This is all installed now in FT so not longer needed in scrambler.

(j) The Wiener-Khintchine pathway to the power spectrum.

In FT Section 32 (c) we derive the continuum version of the W-K theorem as follows:

$$r_{\mathbf{x}}(t) = \int_{-\infty}^{\infty} dt''x(t - t'') x(-t'') \ . \qquad \text{// energy units} \qquad \text{FT (32.7)}$$

$$a(t) = \int_{-\infty}^{\infty} dt'' \ b(t-t'') \ c(t'') \qquad \Leftrightarrow \qquad A(\omega) = B(\omega) \ C(\omega) \ . \qquad \text{FT (3.6)}$$

$$b(t) = x(t) \quad \leftrightarrow \quad B(\omega) = X(\omega)$$
$$c(t) = x(-t) \quad \leftrightarrow \quad C(\omega) = X(-\omega) = X(\omega)^* \qquad \text{// from (7.1) and (7.2)}$$

Thus, the diagonalized frequency domain form $A(\omega) = B(\omega) \ C(\omega)$ is

$$R_{\mathbf{x}}(\omega) = |X(\omega)|^2 \ . \qquad \text{FT (32.8)}$$

Since

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{T} = \frac{|X(\omega)|^2}{2\pi T} \qquad \text{FT (33.23)}$$

where T is the duration of a pulse train, we find that

$$\mathcal{P}(\omega) = 2\pi T \ R_{\mathbf{x}}(\omega) \ . \qquad \text{FT (34. 3)}$$

This says that the spectral power density of a pulse train can be obtained by computing the Fourier Integral Transform of the autocorrelation function and multiplying by $2\pi T$. We can regard this as the Wiener-Khintchine (W-K) theorem, although strictly the W-K theorem is that $R_{\mathbf{x}}(\omega) = |X(\omega)|^2$.

Although this was not addressed in FT, one can obtain a similar W-K theorem for a discrete sequence $\{y_{\mathbf{k}}\}$. In (2.5.33) we define the autocorrelation sequence as

$$r_{\mathbf{s}} \equiv (1/P) \sum_{n=1}^{P} y_{\mathbf{n}} \, y_{\mathbf{n+s}} \ . \qquad \text{s = any integer} \qquad \textit{(2.5.33)}$$

If we regulate our infinite sequence as being (2N+1) copies of the P-symbol subsequence (where N is very large), then we can write the above as

$$r_s \equiv (1/P) \sum_{n=1}^{P} y_n y_{n+s} \approx \frac{1}{P(2N+1)} \sum_{m=-\infty}^{\infty} y_m y_{m+s} = \frac{T_1}{T} \sum_{m=-\infty}^{\infty} y_m y_{m+s} .$$

We continue to represent the infinite duration of the pulse train by the symbol T, knowing that it is going to cancel another T later on, so we write the limit as

$$r_s = \frac{T_1}{T} \sum_{m=-\infty}^{\infty} y_m y_{m+s} = \frac{T_1}{T} \sum_{m=-\infty}^{\infty} y_{-m} y_{-m+s}$$

where for the last expression we took m→-m. We can then construct a Z Transform version of the W-K theorem as follows:

$$r_s = \sum_{m=-\infty}^{\infty} \frac{T_1}{T} y_{s-m} y_{-m}$$

$$a_s = \sum_{m=-\infty}^{\infty} \Delta t \; b_{s-m} c_m \qquad \Leftrightarrow \qquad A''(z) = \Delta t \, B''(z) \, C''(z) \qquad \text{FT (24.5)}$$

$a_s = r_s \quad \leftrightarrow \quad A''(z) = R''(z)$
$b_k = y_k \quad \leftrightarrow \quad B''(z) = Y''(z)$
$c_k = y_{-k} \quad \leftrightarrow \quad C''(z) = Y''(z^{-1}) = Y''(z)^*$ // if the c_k are real
$\Delta t \quad \leftrightarrow \quad (T_1/T)$

Thus, the diagonalized z-domain form $A''(z) = \Delta t \, B''(z) \, C''(z)$ is

$$R''(z) = (T_1/T) \, Y''(z) \, Y''(z)^* = (T_1/T) \, | Y''(z)|^2$$

It is shown at the start of FT Appendix F that

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \, (T_1/T) \, | Y''(z) |^2$$

and thus we may write

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \, R''(z)$$

This is our Z-transform version of the Wiener-Khintchine theorem. We can now use this compute the power spectrum of a sequence composed of repeated P-length subsequences. Consider:

$r_s = \langle a_n^2 \rangle \quad s = NP \quad = \beta$
$r_s = \langle a_n a_{n+s} \rangle \quad s \neq NP \quad = \alpha$ (2.5.35)

The Z transform of this autocorrelation sequence is:

$$R''(z) = \Sigma_{s=-\infty}^{\infty} r_s\ z^{-s} = \beta \Sigma_{s=NP} z^{-s} + \alpha \Sigma_{s\neq NP} z^{-s}$$

The first sum is over s = 0, ±P, ±2P, etc. Let s = rP so then, using $z = e^{i\omega\Delta t}$,

$$\Sigma_{s=NP} z^{-s} = \Sigma_{r=-\infty}^{\infty} z^{-Pr} = \sum_{n=-\infty}^{\infty} (e^{i\omega\Delta t})^{-Pn} = \sum_{n=-\infty}^{\infty} e^{-Pi\omega\Delta tn}$$

Then using FT (13.2) with k = -PΔt

$$\sum_{n=-\infty}^{\infty} e^{ink} = \sum_{m=-\infty}^{\infty} 2\pi\delta(k - 2\pi m) \qquad -\infty < k < \infty \quad . \qquad \text{FT (13.2)}$$

and setting k = -ωPΔt we get

$$\sum_{n=-\infty}^{\infty} e^{-Pi\omega\Delta tn} = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega P\Delta t - 2\pi m)$$

and so

$$\Sigma_{s=NP} z^{-s} = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega P\Delta t - 2\pi m)$$

As for the second sum,

$$\Sigma_{s\neq NP} z^{-s} = \sum_{n=-\infty}^{\infty} z^{-n} - \sum_{n=-\infty}^{\infty} z^{-Pn} = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega\Delta t - 2\pi m) - \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega P\Delta t - 2\pi m)$$

We have then shown that

$$R''(z) = \beta \Sigma_{s=NP} z^{-s} + \alpha \Sigma_{s\neq NP} z^{-s}$$

$$= \beta \left[\sum_{m=-\infty}^{\infty} 2\pi\delta(\omega P\Delta t - 2\pi m) \right] + \alpha \left[\sum_{m=-\infty}^{\infty} 2\pi\delta(\omega\Delta t - 2\pi m) - \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega P\Delta t - 2\pi m) \right]$$

$$= (\beta-\alpha) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega P\Delta t - 2\pi m) \Big] + \alpha \left[\sum_{m=-\infty}^{\infty} 2\pi\delta(\omega\Delta t - 2\pi m) \right.$$

Now from ** we know that $\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\ R''(z)$ so we find that

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \left\{ (\beta-\alpha) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega P\Delta t - 2\pi m) \right] + \alpha \left[\sum_{m=-\infty}^{\infty} 2\pi\delta(\omega\Delta t - 2\pi m) \right\}$$

Now $\Delta t = T_1$ so

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \{ (\beta-\alpha) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega P T_1 - 2\pi m)] + \alpha [\sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \}$$

$$= \mathcal{P}_{pulse}(\omega) \{ (\beta-\alpha)(1/PT_1) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega - 2\pi m/PT_1)] + \alpha (1/T_1) [\sum_{m=-\infty}^{\infty} 2\pi\delta(\omega - 2\pi m/T_1) \}$$

$$= \mathcal{P}_{pulse}(\omega)(1/T_1) \{ (\beta-\alpha)(1/P) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega - m\omega_1/P)] + \alpha [\sum_{m=-\infty}^{\infty} 2\pi\delta(\omega - m\omega_1) \}$$

$$= \mathcal{P}_{pulse}(\omega)\omega_1 \{ (\beta-\alpha)(1/P) \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/P)] + \alpha [\sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1) \}$$