalue equation** $Lu=\lambda u$ where the eigenvalue λ is related to the separation constants. I think in all our cases that the L's so obtained are always of the usual self-adjoint form L = -D(pD) + q (but I am not sure of this). The L operator must be self-adjoint in order to have the possibility of **orthogonal eigenfunctions**. This is easily shown, since the fact (v,Lu) = (Lv,u) is used in the proof of orthogonality, see Stak p I.270. I suspect that a self-adjoint eigenvalue ODE is the key idea of what people call "a Sturm Liouville problem". [endpoints can be regular or singular]

Aside: An operator L is *formally* **self-adjoint** if (v,Lu) = (Lv,u) where you ignore "the parts" of the parts integrations encountered in going from one form to the other. The "q" term in L is obviously self adjoint, so one needs only show that the term D(pD) is formally self-adjoint. Red shows a D about to move:

$$(v,Lu) = \int v[D(pD)u] = -\int (Dv)p(Du)] = +\int D[p(Dv)]u = (Lv,u)$$

This idea is generalized to higher derivatives on Stak p II.9. *Full* self-adjointness requires the parts to vanish as well, so we really do get (v,Lu) = (Lv,u), and Stak p I.269 shows this happens when you have unmixed homogeneous BC's as in his 4.26. When an ODE has a singular endpoint, somehow this BC requirement at that endpoint is magically replaced by the simple requirement that the function be finite at the endpoint, another part of the S-L theory. This is what causes n to be integer in a Legendre situation which requires the z = -1 endpoint.

In the 1D equation one can study the **relationship between curvature and function value**. In some range of the interval of interest, the ODE might approximately say u" ~ +u which says curvature has the same sign as the function. In such a range, you have expo or power behavior of a solution, either blowing up or down. An example would be the $P_n(z)$ functions to the right of z = 1 (they all blow up). But in a different range, one might have u" ~ –u, which says the function wants to curve toward the axis, which means you have oscillatory behavior. An example would be the same $P_n(z)$ in the range (-1,1). The same function (like $P_n(z)$) can show both types of behavior depending on range. This is typical in QM SE problems, by the way. The SE is Laplace plus a potential. The nature (expo vs oscillatory) for the Legendre P case changes because we pass through a regular singular point and that changes the sign of some factor in the equation.

Let us now **review our coordinate systems** in their standard forms with regard to the above discussion:

Cartesians. The atomic forms can be written (always two solutions in each dimension)

$\exp(\pm ik_x x)\ \exp(\pm ik_y y)\ \exp(\pm k'_z z)$ $\qquad k_x^2 + k_y^2 + k_z^2 = 0 \qquad k_{z'} = ik_z$

where we have selected z to be the radial coordinate, while x and y are sinusoidal.

Sphericals. The atomic forms can be written

$[r^n\ ,\ r^{-n-1}]\ [\ P_n^m(z),\ Q_n^m(z)\ ]\ e^{\pm im\varphi}$

where we have made the standard selection m = real = integer for problems with full azimuth. If we have full polar angle range, we are forced to n = integer at cosθ = -1 to stay finite (a S-L thing). So in this case, the two oscillatory dimensions are θ and φ, and then r is the expo radial coordinate. Think $r^n = e^{n\ln(r)}$. The "first kind" $P_n^m(z)$ form orthogonal polynomials in (-1,1). The second-kind Q are "oscillatory far away" instead. That is to say, the Q's oscillate but decay toward ∞.

Cylindricals. The atomic forms can be written

$e^{\pm kz}\ \ [J_m(k\rho),\ N_m(k\rho)]\ e^{\pm im\varphi}$

where we have made the standard selection m = real = integer for problems with full azimuth. For some problems, k will be a continuous real value, other times it will be quantized perhaps to make J be 0 on a cylinder wall, or k might be imaginary. The radial dimension is "z" (for our choice of parameter nature). The "first kind" $J_m(k\rho)$ form orthogonal (oscillatory) functions on (0,∞) The second-kind N are oscillatory

and decay far away, same as the J, but blow up at $\rho = 0$ so cannot be used if $\rho = 0$ is part of the problem space.

Oblate Spheroidals. The atomic forms can be written (Smythe)

$$[P_n^m(i\zeta), Q_n^m(i\zeta)] [P_n^m(\xi), Q_n^m(\xi)] e^{\pm im\varphi}$$

where we have made the standard selection m = real = integer for problems with full azimuth. The radial coordinate is ζ and its functions blow up and down. The $P_n^m(\xi)$ form orthogonal polynomials just as with sphericals, though the argument is bloid angle which becomes polar angle far away. Also, ζ is proportional to spherical r far away.

Ellipticals. The atomic forms can be written (Smythe)

$$[E_m^p(\xi_1), F_m^p(\xi_1)] [E_m^p(\xi_2), F_m^p(\xi_2)] [E_m^p(\xi_3), F_m^p(\xi_3)]$$

Here the three dimensions are on somewhat of an equal footing, as in Cartesians. There is no azimuth coordinate. The ODE is the same for all three dimensions, but the solution nature depends on range of the parameter. Convention is that ξ_1, which labels ellipsoid surfaces, is the radial coordinate. The other two functions will be "oscillatory" in their ranges, meaning they can change sign as the argument varies in its official range. I have never studied a case other than the simple charged ellipsoid problem where the two oscillatory dimensions are constant functions. E are first kind, well behaved for small argument, F are second kind, well behaved for large argument. These are the Lamé functions.

What happens when the origin is placed on a surface of constant potential?

Often an electrostatics problem involves a prescribed potential φ on some surface; this is "the Dirichlet Problem". If the potential is a constant on the surface, the surface can be regarded as a piece of metal. Sometimes that piece of metal will have a charge density on one or both sides (possibly different), though these cannot very well be "prescribed" since the charge is free to move around. But we can imagine a theoretical "sticky surface" (in effect only one "side") where we have some prescribed charge density which does not move around. A charge density is always associated with the normal derivative of the potential at the charge density surface, so one thinks of a problem with prescribed charge density as one where $\partial_n\varphi$ is prescribed on some surface, and this is then "the Neumann Problem". Some problems have combined Dirichlet and Neumann "mixed" boundary conditions, which makes such problems a little harder.

The only problems that can really be solved analytically are those in which the involved surfaces are loci on which one of the three coordinates is a constant, and the two other coordinates vary. The spherical metal bowl problem is an example, or the charged oblate spheroid. So one needs to choose a coordinate system for problem solving that makes this be the case. Thus, you would not try to solve the bowl problem in Cartesian coordinates, and you would not try to solve a box problem in spherical coordinates.

The case of the charged disk has interesting features that took me a while to comprehend. One naturally wants to choose one's origin at the center of this disk, to maximize symmetry. Spherical coordinates comes to mind at once, but there is a problem. If you move away from the origin along the disk which is at constant potential, the functional dependence of the usual atomic form is that you are in

the radial or "power/expo" direction (r^n in this case), you are not moving in an oscillatory direction. Thus, there is no possibility of having orthogonal polynomials in this dimension (r). The implication is that you cannot cause the potential to be constant on a line segment from the origin to the edge of the disk, since you are dealing with non-orthogonal atoms.

When this problem is addressed in cylindrical coordinates, z is the radial direction of expo action, and ρ is an orthogonal functions direction, and in this case you can in fact have things be a function of ρ such that the function is a constant on such a ray (along which now ρ varies). That is to say, you can have a function that is not identically a constant which matches this BV ray, and then does something else and variable when ρ leaves the disk. This is why the charged disk problem is at least "approachable" in cylindrical coordinates, whereas it is intractable in spherical coordinates.

So the upshot is this: if your origin is on a metal surface, the two variables which vary on that surface must be the two oscillatory ones so the constancy of the surface can be matched in both directions. In the disk case, the "oscillatory" φ function is just a constant.

A spherical case where r^n and r^{-n-1} are oscillatory?

This will require some Stak searching. On page 316 we see a certain Mellin-Sine transform that is used in the radial equation for the 2D EV problem. It involves r^{ik} and r^{-ik}. These are e^{iklnr} which are indeed oscillatory, such as cos(ik lnr), obviously as r increases, this is sine-like. But I want to see how this works for the 3D Laplace radial equation. The solutions should be r^{ik} and r^{-ik-1} for the radial equation in its oscillatory mode. This might apply in the treatment of a vertical cone in spherical coordinates where the surface of the cone has a specified Dirichlet function. In this case, we want to be oscillatory in the r and φ directions, and we can be "radial" in the θ dimension. I have never seen such a problem treated anywhere.