WK Attempt #2 PhL 7.21.13

All WK stuff got moved to FT where it belongs. Here I still had it in scrambler in trying to get the MLS spectrum.

Appendix G:

(j) The Z-Transform Wiener-Khintchine Theorem

In FT Section 32 (c) we derive the continuum version of the Wiener-Khintchine theorem as follows, making use of the Convolution Theorem FT (3.6),

$$r_x(t) = \int_{-\infty}^{\infty} dt''x(t - t'') x(-t'') \ . \qquad \text{// energy units} \qquad \text{FT (32.7)}$$

$$a(t) = \int_{-\infty}^{\infty} dt'' \ b(t-t'') \ c(t'') \qquad \Leftrightarrow \qquad A(\omega) = B(\omega) \ C(\omega) \ . \qquad \text{FT (3.6)}$$

$$b(t) = x(t) \quad \leftrightarrow \quad B(\omega) = X(\omega)$$
$$c(t) = x(-t) \quad \leftrightarrow \quad C(\omega) = X(-\omega) = X(\omega)^* \qquad \text{// from FT (7.1) and (7.2)}$$

Thus, the diagonalized frequency-domain form $A(\omega) = B(\omega) \ C(\omega)$ is

$$R_x(\omega) = |X(\omega)|^2 \ . \qquad \text{FT (32.8)}$$

Since

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{T} = \frac{|X(\omega)|^2}{2\pi T} \qquad \text{FT (33.23)}$$

where T is the duration of a pulse train, we find that

$$\mathcal{P}(\omega) = 2\pi T \ R_x(\omega) \ . \qquad \text{FT (34. 3)}$$

This says that the spectral power density of a pulse train can be obtained by computing the Fourier Integral Transform of the autocorrelation function and multiplying by $2\pi T$. We can regard this as the Wiener-Khintchine theorem, although strictly the theorem is that $R_x(\omega) = |X(\omega)|^2$.

We shall now obtain a similar Wiener-Khintchine theorem for an infinite sequence which contains repeated P-length subsequences. In (2.5.33) we define the autocorrelation sequence as

$$r_s \equiv (1/P) \sum_{n=1}^{P} y_n \ y_{n+s} \ . \qquad s = \text{any integer} \qquad \textit{(2.5.33)}$$

If we regulate our infinite sequence as being (2N+1) copies of the P-symbol subsequence (where N is very large), then we can write the above, using $T = (2N+1)PT_1$,

$$r_s \equiv (1/P) \sum_{n=1}^{P} y_n y_{n+s} \approx \frac{1}{P(2N+1)} \sum_{n=-\infty}^{\infty} y_n y_{n+s} = \frac{T_1}{T} \sum_{n=-\infty}^{\infty} y_n y_{n+s} .$$

We continue to represent the infinite duration of the pulse train by the symbol T, knowing that it is going to cancel another T later on, so we write the limit as

$$r_s = \frac{T_1}{T} \sum_{n=-\infty}^{\infty} y_n y_{n+s} = \frac{T_1}{T} \sum_{n=-\infty}^{\infty} y_{-n} y_{-n+s}$$

where for the last expression we took n→-n. We can then construct a Z Transform version of the Wiener-Khintchine theorem as follows:

$$r_s = \sum_{n=-\infty}^{\infty} \frac{T_1}{T} y_{s-n} y_{-n}$$

$$a_s = \sum_{n=-\infty}^{\infty} \Delta t \; b_{s-n} c_n \qquad \Leftrightarrow \qquad A''(z) = \Delta t \, B''(z) C''(z) \qquad \text{FT (24.5)}$$

$a_s = r_s \quad \leftrightarrow \quad A''(z) = R''(z)$
$b_k = y_k \quad \leftrightarrow \quad B''(z) = Y''(z)$
$c_k = y_{-k} \quad \leftrightarrow \quad C''(z) = Y''(z^{-1}) = Y''(z)^*$ // if the c_k are real
$\Delta t \quad \leftrightarrow \quad (T_1/T)$

Thus, the diagonalized z-domain form $A''(z) = \Delta t \, B''(z) C''(z)$ is

$$R''(z) = (T_1/T) \, Y''(z) \, Y''(z)^* = (T_1/T) \, | Y''(z) |^2 \quad . \qquad (2.5.42)$$

It is shown in FT (34.14) that

$$\mathcal{P}(\omega) = T_1 \, \mathcal{P}_{pulse}(\omega) \, (1/T) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^* \, y_n \, e^{i\omega(m-n)T_1} \qquad \text{FT (34.14)}$$

which we easily rewrite as

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \, (T_1/T) \, | Y''(z) |^2 \qquad (2.5.43)$$

Using (2.5.42) we get,

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \, R''(z) \, . \qquad (2.5.44)$$

This says that the spectral power density of repeated P-subsequence infinite pulse train can be obtained by computing the Z Transform of the autocorrelation sequence and multiplying by $\mathscr{P}_{\mathtt{pulse}}(\omega)$. This then is our Z Transform version of the Wiener-Khintchine theorem.

(k) Using Wiener-Khintchine to compute the MLS Spectrum

We shall compute R"(z) and then use the Z transform Wiener-Khintchine theorem (2.5.44) to get $\mathscr{P}(\omega)$. Recall that

$$\begin{aligned} r_s &= \langle a_n^2 \rangle & s &= NP & &= \beta \\ r_s &= \langle a_n a_{n+s} \rangle & s &\neq NP & &= \alpha \ . \end{aligned} \qquad (2.5.35)$$

Therefore

$$\begin{aligned} R''(z) &= \sum_{n=-\infty}^{\infty} r_n z^{-n} = \beta \sum_{n=NP} z^{-n} + \alpha \sum_{n \neq NP} z^{-n} \\ &= \beta \sum_{n=NP} z^{-n} + \alpha \Big(\sum_{n=-\infty}^{\infty} z^{-n} - \sum_{n=NP} z^{-n} \Big) \\ &= (\beta-\alpha) \sum_{n=NP} z^{-n} + \alpha \sum_{n=-\infty}^{\infty} z^{-n} \ . \end{aligned} \qquad (2.5.45)$$

The first sum is over n = 0, ±P, ±2P and so on. We can replace summation index n by index N,

$$\sum_{n=NP} z^{-n} = \sum_{N=-\infty}^{\infty} z^{-NP} = \sum_{n=-\infty}^{\infty} z^{-nP} \qquad (2.5.46)$$

From FT (24.1) we know that z lies on the unit circle in the z-plane and is related to ω by

$$z = e^{i\omega T_1} \qquad \text{FT (24.1)}$$

where T_1 is the duration of a pulse of the pulse train. We then have

$$\sum_{n=NP} z^{-n} = \sum_{n=-\infty}^{\infty} (e^{i\omega T_1})^{-nP} = \sum_{n=-\infty}^{\infty} e^{-i\omega T_1 nP} \ . \qquad (2.5.47)$$

According to FT (13.2) we know how to compute this sum,

$$\sum_{n=-\infty}^{\infty} e^{ink} = \sum_{m=-\infty}^{\infty} 2\pi\delta(k - 2\pi m) \qquad -\infty < k < \infty \ . \qquad \text{FT (13.2)}$$

so setting $k = -\omega T_1 P$ we get

$$\sum_{n=NP} z^{-n} = \sum_{m=-\infty}^{\infty} 2\pi\delta(-\omega T_1 P - 2\pi m) = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 P + 2\pi m) = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 P - 2\pi m) \qquad (2.5.48)$$

where we use $\delta(x) = \delta(-x)$ and in the last step take $m \rightarrow -m$.

Meanwhile, our other sum of interest in (2.5.45) is this one,

$$\sum_{n=-\infty}^{\infty} z^{-n} = \sum_{n=-\infty}^{\infty} (e^{i\omega T_1})^{-n} = \sum_{n=-\infty}^{\infty} e^{-i\omega T_1 n}$$

which is just the previous sum without the P. Thus,

$$\sum_{n=-\infty}^{\infty} z^{-n} = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \quad . \qquad (2.5.49)$$

Inserting (2.5.49) and (2.5.48) into (2.5.45) gives

$$R''(z) = (\beta-\alpha) \sum_{n=NP} z^{-n} + \alpha \sum_{n=-\infty}^{\infty} z^{-n}$$

$$= (\beta-\alpha) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 P - 2\pi m) + \alpha \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)$$

$$= (\beta-\alpha)\,(T_1 P)^{-1} \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 P - 2\pi m/[T_1 P]) + \alpha (T_1)^{-1} \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m/T_1)$$

$$= (2\pi/T_1)\ \{\ (\beta-\alpha)\ (1/P) \sum_{m=-\infty}^{\infty} \delta(\omega T_1 P - 2\pi m/[T_1 P]) + \alpha \sum_{m=-\infty}^{\infty} \delta(\omega T_1 - 2\pi m/T_1)\ \}$$

$$= \omega_1\ \{\ (\beta-\alpha)\ (1/P) \sum_{m=-\infty}^{\infty} \delta(\omega T_1 P - m\omega_1/P) + \alpha \sum_{m=-\infty}^{\infty} \delta(\omega T_1 - m\omega_1)\ \} \qquad (2.5.50)$$

and this concludes our calculation of the Z Transform of the autocorrelation sequence r_n. It only remains to install this into the Wiener-Khintchine theorem (2.5.44)

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\, R''(z)$$

$$= \mathcal{P}_{pulse}(\omega)\omega_1\ \{\ (\beta-\alpha)\ (1/P) \sum_{m=-\infty}^{\infty} \delta(\omega T_1 P - m\omega_1/P) + \alpha \sum_{m=-\infty}^{\infty} \delta(\omega T_1 - m\omega_1)\ \} \qquad (2.5.51)$$

This is in agreement with (2.5.21) which we stole from FT (F.22b) .