Save old section 2.5 (h) PhL 3.26.05

I did this before I learned there was a much simpler expression for the MLS spectrum.
So basically I wasted a lot of Maple time here, but fine. No need to review, just keep for a while.

(h) Plots of MLS spectra for symbos in {0,1}

For a square pulse shape of unit amplitude and width T_1 we have (see FT Sec 9 with $\tau = T_1$ and $A = 1$)

$x_{pulse}(t) = [\ \theta(t + T_1/2) - \theta(t - T_1/2)\ ]$ // time domain FT (9.1)

$X_{pulse}(\omega) = T_1\ \mathrm{sinc}(\omega T_1/2)$. // Fourier Integral Transform FT (9.2)

$$\mathcal{P}_{pulse}(\omega) \equiv \frac{|X_{pulse}(\omega)|^2}{2\pi T_1} = (1/\omega_1)\ \mathrm{sinc}^2(\omega T_1/2) \qquad // \ \omega_1 \equiv 2\pi/T_1 \qquad \text{FT (34.4)}$$

Then (2.5.25) becomes

$$\mathcal{P}(\omega) = \mathrm{sinc}^2(\omega T_1/2) \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P)\ \frac{1}{P}\ (1/4)(1 + 1/P)\ [1\ + 2\pi\ \delta_6(\omega T_1, \frac{P-1}{2})\ ]$$

$$= \frac{1}{P}\ (1/4)(1 + 1/P) \sum_{m=-\infty}^{\infty} \delta(\omega - m[\omega_1/P])\ [\mathrm{sinc}^2(\omega T_1/2)\ +\ \mathrm{sinc}^2(\omega T_1/2)\ 2\pi\ \delta_6(\omega T_1, \frac{P-1}{2})\ ]$$

where

$$2\pi\ \delta_6(\omega T_1, \frac{P-1}{2}) = \frac{1}{P}\ \frac{\sin^2(P\omega T_1/2)}{\sin^2(\omega T_1/2)}$$

The function $2\pi\ \delta_6$ has zeros when $P\omega T_1/2 = N\pi$, or when $\omega = N(\omega_1/P)$, for $N = \pm1, \pm2, \ldots$. Thus, the second term above vanishes except for the $m = 0$ term in the sum over m, where $2\pi\ \delta_6 = P$. Thus we can simplify to get

$$\mathcal{P}(\omega) = (1/4)(1 + 1/P)\ \{\ \sum_{m=-\infty}^{\infty} \delta(\omega - m[\omega_1/P])\ \frac{1}{P}\ \mathrm{sinc}^2(\omega T_1/2)\ +\ 1\ \}$$

Let $K \equiv \frac{1}{P}\ (1/4)(1 + 1/P)$ and rewrite as (we do this so plots have compatible scales)

$$\mathcal{P}(\omega) = \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P)\ (1/K)\{K\mathrm{sinc}^2(\omega T_1/2)\}\ \{K\ [1\ + 2\pi\ \delta_6(\omega T_1, \frac{P-1}{2})\ ]\}.$$

$$= \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P)\ (1/K)\{f_1(\omega)\}\ \{f_2(\omega)\ ]\} = \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_1 m/P) f_3(\omega)$$

What does this line spectrum look like? The half width of the central peak of $\delta_6(k,N)$ is $\Delta k = \pi/N$ which in our case says $T_1\Delta\omega = 2\pi/(P-1)$ or $\Delta\omega = \omega_1/(P-1)$. The half width of the sinc function is $(\Delta\omega)' = 2\pi/T_1 = \omega_1$. Finally, the spacing between the lines of integer m is $(\Delta\omega)'' = \omega_1/P$. Thus,

$\Delta\omega = \omega_1/(P-1)$	half width of central peak of δ_6
$(\Delta\omega)' = \omega_1$	half width of sinc function
$(\Delta\omega)'' = \omega_1/P$	spacing between lines for integer m values

What this says is that there will be a strong central line for m = 0, but all other lines including m = 1 fall outside the central peak of δ_6 and thus only the "1" term in $[1 + 2\pi\ \delta_6]$ contributes to the spectrum. Eventually the sinc^2 function kills off all the lines. The following Maple code is useful to display the spectrum. We first tell; Maple about the δ_6 function

```
d6 := (k,N) -> (2*N+1)^(-1)*sin((N+1/2)*k)^2/ (2*Pi*sin(k/2)^2);
```

$$d6 := (k, N) \to \frac{1}{2}\frac{\sin\left(\left(N+\frac{1}{2}\right)k\right)^2}{(2N+1)\,\pi\,\sin\left(\frac{1}{2}k\right)^2}$$

Next we enter the functions which appear in $\mathcal{P}(\omega)$ above,

```
f1 := K*sinc(omega*T1/2)^2;
```

$$f1 := K\,\text{sinc}\left(\frac{1}{2}\omega\,T1\right)^2$$

```
f2 := K*(1 + 2*Pi*d6(omega*T1,(P-1)/2));
```

$$f2 := K\left(1+\frac{\sin\left(\frac{1}{2}P\,\omega\,T1\right)^2}{P\,\sin\left(\frac{1}{2}\omega\,T1\right)^2}\right)$$

```
f3 := f1*f2/K;
```

$$f3 := K\,\text{sinc}\left(\frac{1}{2}\omega\,T1\right)^2\left(1+\frac{\sin\left(\frac{1}{2}P\,\omega\,T1\right)^2}{P\,\sin\left(\frac{1}{2}\omega\,T1\right)^2}\right)$$

For plotting purposes, we set $T_1 = 2\pi$ to get $\omega_1 = 1$, and we enter a value for P,

```
P := 7; T1 := 2*Pi: omega1 := 2*Pi/T1; K := (1/P)*(1/4)(1+1/P);
```

$$P := 7$$

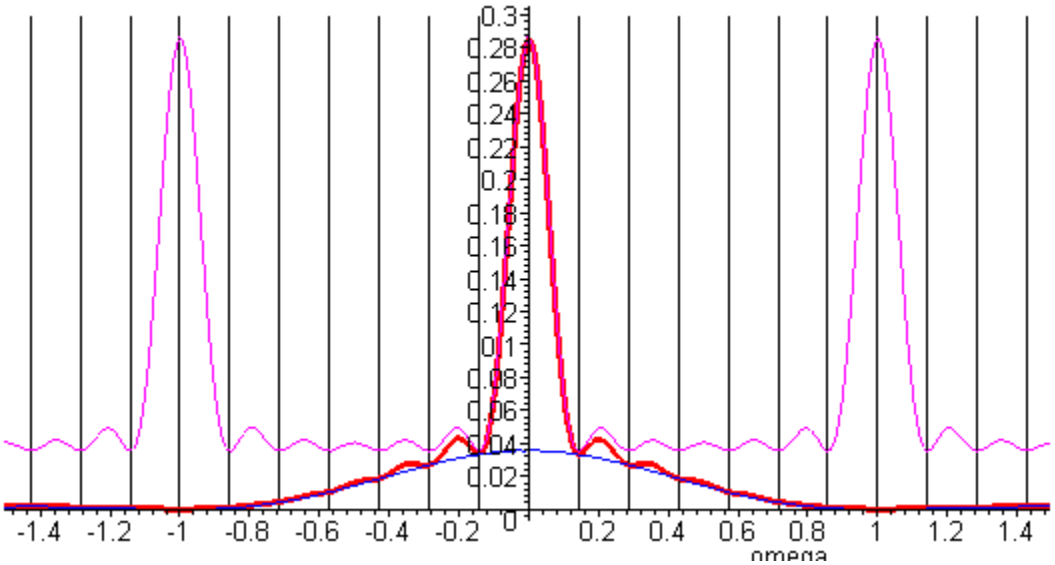
$$\omega 1 := 1$$

$$K := \frac{1}{28}$$

We next plot the three functions $f_i(\omega)$ and a set of vertical lines at $\omega_m = m(\omega_1/P)$ which are the delta function line frequencies,

```
a := 1.5:
p1 := plot(f1,omega = -a..a,color = blue):
p2 := plot(f2,omega = -a..a,color = magenta):
p3 := plot(f3,omega = -a..a, color = red, thickness=3):
p4 := PLOT(seq(CURVES([[N/P,0],[N/P,0.3]]),N=-10..10)):
display(p1,p2,p3,p4);
```

Here are the resulting plots

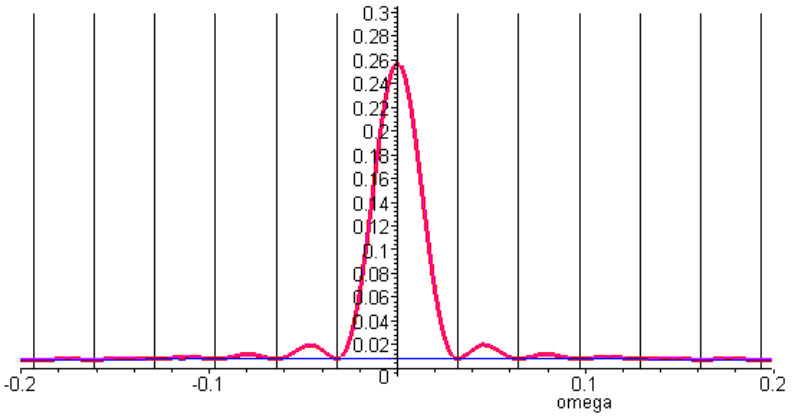


The red curve is $f_3(\omega)$ and this provides the "envelope" function for $\mathcal{P}(\omega)$. The lines are at $\omega_m = m(\omega_1/P)$ which in this case is $\omega_m = m/7$. The delta function amplitudes in $\mathcal{P}(\omega)$ are the points of intersection of the vertical lines with the red curve. The line amplitudes for m = 0 to 10 are given by

```
for m from 0 to 10 do
  omega := omega1*(m+.00000001)/P;
  x[m] := evalf(f3);
od:
for m from 0 to 10 do
  printf("x[%d] = %f\n",m,x[m]);
od;
```

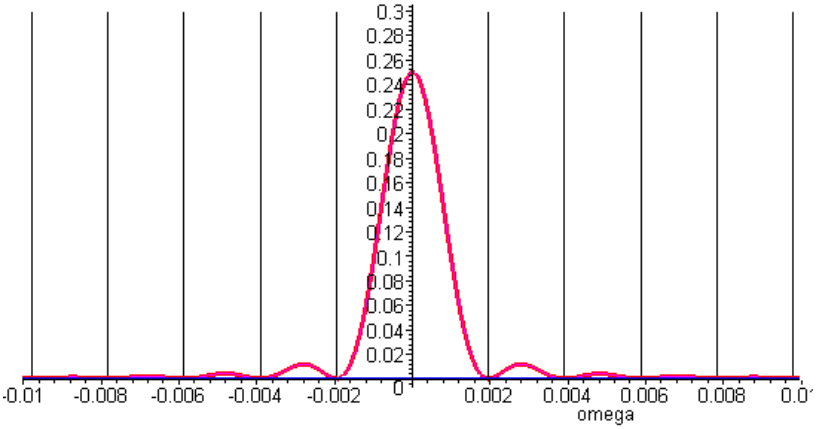
```
x[0] = .285714
x[1] = .033380
x[2] = .027096
x[3] = .018726
x[4] = .010533
x[5] = .004335
x[6] = .000927
x[7] = .000000
x[8] = .000522
x[9] = .001338
x[10] = .001685
```

For larger values of P, the predominance of the central peak become stronger. For P = 31,




```
x[0]  = .258065
x[1]  = .008037
x[2]  = .007955
x[3]  = .007819
x[4]  = .007632
x[5]  = .007398
x[6]  = .007118
x[7]  = .006799
x[8]  = .006445
x[9]  = .006062
x[10] = .005655
```

Now the sinc curve is nearly flat in the region of interest. For P = 511 the spectrum settles in to the expected "white sequence" pattern,




```
x[0]  = .250489
x[1]  = .000489
x[2]  = .000489
x[3]  = .000489
x[4]  = .000489
x[5]  = .000489
x[6]  = .000489
x[7]  = .000489
x[8]  = .000489
x[9]  = .000489
x[10] = .000489
```

It is useful now to compare the above spectra to the spectrum of a "white sequence". This is shown in FT to be

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathbf{pulse}}(\omega) \{ (\beta-\alpha) + \alpha \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m) \} \qquad \text{FT (35.11)}$$

where, from (2.5.34) for Case 1, we have $\alpha = 1/4$, $\beta = 1/2$, and $(\beta-\alpha) = 1/4$ so the above becomes

$$\langle \mathcal{P}(\omega) \rangle = \mathcal{P}_{\mathbf{pulse}}(\omega)\, (1/4) \{ 1 + \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m) \}$$

Installing our same square wave pulse with $\mathcal{P}_{\mathbf{pulse}}(\omega) = (1/\omega_1)\, \mathrm{sinc}^2(\omega T_1/2)$ we find

$$\langle \mathcal{P}(\omega) \rangle = (1/\omega_1)\, \mathrm{sinc}^2(\omega T_1/2)\, (1/4) \left[1 + \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1) \right]$$

$$= (1/\omega_1)\, (1/4) \left[\mathrm{sinc}^2(\omega T_1/2) + \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1)\, \mathrm{sinc}^2(m\pi) \right]$$

$$= (1/\omega_1)\,(1/4)\,[\mathrm{sinc}^2(\omega T_1/2) + \omega_1\,\delta(\omega)\,]$$

$$= (1/4)\,[(1/\omega_1)\,\mathrm{sinc}^2(\omega T_1/2) + \delta(\omega)\,]$$

We now have an m = 0 line with amplitude (1/4) = .2500000 and the rest is a continuous spectrum which we now plot assuming, as in earlier plots, $T_1 = 2\pi$ and $\omega_1 = 1$.