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2.5 The MLS Spectral Power Density and Autocorrelation Sequence

Before looking at the MLS spectra, it is useful to review the characteristics of a white power spectrum.

(a) Spectral Power Density of a White Sequence

The spectral power density of an infinite uncorrelated statistical pulse train which has amplitudes taken from the set {A,B} is shown in FT (35.37) to be

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathbf{pulse}}(\omega) \left[\sigma^2 + \mu^2 \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m) \right]$$

where (2.5.1)

$$\sigma^2 = [\, p(1-p)]\, (A-B)^2 \qquad \sigma^2 = (\beta-\alpha)$$
$$\mu = [p]A + [1-p]B \qquad \mu^2 = \alpha$$

Here p is the probability that y_n takes value A, and p-1 is the probability that y_n takes value B.

A white sequence is such a sequence in which p = 1/2, so that

$$\sigma^2 = (1/4)(A-B)^2$$
$$\mu^2 = (1/4)(A+B)^2 \qquad (2.5.2)$$

so our general white sequence $\langle \mathcal{P}(\omega) \rangle$ is given by

$$\langle \mathcal{P}(\omega) \rangle_{\mathbf{white}} = \mathcal{P}_{\mathbf{pulse}}(\omega) \left[(1/4)(A-B)^2 + (1/4)(A+B)^2 \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m) \right] . \qquad (2.5.3)$$

If {A,B} = {1,0} we find the following mixed spectrum (partially continuous, partially discrete),

$$\langle \mathcal{P}(\omega) \rangle_{\mathbf{white}} = \mathcal{P}_{\mathbf{pulse}}(\omega) \left[(1/4) + (1/4) \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m) \right] \qquad // \{1,0\} \qquad (2.5.4)$$

and {A,B} = {1,-1} produces instead the following purely continuous spectrum,

$$\langle \mathcal{P}(\omega) \rangle_{\mathtt{white}} = \mathcal{P}_{\mathtt{pulse}}(\omega) \ . \qquad \text{// } \{1,-1\} \qquad (2.5.5)$$

If the pulse is a square pulse of amplitude 1 and width T_1, then

$$\mathcal{P}_{\mathtt{pulse}}(\omega) = (1/\omega_1)\, \mathrm{sinc}^2(\omega T_1/2) \qquad \omega_1 = 2\pi/T_1 \qquad \text{FT (36.1)}$$

and one then finds for the {1,0} case that

$$\langle \mathcal{P}(\omega) \rangle_{\mathtt{white}} = (1/\omega_1)\, \mathrm{sinc}^2(\pi \frac{\omega}{\omega_1}) \, [(1/4) + (1/4) \sum_{m=-\infty}^{\infty} \delta(\frac{\omega}{\omega_1} - m) \,]$$

$$= (1/\omega_1)\, \mathrm{sinc}^2(\pi \frac{\omega}{\omega_1}) \, [(1/4) + (1/4)\, \delta(\frac{\omega}{\omega_1}) \,] \qquad \text{// } \{1,0\} \qquad (2.5.4a)$$

since the sinc function kills off the lines for $m \neq 0$. For the {1,-1} case

$$\langle \mathcal{P}(\omega) \rangle_{\mathtt{white}} = (1/\omega_1)\, \mathrm{sinc}^2(\pi \frac{\omega}{\omega_1}) \ . \qquad \text{// } \{1,-1\} \qquad (2.5.5a)$$

The first spectrum has a DC line and a continuous sinc^2 spectrum as shown. The second spectrum has no DC line of course and has four times the continuous spectrum since the peak-to-peak swing is double that of the {1,0} case.

(b) The general power formula for an infinite P-repeated sequence

Consider an infinite pulse train with underlying pulse $x_{\mathtt{pulse}}(t)$ whose amplitudes $y_{\mathtt{n}}$ are an MLS sequence of period P,

$$x(t) = \sum_{n=-\infty}^{\infty} y_{\mathtt{n}} \ x_{\mathtt{pulse}}(t - t_{\mathtt{n}}) \qquad \text{// infinite pulse train}$$

We are interested in determining the spectral power density $\mathcal{P}(\omega)$ of this pulse train.

First, we define the **autocorrelation sequence** $r_{\mathtt{s}}$ of a sequence as the following horizontal average across the sequence:

$$r_{\mathtt{s}} \equiv \langle y_{\mathtt{n}}\, y_{\mathtt{n+s}} \rangle_{\mathtt{1}} \qquad (2.5.6)$$

For an infinite sequence this can be written

$$r_{\mathtt{s}} \equiv \lim_{\mathtt{N}\to\infty} [\frac{1}{(2N+1)} \sum_{n=-N}^{N} y_{\mathtt{n}}\, y_{\mathtt{n+s}} \,] \ . \qquad \text{FT (F41.a)}$$

If the sequence y_n is periodic with period P, then it is shown in FT that

$$r_s = \frac{1}{P} \sum_{n=0}^{P-1} y_n y_{n+s} \qquad \text{FT (F41.b)} \qquad (2.5.7)$$

Now, in FT Appendix F it is shown that for *any* pulse train constructed from a repeating subsequence of length P, the following theorem applies:

Theorem : For an infinite statistical sequence made of repeated subsequences of length P : FT (F.54)
If the following is found to be true concerning the autocorrelation sequence elements,

$$\langle y_m y_n \rangle_1 = \alpha \quad \text{for } m \neq n + NP \quad N = \text{any integer} \qquad (2.5.8)$$
$$\langle y_m y_n \rangle_1 = \beta \quad \text{for } m = n + NP$$

then the spectral power density is given by :

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\ \omega_1 \sum_{m=-\infty}^{\infty} [\ (\beta-\alpha)\ \frac{1}{P}\ \delta(\omega - \omega_1 m/P) + \alpha\ \delta(\omega - \omega_1 m)\ ]\ . \qquad (2.5.9)$$

One can break out the DC term in (2.5.9) and write this alternative form (FT (F.52d))

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\ \omega_1 [\ (\beta-\alpha)\frac{1}{P} \sum_{m \neq 0} \delta(\omega - \omega_1 m/P) + \alpha \sum_{m \neq 0} \delta(\omega - \omega_1 m) + \{(\beta-\alpha)/P + \alpha\}\ \delta(\omega)\ ] \qquad (2.5.9a)$$

where m≠0 means that both positive and negative integers should be included in the sums.

We will now show that the above Fact applies to any MLS sequence of period P.

(c) Case 1: The Spectral Power Density of an MLS Sequence with symbols in {1,0}.

Consider once again our "compressed" chart (3.4.8) which collected facts about the product of a specific MLS sequence and a shifted version of itself,

The Counts	Column 3	Column 4	Shifted n's	Unshifted n's	
n_1 = number of (0,0)'s	sum = 0	product = 0	$(2^k - 1) - 3 \cdot 2^{k-2}$	$2^{k-1} - 1$	
n_2 = number of (0,1)'s	sum = 1	product = 0	2^{k-2}	0	
n_3 = number of (1,0)'s	sum = 1	product = 0	2^{k-2}	0	
n_4 = number of (1,1)'s	sum = 0	product = 1	2^{k-2}	2^{k-1}	(2.5.10)

where $P = 2^k - 1$. First of all, the mean value for an MLS sequence is given by (see (2.4.5)')

$$\langle y_n \rangle_1 = \text{Prob}(1)*1 + \text{Prob}(0)*0 = \text{Prob}(1) = [\text{number of 1's}]/P = 2^{k-1}/P = (1/2)(1+1/P)\ .$$

If the shifted version of our MLS sequence is shifted by s bits, then for $s \neq 0$,

$$\langle y_n y_{n+s} \rangle_1 = 1 * 1 * \text{Prob}(1,1) + 1*0* \text{Prob}(1,0) + \text{etc} = \text{Prob}(1,1) = n_4/P = 2^{k-2}/P.$$

Only facing bits of the pattern (1,1) contribute to $\langle y_n y_{n+s} \rangle_1$, and the count of those is $n_4 = 2^{k-2}$ as shown in the chart above (Shifted n's column), while the total number of bit pairs is P. Now

$$2^{k-2}/P = (1/4)2^k/P = (1/4)(P+1)/P = (1/4)(1 + 1/P)$$

so we have just shown that

$$\langle y_n y_{n+s} \rangle_1 = (1/4)(1 + 1/P) \qquad \text{for } s \neq 0 \quad . \tag{2.5.11}$$

For s = 0, there is no shift and, since there are 2^{k-1} one's in any MLS sequence, this is also the number of (1,1) matching pairs and $n_4 = 2^{k-1}$ as shown in the last column of the chart, bottom row. Thus

$$\begin{aligned} \langle y_n y_{n+0} \rangle_1 &= 1 * 1 * \text{Prob}(1,1) + \text{etc} = \text{Prob}(1,1) = 2^{k-1}/P = 2\,[2^{k-2}/P] \\ &= (1/2)(1 + 1/P) \, . \end{aligned} \tag{2.5.12}$$

We have now established that

$$r_s = \langle y_n\, y_{n+s} \rangle_1 = \begin{cases} \alpha & s \neq NP \\ \beta & s = NP \end{cases} \qquad N = \text{ any integer}$$

where (2.5.13)

$$\alpha = (1/4)(1 + 1/P)$$
$$\beta = (1/2)(1 + 1/P) \qquad \Rightarrow \qquad (\beta-\alpha) = \alpha = (1/4)(1 + 1/P)$$

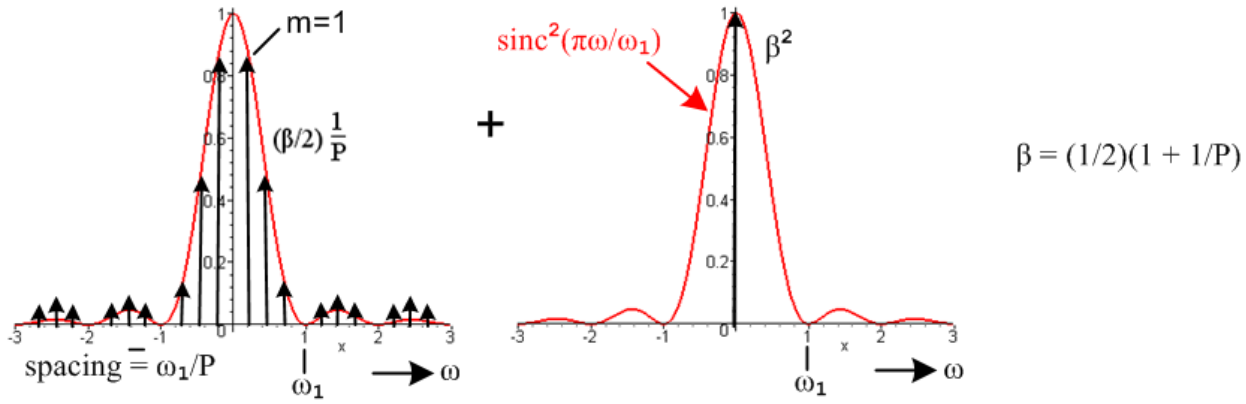
Since this meets the condition (2.5.8), the spectral power density of a pulse train which uses this MLS sequence as its amplitudes is given by (2.5.9a)

$$\begin{aligned} \mathcal{P}(\omega) &= \mathcal{P}_{pulse}(\omega)\, \omega_1 \left[(\beta-\alpha) \frac{1}{P} \sum_{m \neq 0} \delta(\omega - \omega_1 m/P) + \alpha \sum_{m \neq 0} \delta(\omega - \omega_1 m) + \{(\beta-\alpha)/P + \alpha\}\, \delta(\omega) \right] \\ &= \mathcal{P}_{pulse}(\omega)\, \omega_1 \left[\alpha \frac{1}{P} \sum_{m \neq 0} \delta(\omega - \omega_1 m/P) + \alpha \sum_{m \neq 0} \delta(\omega - \omega_1 m) + \alpha\{1+1/P\}\, \delta(\omega) \right] \\ &= \mathcal{P}_{pulse}(\omega)\, \omega_1 \left[(\beta/2) \frac{1}{P} \sum_{m \neq 0} \delta(\omega - \omega_1 m/P) + (\beta/2) \sum_{m \neq 0} \delta(\omega - \omega_1 m) + \beta^2\, \delta(\omega) \right] . \end{aligned} \tag{2.5.14}$$

where $\beta = (1/2)(1 + 1/P)$. In the case of a box-shaped pulse $\mathcal{P}_{pulse}(\omega)\, \omega_1 = \text{sinc}^2(\pi \frac{\omega}{\omega_1})$, the entire second sum is killed off by the zeros of the sinc function leaving this result

$$\mathcal{P}(\omega) = (\beta/2)\,\mathrm{sinc}^2(\pi\frac{\omega}{\omega_1})\,\frac{1}{P}\sum_{m\neq 0}\delta(\omega - \omega_1 m/P) + \beta^2\,\delta(\omega), \qquad \beta = (1/2)(1 + 1/P) \qquad (2.5.15)$$

Here is a graphical representation of this MLS $\mathcal{P}(\omega)$ drawn for P = 4 :



Spectrum of an MLS sequence with symbols in {1,0} and a box pulse shape. Fig 2.8

Each vertical arrow represents a spectral δ line. The height of the arrow is the value of the red envelope curve times the quantity shown. The little arrows are drawn too long so they can be seen. As P gets larger, the density of arrows on the left increases, but the height of the arrows $(\beta/2)\,\frac{1}{P}\,\mathrm{sinc}^2(\pi m/P)$ decreases. Remember that each arrow represents a delta function which is infinite in "height", so this infinite height is being scaled down by 1/P as P increases. In the limit P→∞, FT (F.38) shows that

$$\lim_{P\to\infty}\left[\frac{\omega_1}{P}\sum_{m=-\infty}^{\infty}\delta(\omega - \omega_1 m/P)\right] = 1 \qquad \text{FT (F.38)}$$

so the P→∞ limit of the above MLS power spectrum becomes

$$\mathcal{P}(\omega) = (1/4)\,\mathrm{sinc}^2(\pi\frac{\omega}{\omega_1})\,(1/\omega_1) + (1/4)\,\delta(\omega), \qquad \beta = (1/2) \qquad (2.5.16)$$

which agrees with the white power spectrum (2.5.4a). Thus, the densely packed lines have coalesced into the continuous sinc² spectrum shown.

More generally, for any pulse shape it is easy to verify from our equations above that

$$\lim_{P\to\infty}[\mathcal{P}(\omega)_{MLS}] = \langle\mathcal{P}(\omega)\rangle_{white}. \qquad // \{1,0\} \qquad (2.5.17)$$

(d) Case 2: The Spectral Power Density of an MLS Sequence with symbols in {1,-1}

Our starting point to carry out this program is to consider once again our "compressed" chart which collected facts about the product of a specific MLS sequence and a shifted version of itself. However, we now draw the chart for our new choice of symbols: (the right two columns are unchanged)

The Counts	Column 3	Column 4	Shifted n's	Unshifted n's
n_1 = number of (-1,-1)'s	sum = -2	product = 1	$(2^k - 1) - 3\bullet 2^{k-2}$	$2^{k-1} - 1$
n_2 = number of (-1,1)'s	sum = 0	product = -1	2^{k-2}	0
n_3 = number of (1,-1)'s	sum = 0	product = -1	2^{k-2}	0
n_4 = number of (1,1)'s	sum = 2	product = 1	2^{k-2}	2^{k-1}

(2.5.10)'

First of all, the mean value for an MLS sequence of this type is given by (see (2.4.5)')

$$\langle y_n \rangle_1 = \text{Prob}(1)*1 + \text{Prob}(-1)*(-1) = [\text{number of 1's}]/P - [\text{number of -1's}]/P$$

$$= [2^{k-1}]/P - [(2^{k-1} - 1)]/P = 1/P .$$

Our new computation for $\langle y_n y_{n+s} \rangle_1$ goes like this, assuming $s \neq 0$,

$$\langle y_n y_{n+s} \rangle_1 = 1*1*\text{Prob}(1,1) + 1*(-1)*\text{Prob}(1,-1) + (-1)*1*\text{Prob}(-1, 1) + (-1)*(-1)\text{Prob}(-1,-1) .$$

For $s \neq 0$, the "shifted n's" column tells us that

$$\text{Prob}(1,1) = \text{Prob}(-1,1) = \text{Prob}(1,-1) = 2^{k-2}/P$$
$$\text{Prob}(-1.-1) = [(2^k - 1) - 3\bullet 2^{k-2}]/P$$

Then for $s \neq 0$,

$$\langle y_n y_{n+s} \rangle_1 = 2^{k-2}/P - 2^{k-2}/P - 2^{k-2}/P + [(2^k - 1) - 3\bullet 2^{k-2}]/P$$

$$= - 2^{k-2}/P + [(2^k - 1) - 3\bullet 2^{k-2}]/P = [(2^k - 1) - 4\bullet 2^{k-2}]/P$$

$$= [P - (4/4)\bullet 2^k]/P = [P - (P+1)]/P$$

$$= -1/P . \qquad (2.5.11)'$$

For $s = 0$ we get instead

$$\langle y_n y_{n+0} \rangle_1 = 1*1*\text{Prob}(1,1) + 1*(-1)*\text{Prob}(1,-1) + (-1)*1*\text{Prob}(-1, 1) + (-1)*(-1)\text{Prob}(-1,-1)$$

$$= \text{Prob}(1,1) + \text{Prob}(-1,-1) = 2^{k-1}/P + (2^{k-1}-1)/P = (2^k-1)/P = P/P = 1 . \qquad (2.5.12)'$$

This result is fairly obvious since $\langle y_n^2 \rangle = 1$ for both y_n in $\{1,-1\}$.

We have now established that

$$r_s = \langle y_n\, y_{n+s} \rangle_1 = \begin{cases} \alpha & s \neq NP \\ \beta & s = NP \end{cases} \qquad N = \text{ any integer}$$

where (2.5.13)'

$$\alpha = -1/P$$
$$\beta = 1 \quad \Rightarrow \quad (\beta-\alpha) = (1+1/P)$$

Since this meets the condition (2.5.8), the spectral power density of a pulse train which uses this MLS sequence as its amplitudes is given by (2.5.9a)

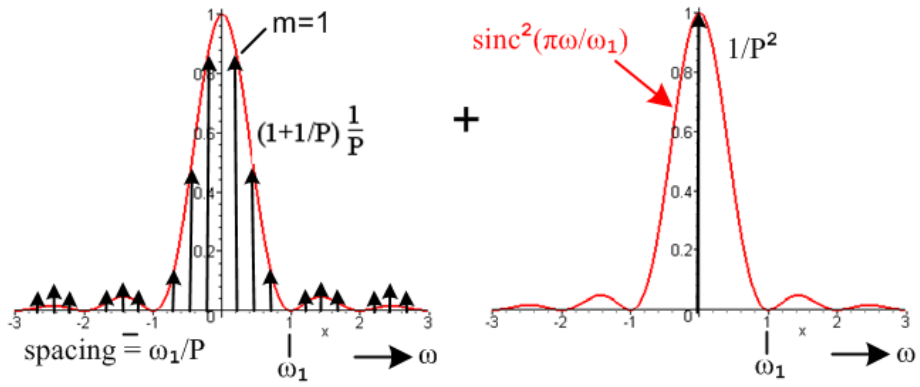
$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\, \omega_1 \left[(\beta-\alpha)\frac{1}{P} \sum_{m \neq 0} \delta(\omega - \omega_1 m/P) + \alpha \sum_{m \neq 0} \delta(\omega - \omega_1 m) + \{(\beta-\alpha)/P + \alpha\}\, \delta(\omega) \right]$$

$$= \mathcal{P}_{pulse}(\omega)\, \omega_1 \left[\left(1+\frac{1}{P}\right)\frac{1}{P} \sum_{m \neq 0} \delta(\omega - \omega_1 m/P) - \frac{1}{P} \sum_{m \neq 0} \delta(\omega - \omega_1 m) + \frac{1}{P^2}\, \delta(\omega) \right] \qquad (2.5.14)'$$

In the case of a box-shaped pulse $\mathcal{P}_{pulse}(\omega)\, \omega_1 = \text{sinc}^2\left(\pi \frac{\omega}{\omega_1}\right)$, the entire second sum is again killed off by the zeros of the sinc function leaving this result

$$\mathcal{P}(\omega) = \text{sinc}^2\left(\pi \frac{\omega}{\omega_1}\right)\left[\left(1+\frac{1}{P}\right)\frac{1}{P} \sum_{m \neq 0} \delta(\omega - \omega_1 m/P) + \frac{1}{P^2}\, \delta(\omega) \right] \qquad (2.5.15)'$$

Here is a graphical representation of this MLS $\mathcal{P}(\omega)$ drawn for P = 4 :



Spectrum of an MLS sequence with symbols in {1,-1} and a box pulse shape. Fig 2.9

Using the same FT (F.38) limit $\lim_{P\to\infty} \left[\frac{\omega_1}{P} \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P) \right] = 1$, we find for $P \to \infty$

$$\mathcal{P}(\omega) = (1/\omega_1)\ \text{sinc}^2\left(\pi \frac{\omega}{\omega_1}\right) \qquad (2.5.16)'$$

which agrees with the white power spectrum (2.5.5a). The DC line has gone away and the closely spaced lines on the left have again coalesced into the continuous sinc^2 spectrum shown.

More generally, for any pulse shape it is easy to verify from our equations above that

$$\lim_{P\to\infty} [\mathcal{P}(\omega)_{MLS}] = \langle \mathcal{P}(\omega)\rangle_{white}. \qquad // \{1,-1\} \qquad (2.5.17)'$$

(e) Summary, MLS correlation, and graphs of the autocorrelation sequences

We have seen from (2.5.6) and (2.5.7) that the autocorrelation sequence is given by

$$r_s = \frac{1}{P}\sum_{n=0}^{P-1} y_n\, y_{n+s} = \langle y_n\, y_{n+s}\rangle_1 , \qquad y_{n+NP} = y_n$$

We may gather up our results of the last three sections to construct this table

Case 1 {1,0}	$\langle y_n\rangle = \mu$	$r_0 = \beta = \langle y_n^2\rangle$	$r_{m-n} = \alpha = \langle y_m y_n\rangle$	
uncorrelated	μ	μ	μ^2	$0 \le \mu \le 1$
MLS	(1/2)(1+1/P)	(1/2)(1+1/P)	(1/4)(1+1/P)	
white	1/2	1/2	1/4	

Case 2 {1,-1}	$\langle y_n\rangle = \mu$	$r_0 = \beta = \langle y_n^2\rangle$	$r_{m-n} = \alpha = \langle y_m y_n\rangle$		
uncorrelated	μ	1	μ^2	$-1 \le \mu \le 1$	
MLS	1/P	1	-1/P		
white	0	1	0		(2.5.18)

Is the MLS sequence correlated or uncorrelated?

For {1,0} here is what we know (see FT Appendix G (c)) :

$$\sigma^2(Y_n) = E(Y_n^2) - E(Y_n)^2 = \langle y_n^2\rangle - \langle y_n\rangle^2 = (1/2)(1+1/P) - (1/2)^2(1+1/P)^2 \qquad \text{FT (G.30)}$$

$$= (1/4)(1-1/P^2)$$

$$\mathrm{cov}(Y_n,Y_m) = E(Y_nY_m) - E(Y_n)E(Y_m) = \langle y_n y_m\rangle - \langle y_n\rangle^2 = (1/4)(1+1/P) - (1/2)^2(1+1/P)^2$$

$$= -(1/4)(1/P)(1+1/P) \qquad \text{FT (G.24b)}$$

Therefore

$$\mathrm{corr}(Y_n,Y_m) = \frac{\mathrm{cov}(Y_n, Y_m)}{\sigma(Y_n)\,\sigma(Y_m)} = \frac{\mathrm{cov}(Y_n, Y_m)}{\sigma^2(Y_n)} = \frac{-(1/4)(1/P)(1+1/P)}{(1/4)(1-1/P^2)} = \frac{-1/P}{1+1/P} \qquad \{1,0\}$$

Since the result is not zero, the {1,0} MLS sequence is correlated. As $P\to\infty$ it becomes uncorrelated.

For {1,-1} we find instead:

$$\sigma^2(Y_n) = E(Y_n^2) - \quad E(Y_n)^2 = \langle y_n^2 \rangle - \langle y_n \rangle^2 \; = 1 \; - (1/P)^2$$

$$\mathrm{cov}(Y_n, Y_m) = E(Y_n Y_m) - E(Y_n)E(Y_m) = \langle y_n y_m \rangle - \langle y_n \rangle^2 = -1/P - \; (1/P)^2 = -(1/P)(1+1/P)$$

Therefore

$$\mathrm{corr}(Y_n, Y_m) = \frac{\mathrm{cov}(Y_n, Y_m)}{\sigma(Y_n)\,\sigma(Y_m)} = \frac{\mathrm{cov}(Y_n, Y_m)}{\sigma^2(Y_n)} = \frac{-(1/P)(1+1/P)}{1 \; - (1/P)^2} = \frac{-1/P}{1-1/P} \qquad \{1,-1\}$$

Since the result is not zero, the {1,-1} MLS sequence is correlated. As P→∞ it becomes uncorrelated.

We now plot the six autocorrelation sequences implied by (2.5.18). The sequence values are shown as black dots, and the values are linearly interpolated by a red line as a cosmetic guide. In these drawings it is assumed that P > 4 so we don't see the autocorrelation sequences peaking every P integers, but one should keep in mind that they do just that.

In the process of acquiring and tracking a spread spectrum signal, where one seeks to align the receiving clock with the transmitting clock, one reverts to a continuous (analog) autocorrelation function and then these red lines are more than cosmetic. See Comments at the start of Section 3.5 below.

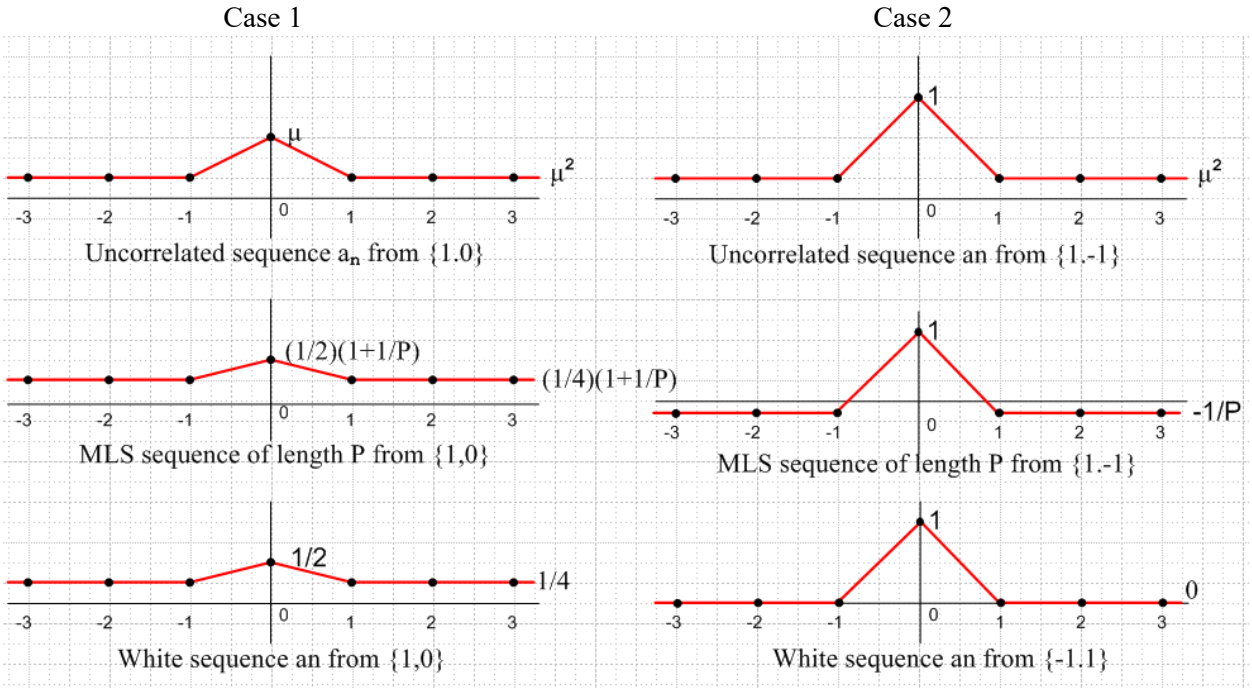


Fig 2.10