This is the Title PhL 3.26.05

Diophantine stuff ended up in GA, not in scrambler.

Lemma: In the integer equation ax = by, the smallest solution for x is x = b/GCD(a,b). (4.22)

Proof: A candidate solution of ax = by is x = b and y = a. If d ≡ GCD(a,b), then we know that an improved solution is x' = b/d and y' = a/d since x' and y' are both integers and a[b/d] = b[a/d]. Since by definition d is the largest common divisor of b and a, th

Lemma: In the integer equation ax = by with a,b > 0, the smallest solution for x is x = b/GCD(a,b). (4.22)

Proof: ax = by ⇒ a/y = b/x ≡ β . The most general solution of ax = by then has this form:

x = by/a = b/[a/y] = b/β

y = ax/b = a/[b/x] = a/β

We know that x = b and y = a is *a* solution of ax = by, and we seek solutions with x < b, so only values of β > 1 are of interest.

In the reals, the any solution of ax = by would have the form x = b/β and y = a/β.

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Lemma: Consider the Diophantine [see App. G] equation ax = by where a and b are positive integers and we seek solutions where x and y are positive integers. The smallest solution for x is x = b/GCD(a,b). (4.22)

Proof: Let d ≡ GCD(a,b). A candidate smallest solution for x is then x = b/d with y = a/d since these x and y values are both integers and since a[b/d] = b[a/d]. Suppose there were some solution pair x',y' such that x' < x. Since ax = by and ax' = by', if x' < x we must have y' < y. Then we can write x' = x - I and y' = y - J where both I and J are positive integers. Then

ax' = by' ⇒ a[x-I] = b[y-J] ⇒ a[b/d - I] = b[a/d -J] ⇒ aI = bJ

Then x = I and y = J is another solution of ax = by.

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Consider ax = by. Suppose d = gcd(a,b). Then divide ax=by by d to get

Ix = J y. where I= a/d and J = b/d and gcd(I,J) = 1.

Now, what is the smallest-x solution of Ix = Jy ? One solution is x = J and y = I. Suppose there were some other solution x = αJ and y = αI where α < 1.

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Consider the simpler problem:

Suppose ax = by and gcd(a,b) =1. What is the smallest solution x ?

We know that x = b and y = a is at least *a* solution.

Why can't we have a solution of the form x = αb and y = αa for some fraction α like α = 2/3 ?

Try to find a good example. a = 9 and b = 4. Then want x = α4 and y = α9. Try fraction α = I/J, then have

x = (I/J)4 y = (I/J)9

How do I know that no I,J exist which make x and y integers? Try J = 4 for starters so x = I. Then we have to have y = I(9/4) but we need I < J so I < 4. But I has to cancel the 4. Remember theorem:

Fact 2: Suppose GCD(N_1,N_2) = 1 and N_1A = N_2B. Then N_2 divides A and N_1 divides B. (G.2)

How does this apply to ax = by? Write as

Fact 2: Suppose GCD(a,b) = 1 and ax = by. Then b divides x and a divides y. (G.2)

This tells us that x/b is an INTEGER. And y/a is an integer as well. They are the same integer.
This is then new information.

x/b = I = y/a

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Consider the simpler problem:

Suppose ax = by and gcd(a,b) =1. What is the smallest solution x ?

First of all, we can write

$ax = by \quad \Rightarrow \quad y/a = x/b \equiv \alpha$

Then we can write

$x = by/a = \alpha\, b$
$y = ax/b = \alpha\, a$

The smallest-x solution will be the smallest-α solution. But G.2 says that α must be a positive integer. Thus the smallest x solution is x = b.

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Fact 15: Consider the Diophantine equation ax = by where a,b > 0. Consider the set of solutions (x,y) in which x,y > 0. The solution with the smallest value of x is (x = b/d , y = a/d) where d = GCD(a,b).

Proof: Let N_1, N_2 be the positive integers $N_1 = a/d$, $N_2 = b/d$. From **Fact 1**, $GCD(N_1,N_2) = 1$. Also, we see that ax = by $\Leftrightarrow N_1x = N_2y$. The solutions (x,y) of these two equations are the same, so the smallest-x solution will be the same. We shall then attempt to determine the smallest-x solution of $N_1x = N_2y$.

According to **Fact 2** with A = x and B = y, we know, since $GCD(N_1,N_2) = 1$ and $N_1x = N_2y$, that $x/N_2 = y/N_1 = I$, a positive integer, Thus, solutions of $N_1x = N_2y$ are $x = IN_2$ and $y = IN_1$ where I is a positive integer. The smallest-x solution must then be $x = N_2$ and $y = I_1$. This is then also the smallest-x solution of equation ax = by. Thus, the smallest-x solution of ax = by is x = b/d and y = a/d. **QED**

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