How many independent solutions does Ly=0 have? PhL 11.24.04

Consider $L = D^n + b_1 D^{n-1} + ... + b_n\ 1$ as a differential operator of degree n with constant coefficients. I can see that there are n constants here.

Question: How do you know there are n independent solutions to the equation Lf = 0 ?

I know that I can factor the thing and that if all the roots are different, I can find n different solutions in that manner -- they are little exponentials like $\phi_k(x) = \exp(r_k x)$. And I guess it is clear that these functions are independent. But how do I know there are not some *other* solutions to this equation Af = 0? How do I know that the nullity = dimension of the nullspace of L = n ?

This seems like a very trivial question, but today I cannot seem to locate the answer. I did lots of reading in Stakgold, but I could not find the answer there, maybe I just did not recognize it.

Where might I look for the answer to this question? I got a few ODE books from the SLC library today, thinking maybe they would have the answer somewhere.

Lothar Collatz's book (SLC library) has a possible answer to my question on his page 90. There is a lot of stuff going on here, and the answer is not completely trivial. I will try to build up the various points that are needed in the proof.

1. Consider the arbitrary first order ODE y' = f(x,y). In some range of x, you want to know (for reasons to be seen later) WHEN there is a unique solution y(x). This is pretty much the most general 1st order ODE, as we saw in Scheid. It turns out that you must require a certain **Lipschitz condition** to be true, or the solution is NOT unique. That condition is this (two different versions with same results)

$[f(x,y) - f(x,y^*)] / (y - y^*) \leq k$ // over a region of interest in the x,y plane

$\partial f(x,y)/\partial y \leq k$ // over a region of interest in the x,y plane

The example Lothar gives is $f(x,y) = 3\ y^{2/3}$. In regions including y=0, the L condition is violated because we get some inverse power of y (let $y^* = 0$ in the first version above, for example). The solution is not unique because you can "ride along" the x axis. This situation was in fact mentioned on page 199 of Scheid.

2. Given that f(x,y) respects the **Lipschitz condition** , then for any initial condition $y(x_0) = y_0$, there is in fact a unique solution to this ODE. You may also need f(x,y) to be bounded and continuous. I don't know quite where Lother shows this in his book, but I think it is pretty reasonable and I could find this elsewhere.

2. Now, we go to page 33 of Lothar. Here he is considering *a system* of 1st order ODE's (need not be linear). Each equation has the form shown in 1 above, namely, $y_i' = f_i(x,\text{all the } y_k)$. He has i = 1 to s so there are s equations. He shows here that if

a) the f_i are all continuous and bounded and satisfy a certain multi-variable Lipschitz,

b) the initial conditions $y_i(x_0)$ are all specified at a given point x_0 (there are s conditions then)

then the solution of the system of equation is unique.

3. Next, on page 26 Lothar considers the general n_{th} degree ODE

$$y^{(n)} = f [x, y, y', y'', ... y^{(n-1)}]$$

and he shows that this can be thought of as a set of n 1st order ODEs. Here they are:

$$y_{(1)} = y'(x) = Dy = Dy_{(0)} \qquad y_{(2)} = y''(x) = \quad D\, y_{(1)} \qquad \text{and so on}$$

and then label $y_{(0)} = y$ for a uniform notation. Then we have these equations:

$$y'_{(0)} = y_{(1)}$$
$$y'_{(1)} = y_{(2)}$$
$$...$$
$$y'_{(n-1)} = y_{(n)} = f [x, y, y', y'', ... y^{(n-1)}]$$

which is n first order ODEs. Notice that this is a system of n first order ODE's, and in fact there is only one f function which is interesting (and possibly nonlinear), all the other f functions are trivial. We can now apply the result of item 2 above to conclude that if the function y and all its derivatives up to the n-1th are specified at a point x_0,

$$y_{(0)}(x_0) = y(x_0) = k_0$$
$$y_{(1)}(x_0) = y' (x_0) = k_1$$
$$....$$
$$y_{(n-1)}(x_0) = y'''' (x_0) = k_{n-1}$$

and if our one interesting f function is bounded, continuous and Lipschitz, THEN the system of equations has a unique solution, and therefore, the n^{th} degree ODE has a unique solution! Note that determination of the unique solution requires the specification of n constants!

4. Now, suppose you are given an ODE $y^{(n)} = f [x, y, y', y'', ... y^{(n-1)}]$ of degree n. If the ODE is *linear*, then the ODE must have the specific form of either a homogeneous or inhomogeneous equation with general function coefficients.

(4a) We know from above that the *homo* equation must have a *unique* solution that meets the n boundary conditions, since this is true for any ODE of order n, as shown above, given the right conditions. Thus, we know that there must be exactly n linearly independent solutions $\phi_i (x)$. The reason is this: if there are n $\phi_i (x)$, then we can write the general homo solution as $y(x) = \Sigma^n a_i \phi_i (x)$ with n constants a_i . When we require the BC's to be met, this sets all n of these coefficients. If there were less than n functions $\phi_i (x)$, we could not meet the boundary conditions, and then we would not have any solution at all. If there were more than n, the solution would not be unique. Therefore, since we know the solution *is* unique, there must be n solutions.

We have therefore and finally proved this theorem:

Theorem: The homogeneous order-N ODE with general function coefficients has a nullspace of dimension N. That is to say, let $L = D^N + b_1 D^{N-1} + ... + b_N 1$ where the b_i can be functions of x. This is an operator in L_2. Consider the equation Ly = 0, which is our homo equation. The space of solutions y has dimension N. This space is the nullspace of L. So the nullity is N. This is of course true in the special case that the coefficients are just constants. In that special case, we know exactly how to find the independent basis functions.

(4b) Now consider the inhomo equation. Again, there is a unique solution that meets the n boundary conditions. We know we can write this as:

$$y(x) = \Sigma^n a_i \phi_i(x) + \psi(x)$$

where $\psi(x)$ is "a particular solution" of the inhomo without regard to boundary conditions. We know that imposing the boundary conditions then sets the a_i coefficients.

The above argument is definitely not simple! Let's do it again:

1. General first order ODE with one BC has a unique solution if certain conditions are met.
2. The n^{th} order ODE can be cast as n 1^{st} order ODEs. This shows that the n normal BC's of the n^{th} order ODE are just the simple BC's of this system, so this shows that the n^{th} order ODE has a unique solution that meets the n BC's.
3. The Ly=0 n^{th} order ODE must therefore have n independent solutions. Only then are there exactly the right number of free constants to be able to meet the n boundary conditions.