Regular Singular Points etc PhL 5.2.09

1. Frobenius n = 2. There are many ways to write an ODE for n=2. One of them is the Frobenius form:

$$L_f u = (z-z_0)^2 u''(z) + (z-z_0)\,P(z)u'(z) + Q(z)u(z)$$

As shown in W&W and in my ODE1 notes, if P and Q are analytic at $z = z_0$, then we can find two independent solutions to $L_f u = 0$ using "the Frobenius method". We expand everything in sight in a power series about $z = z_0$ and we match powers. The highest power match gives "the indicial equation" which is a polynomial of degree n = 2 which we solve for roots r_1 and r_2 . Regardless of the values of these two roots, the Frobenius method provides us with two independent solutions to this problem. We don't care about boundary conditions in doing the Frobenius solutions, that would come later where we then assume that $u = Au_1 + Bu_2$. There are three cases of interest for the r_1, r_2 , but in all three cases, we have our solutions. Here are the three cases:

Case 1: general case, $r_1 - r_2 \neq$ integer: The A's are specific coefficients:

$$w_1(z) = (z-z_0)^{r1}\,[\,1 + \Sigma_{n=1}^{\infty} A_n^{(1)}\,(z-z_0)^n]$$
$$w_2(z) = (z-z_0)^{r2}\,[\,1 + \Sigma_{n=1}^{\infty} A_n^{(2)}\,(z-z_0)^n]$$

Notice that in general these solutions have a branch cut at z_0 due to the power factor.

Case 2: $r_1 = r_2$ Here, the $A_n^{(1)}$ are the same as above, and the h_n are some other coefficients:

$$w_1(z) = (z-z_0)^{r}\,[\,1 + \Sigma_{n=1}^{\infty} A_n^{(1)}\,(z-z_0)^n]$$
$$w_2(z) = (z-z_0)^{r}\;\;[\Sigma_{n=1}^{\infty} h_n\,(z-z_0)^n\,] \;+ w_1(z)\,\ln(z-z_0)$$

Now the second solution has a fancier branch point at $z = z_0$ of the form $(z-z_0)^r \ln(z-z_0)$.

Case 3: $r_1 = r_2 + s$ s = integer > 0

$$w_1(z) = (z-z_0)^{r1}\,[\,1 + \Sigma_{n=1}^{\infty} A_n^{(1)}\,(z-z_0)^n]$$
$$w_2(z) = (z-z_0)^{r2}\;\;[\,-1/s + \Sigma_{n=1}^{\infty} h_n\,(z-z_0)^n\,] \;+ g_s\,w_1(z)\,\ln(z-z_0)$$
$$w_2(z) = (z-z_0)^{r2}\;\;[\,1 + \Sigma_{n=1}^{\infty} (-sh_n)\,(z-z_0)^n\,] \;+(-sg_s)\,w_1(z)\,\ln(z-z_0)$$

Here we write the w_2 in two alternate forms where the second form puts a 1 as the leading term, and where s is our positive integer. By convention we assume r_1 is the larger root. The constant g_s is a function of the ODE and could be 0. If it is zero, then I think $-sh_n = A_n^{(2)}$. I don't think there is an obvious simple method to find the value of g_s.

2. Definitions: If P and Q are analytic at $z = z_0$ then the point z_o is "nice" because the Frobenius solutions exist and we know a lot about our problem. In the general case, we know that the Frobenius solutions will have branch cuts at $z = z_0$. If at least one of the solutions has a branch cut at $z = z_0$, then z_0

is called a "**regular singular point**". If the roots are such that neither solution has a branch cut, then z_0 is called an "**ordinary point**". If either P or Q is not analytic at $z = z_0$, then z_0 is called an "**irregular singular point**." Notice that ordinary point means all solutions are analytic at $z = z_0$.

2. Frobenius n = n. // think $g_n = 1$

$$L_f u = (z-z_0)^n u^{(n)}(z) + (z-z_0)^{n-1} g_{n-1}(z)\, u^{(n-1)} + .. (z-z_0)^i g_i(z)\, u^{(i)} + ... + g_0(z)u(z)$$

There are n functions $g_i(z)$ which we assume are analytic at $z = z_0$. We expand everything in sight about $z = z_0$ and match powers. We get an indicial polynomial of degree n, and find n roots r_1, r_2 ... r_n. In the "general " case 1 situation where the r_i are all sort of random numbers, we will get solutions:

$$w_i(z) = (z-z_0)^{ri} [\, 1 + \Sigma_{n=1}^{\infty} A_n^{(i)} (z-z_0)^n] \qquad i = 1,2...n$$

No doubt there are lots of special cases where pairs of indices are the same or differ by integers, as in the n = 2 case. Somehow you can handle all these cases and there will still be n independent solutions, but I have never read anywhere the details of how that is done. We really only care about n = 2, but I wanted to get this generalization here just for the record.

The definitions for the three kinds of "points" given above can be applied here.

- if P Q are analytic at z_0 and no solutions have branch cuts at $z = z_0$, then z_0 is an ordinary point
- if P Q are analytic at z_0 and at least one solution has a branch cut at $z = z_0$, then z_0 is an regular singular point.
- if either P or Q is not analytic at z_0 then z_0 is an irregular singular point.

3. Basic Theorem for n=n.

Consider this differential equation of order n with n coefficient functions: // think $f_n = 1$

$$L_f u = u^{(n)}(z) + f_{n-1}(z)\, u^{(n-1)} + ... f_i(z)\, u^{(i)} + ... + f_0(z)u(z)$$

If the coefficient functions are all analytic at $z = z_0$, then the equation $L_f u = 0$ with *initial-value* boundary conditions $u^{(n)}(z_o) = C_n$ will have a solution which is unique and which is analytic at $z = z_0$. This certainly seems reasonable, the proof may be somewhere in Stakgold but I don't know where. Such a value of z_0 is an ordinary point as defined earlier: all solutions are analytic at $z = z_0$.

Now compare this to our Frobenius form above and we have:

$$f_i(z) = g_i(z) / (z-z_0)^{n-i} \qquad i = 0,1..n$$

$$f_n(z) = g_n(z) = 1$$
$$f_{n-1}(z) = g_{n-1}(z) / (z-z_0)$$
$$f_{n-2}(z) = g_{n-2}(z) / (z-z_0)^2$$
...
$$f_0(z) = g_0(z) / (z-z_0)^n$$

Now we make this claim: The Frobenius solutions exist for $z = z_0$ if all these are true:

$f_{n-1}(z)$ has at worst a single pole at $z = z_0$
$f_{n-1}(z)$ has at worst a double pole at $z = z_0$
$f_{n-2}(z)$ has at worst a triple pole at $z = z_0$
...
$f_0(z)$ has at worst an n-th order pole at $z = z_0$

In this case, if none of those Frobenius solutions has a cut at $z = z_0$, then z_0 is an ordinary point.
If at least one of the Frobenius solutions has a cut, then z_0 is a regular singular point.
If at least one of the functions f_i shown above has a worse singularity at $z = z_0$ than we show in the list, then z_0 is an irregular singular point. This would be the case if some f_i has a pole of too high an order, or if it has a cut, or there is some pathological thing going on at $z = z_0$.

3. Basic Theorem for n=2

$L_4u = u'' + p_1u' + p_0u$

If the coefficient functions are all analytic at $z = z_0$, then the equation $L_fu = 0$ with *initial-value* boundary conditions $u^{(n)}(z_o) = C_n$ will have a solution which is unique and which is analytic at $z = z_0$. This certainly seems reasonable, the proof may be somewhere in Stakgold but I don't know where. Such a value of z_0 is an ordinary point as defined earlier: all solutions are analytic at $z = z_0$. [Stakgold quotes this theorem on page 58 as far as existence and uniqueness are concerned, not analyticity.]

Now compare this to our Frobenius form above,

$L_fu = (z-z_0)^2u''(z) + (z-z_0)\,P(z)u'(z) + Q(z)u(z)$

and we have

$$p_1(z) = P(z) / (z-z_0)$$
$$p_2(z) = Q(z) / (z-z_0)^2$$

Now we make this claim: The Frobenius solutions exist for $z = z_0$ if all these are true:

$p_1(z)$ has at worst a single pole at $z = z_0$
$p_2(z)$ has at worst a double pole at $z = z_0$

In this case, if none of those Frobenius solutions has a cut at $z = z_0$, then z_0 is an ordinary point.
If at least one of the Frobenius solutions has a cut, then z_0 is a regular singular point.
If at least one of the functions p_i shown above has a worse singularity at $z = z_0$ than we show in the list, then z_0 is an irregular singular point. This would be the case if some p_i has a pole of too high an order, or if it has a cut, or there is some pathological thing going on at $z = z_0$.

Definition: If there is no z_0 at which an ODE has an irregular singular point (including $z_0 = \infty$), then it has at worst only regular singular points (and perhaps none of these). In this case, one says that the ODE is **Fuchsian**.

Comments:

(1) We have seen that for ordinary points and for regular singular points, the Frobenius solutions exist and things are happy. For irregular singular points, we have no Frobenius information about solutions, but presumably solutions still exist.

(2) Nothing whatsoever is said above about the "interval" on which an ODE system is defined. We only make reference to a single point at a time, such as z_0, and we talk about initial-value BC's at that point.

(3) Stakgold talks about $L_{s1}u = a_0u'' + a_1u' + a_2u$ or

$$(1/a_0)\, L_{s1}u = u'' + (a_1/a_0)u' + (a_2/a_0)u \qquad = u'' + p_1u' + p_0u$$

so we identify

$$p_1 = (a_1/a_0) \qquad\qquad p_2 = (a_2/a_0)$$

The for L=L* situations we have $a_1 = a_0'$ so we then write $a_0 = -p$ and $a_2 = q$:

$$p_1 = (a_0'/a_0) = (\ln a_0)' \qquad\qquad p_2 = (a_2/a_0)$$

$$p_1 = (p'/p) = (\ln p)' \qquad\qquad p_2 = (-q/p)$$

In what Stakgold calls the **"regular boundary value problem"**, we have L = L* ,and we have a finite interval (a,b), and we have p and q real with $p > 0$ and we have a $-s\lambda$ term included in L_λ with $s>0$ and we have an unmixed BC at each end of the interval. The various conditions listed are all "on the interval".

If any of these conditions is violated, we have a **"singular boundary value problem"**.

(4) So one Stak use of the terms "regular" and "singular" is in reference to Boundary Value Problems, not to bare ODE's. He talks about certain points being "singular" such as if $b = \infty$ as the upper endpoint, or $p(b) = 0$ perhaps at an endpoint b. I now think his use of the word "singular" here has nothing whatsoever to do with the terms "regular singular point" or "irregular singular point" of an ODE. Stakgold's singular points are with respect to a certain ODE System including its BC's, whereas the Frobenius singular points have nothing at all to do with boundary values or endpoints or even intervals. Similarly, terms like limit-point and limit-circle have no place in the Frobenius analysis. Stakgold in his Chapter 4 analysis requires one unmixed BC at each end of the interval (a,b), whereas the Frobenius stuff uses two unmixed BC's at a single point z_0 (ie, Frobenius involves the initial-value BC's). In Chapter 1 Stak did talk about completely general BC's including the initial-value BC's.

Stakgold's main interest is solving $L_\lambda u = f$ for a set of BC's, and he always wants to do this with a Green's function, since that gives us a solution "for any f" and the BC's are always met. This is different

from the subject matter of books or sections of books about "special functions". Stakgold's book is not entitled "ordinary differential equations", it is entitled "boundary value problems".

So basically it's **apples and oranges.**

(5) Stak *also* uses the words "regular" and "singular" when discussing abstract Hilbert Space operators. This is yet another meaning for him of these two words, again unrelated to Frobenius.

Definition: **Operator A is regular**. Recall that a E_n operator was either regular or singular, but now things are messier. In ∞ dimensions, to be regular we need 3 conditions to be met:

(a) $Ax=0 \Rightarrow x=0$, as with matrices; and this implies that A^{-1} exists;

(b) the range must be *the entire* Hilbert space, so $Ax = f$ has a solution for any f in H;

(c) A^{-1} must be bounded (= continuous).

If an operator is not regular, it is singular.

So this notion of regular and singular refers to operators. It is true that L and L_λ are differential operators. In general, whether the operator $B = A-\lambda$ is regular or singular depends on the value of parameter λ.

Classifying Closed Operators. These are the only possibilities, one regular and three singular:

(1) B is regular, meaning B^{-1} exists, $\overline{R_B} = A$, and B^{-1} is bounded

(2) B^{-1} does not exist, nontrivial $Bx=0$;

(3) B^{-1} exists *and* B^{-1} is unbounded and $R_B \neq A$ but $\overline{R_B} = A$.

(4) B^{-1} exists *and* R_B does not close to A so there are points in A which lie outside our range.

Case 1: B is regular, meaning $Bx=0$ has only trivial and $\overline{R_B} = A$ and B^{-1} is bounded. The λ in this situation comprise the **resolvent set** of A, which is not part of the singular spectrum.

Case 2: Consider $Bx=0$ or $Ax_i = \lambda_i x_i$. The eigenvectors x_i are non-trivial solutions for the eigenvalues of this equation, λ_i are places where B is singular and has no inverse [(2) above] , and we have usually seen these values to be isolated points, so λ_i make up the **point spectrum** of A. [The eigenfunctions are solutions of $Bx = 0$, so B has a nullspace, so B^{-1} cannot exist (not 1 to 1).

Case 3: Consider λ where $Bx = 0$ has only trivial solution and B^{-1} is unbounded and $R_B \neq A$ but $\overline{R_B} = A$, which is case 3. Such λ belong to the **continuous spectrum** of A. Not sure why it is continuous.

Case 4: This leaves only <u>case 4</u> where $Bx=0$ has only trivial, but $\overline{R_B} < A$ so $(\overline{R_B})^{\perp}$ has dim > 0 and this *dimension* is called the **deficiency of λ.** So deficiency = dim $(\overline{R_B})^{\perp}$ = some integer. It is something like nullity. When λ fall into this case, you have the **residual spectrum** of A. So this case can only occur when the range of B does not fill the whole space A. Note that for matrices E^n, we only had cases 1 and 2, regular or point spectrum. Nothing is every unbounded (A or A^{-1} when it exists), so no continuous spectrum. And range is always full, so no residual spectrum.

Thus, we come to talk about a "value of λ as being regular or singular", by which we mean that the operator $B = A-\lambda$ is regular or singular. A value of λ might be in the resolvent set since B is regular, or it might be in the point or continuous spectrum, or even the residual spectrum.

My main point is this: All this stuff of operators L_λ (or K_λ in the integral equations world) being regular or singular relates to the value of λ being regular or singular, and singular values are "the spectrum" of L. None of this has anything whatsoever to do with an ODE at its "regular singular points". Those are points of the variable z in which we write the ODE, they have nothing to do with any λ parameter.