This was something I was going to put inside Scrambler. Not sure how it ended up, but at least lets rename this to be scrambler_WK and file it in the scrambler folder. // Actually, all this stuff got put in a new release of FT, so I didn't need to put it into scrambler.

(j) Using the Z-transform Wiener-Khintchine to compute the MLS Spectrum

Here we show an alternate method for computation of the spectral power density $\mathcal{P}(\omega)$ of an MLS sequence.

Equation FT (32.20) states what is essentially the Z Transform Weiner-Khintchine theorem,

$$\mathcal{P}(\omega) = \mathcal{P}_{\mathtt{pulse}}(\omega)\, R''(z)\ , \qquad \text{FT (32.20)} \qquad (2.5.42)$$

which allows one to compute $\mathcal{P}(\omega)$ from the Z transform R"(z) of the autocorrelation sequence r_s.

We are interested in an infinite pulse train which is periodic with period P and consists of a repeating subsequence of length P. In (2.5.33) we defined the autocorrelation sequence for this subsequence as

$$r_s \equiv (1/P) \sum_{n=1}^{P} y_n\, y_{n+s} = \frac{1}{P} \sum_{n=0}^{P-1} y_n\, y_{n+s} = \langle y_n\, y_{n+s} \rangle \qquad (2.5.33)$$

where $\langle y_n\, y_{n+s} \rangle$ is an expectation value over the P-length subsequence. We can rewrite r_s by averaging over 2M+1 copies of the repeated subsequence,

$$r_s = \frac{1}{(2M+1)P} \sum_{n=-MP}^{MP+(P-1)} y_n\, y_{n+s} \approx \frac{1}{2MP} \sum_{n=-MP}^{MP} y_n\, y_{n+s} \qquad (2.5.xx)$$

where on the right we assume M is very large. Defining large N = MP this can be written

$$r_s \approx \frac{1}{2N} \sum_{n=-N}^{N} y_n\, y_{n+s} \approx \frac{1}{(2N+1)} \sum_{n=-N}^{N} y_n\, y_{n+s}\ . \qquad (2.5.xx)$$

For an infinite sequence we take $N \to \infty$ so that

$$r_s = \lim_{N\to\infty} \left[\frac{1}{(2N+1)} \sum_{n=-N}^{N} y_n\, y_{n+s} \right] = \langle y_n\, y_{n+s} \rangle \qquad (2.5.xx)$$

This last equation is the definition of r_s that appears in FT (32.16) and which applies to any infinite sequence, not just a periodic one.

Our task now is to compute R"(z) for our MLS sequence and then to insert it into (2.5.xx) above to obtain $\mathcal{P}(\omega)$. The staring point is,

$$\begin{aligned} r_s &= \langle a_n^2 \rangle \qquad s = NP \qquad = \beta \\ r_s &= \langle a_n a_{n+s} \rangle \qquad s \neq NP \qquad = \alpha \ . \end{aligned} \tag{2.5.35}$$

Therefore

$$\begin{aligned} R"(z) &= \sum_{n=-\infty}^{\infty} r_n z^{-n} = \beta \sum_{n=NP} z^{-n} + \alpha \sum_{n \neq NP} z^{-n} \\ &= \beta \sum_{n=NP} z^{-n} + \alpha \Big(\sum_{n=-\infty}^{\infty} z^{-n} - \sum_{n=NP} z^{-n} \Big) \\ &= (\beta-\alpha) \sum_{n=NP} z^{-n} + \alpha \sum_{n=-\infty}^{\infty} z^{-n} \ . \end{aligned} \tag{2.5.45}$$

The first sum is over n = 0, ±P, ±2P and so on. We can replace summation index n by index N,

$$\sum_{n=NP} z^{-n} = \sum_{N=-\infty}^{\infty} z^{-NP} = \sum_{n=-\infty}^{\infty} z^{-nP} \ . \tag{2.5.46}$$

From FT (24.1) we know that z lies on the unit circle in the z-plane and is related to ω by

$$z = e^{i\omega T_1} \tag{FT (24.1)}$$

where T_1 is the duration of a pulse of the pulse train. We then have

$$\sum_{n=NP} z^{-n} = \sum_{n=-\infty}^{\infty} (e^{i\omega T_1})^{-nP} = \sum_{n=-\infty}^{\infty} e^{-i\omega T_1 nP} \ . \tag{2.5.47}$$

According to FT (13.2) we know how to compute this sum,

$$\sum_{n=-\infty}^{\infty} e^{ink} = \sum_{m=-\infty}^{\infty} 2\pi\delta(k - 2\pi m) \qquad -\infty < k < \infty \ . \tag{FT (13.2)}$$

so setting $k = -\omega T_1 P$ we get

$$\sum_{n=NP} z^{-n} = \sum_{m=-\infty}^{\infty} 2\pi\delta(-\omega T_1 P - 2\pi m) = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 P + 2\pi m) = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 P - 2\pi m) \tag{2.5.48}$$

where we use $\delta(x) = \delta(-x)$ and in the last step take $m \rightarrow -m$.

Meanwhile, our other sum of interest in (2.5.45) is this one,

$$\sum_{n=-\infty}^{\infty} z^{-n} = \sum_{n=-\infty}^{\infty} (e^{i\omega T_1})^{-n} = \sum_{n=-\infty}^{\infty} e^{-i\omega T_1 n}$$

which is just the previous sum without the P. Thus,

$$\sum_{n=-\infty}^{\infty} z^{-n} = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m) \quad . \tag{2.5.49}$$

Inserting (2.5.49) and (2.5.48) into (2.5.45) gives

$$R''(z) = (\beta-\alpha) \sum_{n=NP} z^{-n} + \alpha \sum_{n=-\infty}^{\infty} z^{-n}$$

$$= (\beta-\alpha) \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 P - 2\pi m) + \alpha \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega T_1 - 2\pi m)$$

$$= (\beta-\alpha)\,(T_1 P)^{-1} \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega - 2\pi m/[T_1 P]) + \alpha (T_1)^{-1} \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega - 2\pi m/T_1)$$

$$= (2\pi/T_1)\ \{\ (\beta-\alpha)\,(1/P) \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m/[T_1 P]) + \alpha \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi m/T_1)\ \}$$

$$= \omega_1\ \{\ (\beta-\alpha)\,(1/P) \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/P) + \alpha \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1)\ \} \tag{2.5.50}$$

and this concludes our calculation of the Z Transform of the autocorrelation sequence r_n. It only remains to install this into the Z Transform Wiener-Khintchine theorem (2.5.44)

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\, R''(z)$$

$$= \mathcal{P}_{pulse}(\omega)\, \omega_1\ \{\ (\beta-\alpha)\,(1/P) \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/P) + \alpha \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1)\ \} \quad . \tag{2.5.51}$$

This is in agreement with (2.5.21) which we stole from FT (F.22b) and upon which we based all our MLS spectral results in sections (h) and (i).

Comment: Our initial computation of the MLS spectrum including the work of FT Appendix F made no use whatsoever of the autocorrelation sequence, much less its Z transform or the Wiener-Khintchine theorem. For an MLS spectrum these concepts just provide an alternate pathway to computing the

spectrum, and of course it is gratifying to see the result come out the same by either method. Deep down, and perhaps not really *that* deep down, both methods are the same method with a different order of calculation, and the intermediate objects have different names or in some cases no names at all.