Linkage between Scrambler paper and Fourier Transform paper PhL 7.16.13

This is now all resolved, I had to clarify meaning of "statistical pulse train" in FT, release new version and so on.

Issue #1. In FT, I talk about a statistical ensemble of general pulse trains. But in Scrambler I talk about a single MLS sequence of length P. In FT the <...> notation refers to a statistical average over an ensemble of I pulse trains. So it is not obvious how I apply FT equations to Scrambler applications.

[Resolution: don't use any FT results for statistical pulse trains, go back earlier as shown below and use results for a single pulse train. Those results certainly apply to pulse train = MLS sequence.]

[But I ended up doing just the opposite of the above "resolution" !]

Issue #2. In FT my general α β formula for P(ω) requires uncorrelated pulse trains, but an MLS pulse train is correlated. [Will soften the requirement in FT.]

Issue #3. What happens with Weiner-Kichinte?

Playing with Issue #1.

[I wrote these notes before Appendix F existed. But doing all this led to Appendix F. I had already conjectured the answer in my Exercises for the Reader comments. But I did not yet have the little shifted ensemble idea.]

So really in Scrambler I am not talking about statistical pulse train, I am talking about some specific MLS sequence associated with some primitive polynomial. I think FT Section 34 for single pulse train would apply.

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{T} = \frac{|X(\omega)|^2}{2\pi T} \qquad (33.23)$$

$$\mathcal{P}_{pulse}(\omega) \equiv \frac{\mathcal{E}_{pulse}(\omega)}{T_1} = \frac{|X_{pulse}(\omega)|^2}{2\pi T_1} \qquad (33.24)$$

$|X(\omega)|^2 = R_x(\omega)$,

In section (b) I could think about the finite pulse train which has 2N+1 elements, an odd number. The MLS sequence has 2^k- 1 elements, also odd (lucky for me). This result I think applies:

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{T} = (1/2\pi T)\,|\,X_{pulse}(\omega)\,|^2 \sum_{n=-N}^{N} \sum_{m=-N}^{N} a_m^*\, a_n\, e^{i\omega(m-n)T_1} \qquad (34.12)$$

where $T = (2N+1)T_1$ $\{a_n\}$ = my desired MLS sequence.

But now we have another confusion. So I want to talk about an isolated MLS sequence surrounded by 0-'s, or do I want to talk about an infinite periodic sequence which is the real MLS sequence?

Let's go with the infinite pulse train and switch back to

$$\mathcal{P}(\omega) = T_1 \mathcal{P}_{pulse}(\omega)\,(1/T) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^* y_n e^{i\omega(m-n)T_1} \tag{34.14}$$

where my a_n just happens to be periodic with period P. Well, I can go on an assume my repeated sequence result is correct and I got

(b) If the repeating sequence is $A_0, A_1 .... A_{M-1}$ show that

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \frac{1}{M^2} \left| \sum_{k=0}^{M-1} A_k e^{-ik\omega T_1} \right|^2 \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/M)\,. \tag{34.30}$$

For now on, let's think of the MLS sequence as $\{a_0, a_1 ..... a_{P-1}\}$, then the above becomes

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \frac{1}{P^2} \left| \sum_{k=0}^{P-1} a_k e^{-ik\omega T_1} \right|^2 \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/P)\,. \tag{34.30}$$

If my derivation of this thing was good, I think it applies exactly to the current situation. But I go on to relate this to the Z-transform!

$$X''_{seq}(z) = \Sigma_{k=0}^{M-1} a_k z^{-k}$$

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \frac{1}{P^2} \left| X''_{seq}(z) \right|^2 \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/P) \tag{34.31}$$

Now earlier in Scrambler, I do a lot with Z transforms, it is my z-space throughout! Do I know anything about the Z transform of an MLS sequence? In Scrambler Chap 2 I have a lot to say about the MLS sequence in the time domain, I call it $\{o_i\}$ there.

solution = $\{o_j\} = \{ o_0, o_1, o_2, ... \}$

where it is assumed this vanishes to the left and goes on forever to the right. Well OK, suppose I have the above sequence. Then I know that

$O(z) = \Sigma_{n=0}^{\infty} o_n z^{-n}$ from (1.1.1)

Now I showed that

$\{o_i\} = \{\mathbf{c},\mathbf{c},\mathbf{c},\mathbf{c} ...\}$ where each c has n components c_0 through c_n (n+1 coefficients).

This means that

$$o_{i+nI} = o_i$$

but I never state it this way. This means you can write

$$O(z) = \Sigma_{n=0}^{\infty} o_n z^{-n} = [\, c_0 + c_1 z^{-1} + \ldots + c_n z^{-n} \,] + z^{-n}[\, c_0 + c_1 z^{-1} + \ldots + c_n z^{-n} \,]$$

$$+ z^{-2n}[\, c_0 + c_1 z^{-1} + \ldots + c_n z^{-n} \,] + \ldots$$

$$= \Sigma_{i=0}^{\infty} z^{-in} [\, \Sigma_{j=0}^{n} c_j z^{-j} \,]$$

Let's call the inside sum C(z) and it does not depend on index I so

$$= C(z) \Sigma_{i=0}^{\infty} z^{-in}$$

$$= C(z) [\, 1 + z^{-n} + (z^{-n})^2 + \ldots. \,] \qquad 1 + x + x^2 \ldots. = 1/(1-x)$$

$$= C(z) \frac{1}{1-z^{-n}} = C(z) \frac{z^n}{z^n - 1}$$

So this then is the Z transform of an MLS sequence! If correct, I need to work that in somewhere.

$$O(z) = C(z) \frac{z^n}{z^n - 1} \qquad C(z) \equiv \Sigma_{j=0}^{n} c_j z^{-j} = c_0 + c_1 z^{-1} + \ldots + c_n z^{-n}$$

$n = 2^k - 1$ = cyclic code length.

I presume that I can now rewrite as

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \frac{1}{P^2} |\, C(z) \,|^2 \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/P) \qquad (34.31)$$

I think this shows some progress. For MLS, the c_i are what I have been calling a_i and then

$$C(z) = \Sigma_{j=0}^{n} a_j z^{-j} \qquad n = 2^k - 1 = P.$$

So I have to square this thing, but I have done that many times before. Looking at Section 34 I think I could write

$$\mathcal{P}(\omega) = T_1 \mathcal{P}_{pulse}(\omega) (1/T) \sum_{n=-N}^{N} \sum_{m=-N}^{N} y_m^* y_n e^{i\omega(m-n)T_1} \qquad (34.14)$$

where I can regard the y_m as a shifted version of the a_m to get things centered at the origin. I think I could just deal with a shifted-MLS sequence and have 2N+1 = P = 2^k -1 so then 2N+1 = 2^k -1

$2N+2 = 2^k$ $\quad$ $N+1 = 2^{k-1}$ $\quad$ $N = 2^{k-1} - 1$

So far everything is exact! I then start doing my "repeated sequences" section ending with (34.31) as shown above. So I am not there yet!

$$C(z) = \Sigma_{j=0}^{P} a_j z^{-j}$$

$$| C(z) |^2 = \Sigma_{j=0}^{P} a_j z^{-j} \Sigma_{i=0}^{P} a_i z^{-i} = \Sigma_{i,j} a_j a_i z^{-(i+j)}$$

This is NOT something I have already computed for MLS. So I think this is the "hard path" to getting an answer. Maybe later when I have perhaps some kind of more concrete realization of the a_i (matrices?) I can try to compute this sum for an MLS sequence. I do know that $a_i = c_i$ are coefficients of a cyclic code word, they are not just pulled out of my proverbial hat.

Playing with Issue #3. the W-K business.

I discuss this in FT Section 32 (c). The idea there was this:

$$r_x(t) = \int_{-\infty}^{\infty} dt" x(t - t") x(-t") \ . \qquad \text{// energy units} \qquad (32.7)$$

$$a(t) = \int_{-\infty}^{\infty} dt" \ b(t-t") c(t") \qquad \Leftrightarrow \qquad A(\omega) = B(\omega) C(\omega) \ . \qquad (3.6)$$

$b(t) = x(t) \quad \leftrightarrow \quad B(\omega) = X(\omega)$
$c(t) = x(-t) \quad \leftrightarrow \quad C(\omega) = X(-\omega) = X(\omega)^*$ $\quad$ // from (7.1) and (7.2)

Thus, the diagonalized frequency domain form $A(\omega) = B(\omega) C(\omega)$ is

$$R_x(\omega) = |X(\omega)|^2 \ . \qquad (32.8)$$

Now let's try to do this for discrete stuff! Write

$$r_k \equiv (1/P) \sum_{n=1}^{P} a_n a_{n+k}$$

Suppose I go ahead and say that $\{a_k\}$ is just my c_k sequence that only exists in the finite range shown. Then with implied theta functions on a_n I can write

$$r_k \equiv (1/P) \sum_{n=-\infty}^{\infty} a_n a_{n+k}$$

Now mimic the next several lines from the continuous case

$$a_n = \sum_{m=-\infty}^{\infty} \Delta t \; b_{n-m} c_m \qquad \Leftrightarrow \qquad A''(z) = \Delta t \, B''(z) C''(z) \qquad \textit{(24.5)}$$

$a_n = r_n \quad \leftrightarrow \quad A''(z) = R''(z)$
$b_k = x_k \quad \leftrightarrow \quad B''(z) = X''(z)$
$c_k = x_{-k} \quad \leftrightarrow \quad C''(z) = X''(z^{-1}) = X''(z)^*$ // if the c_k are real

$R''(z) = \Delta t \, X''(z) \, X''(z)^* = \Delta t \, | X''(z)|^2$

Now set $\Delta t = 1/P$ and find

$R''(z) = (1/P) \, | X''(z)|^2$

So now X''(z) is my C(z) still, so I then have

$| C(z) |^2 = P \, R''(z)$

where R''(z) is the Z transform of the autocorrelation sequence, but a little fuzzy. I had to θ-restrict a_i to make this viable, which is OK since c_i really. So I have a similarly restricted thing from Scrambler,

$$r_k \equiv (1/P) \sum_{n=1}^{P} a_n a_{n+k} \qquad (2.5.2)$$

But I claim to KNOW all about r_k so computing R''(z) ought to be simple.

But do I REALLY know r_k or was that all wrong? I plotted the white sequence r_k but I never plotted anything relating to the MLS thing (yet).

But in (2.6.1) I do show that $\langle a_m a_n \rangle$ is independent of m-n.

$r_{m-n} = \langle a_m a_n \rangle = (1/4)(1+1/P) \equiv Q$

$r_0 = (1/2)(1+1/P) = 2Q$

So I really DO know the autocorrelation sequence and it really is simple. So \

$$R''(z) = \sum_{n=-\infty}^{\infty} r_n z^{-n} = r_0 + \Sigma_{n=-\infty}^{-1} r_n z^{-n} + \Sigma_{n=1}^{\infty} r_n z^{-n}$$

$$= r_0 + \Sigma_{n=1}^{\infty} r_{-n} z^{n} + \Sigma_{n=1}^{\infty} r_n z^{-n} = r_o + \Sigma_{n=1}^{\infty} [r_{-n} z^{n} + r_n z^{-n}]$$

But r_k is symmetric, so this becomes

$$= r_o + \Sigma_{n=1}^{\infty} r_n [z^{n} + z^{-n}]$$

Furthermore, r_n is a constant and we then have

$$R''(z) = 2Q + Q \Sigma_{n=1}^{\infty} [z^{n} + z^{-n}]$$

Now use

$$1 + x + x^2 + ... = 1/(1-x)$$

$$x + x^2 + ... = 1/(1-x) - 1 = x/(1-x)$$

twice to get

$$R''(z) = 2Q + Q \left[\frac{z}{1-z} + \frac{z^{-1}}{1-z^{-1}} \right] = 2Q + Q \left[\frac{z}{1-z} + \frac{1}{z-1} \right] = 2Q + Q \left[\frac{-z}{z-1} + \frac{1}{z-1} \right]$$

$$= 2Q + Q \left[\frac{1-z}{z-1} \right] = 2Q - Q = Q = \text{constant !!!}$$

So barring errors, I get the strange result that

$$R''(z) = Q = (1/4)(1+1/P)$$

And then

$$| C(z) |^2 = P R''(z) = P (1/4)(1+1/P)$$

And then

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \frac{1}{P^2} | C(z) |^2 \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/P) \qquad (34.31)$$

$$= \mathcal{P}_{pulse}(\omega) \frac{1}{P^2} P (1/4)(1+1/P) \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/P)$$

$$= \mathcal{P}_{pulse}(\omega)\ \frac{1}{P}\ (1/4)(1+1/P)\ \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/P)$$

I don't believe it. Why would the 1/P factor be there? In any event, install the square pulse NRZ

$$X_{pulse}(\omega) = (VT_1)\ \mathrm{sinc}(\omega T_1/2)$$

$$\mathcal{P}_{pulse}(\omega) = \frac{|X_{pulse}(\omega)|^2}{2\pi T_1} = (VT_1)^2\ \mathrm{sinc}^2(\omega T_1/2)/(2\pi T_1) = (1/2\pi)\ V^2 T_1\ \mathrm{sinc}^2(\omega T_1/2) \tag{36.1}$$

Then my claimed result is

$$\mathcal{P}(\omega) = (1/2\pi)\ T_1\ \mathrm{sinc}^2(\omega T_1/2)\ \frac{1}{P}\ (1/4)(1+1/P)\ \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/P)$$

So any of the lines vanish?

$$\mathrm{sinc}(m\omega_1/P * T_1/2) = \mathrm{sinc}(m2\pi/T_1 P * T_1/2) = \mathrm{sinc}(m\pi/P)$$

Well m = 0 contributes, and m = P gives 0 so certain lines do vanish. Write as

$$\mathcal{P}(\omega) = (1/2\pi)\ T_1\ \frac{1}{P}\ (1/4)(1+1/P)\ \omega_1 \sum_{m=-\infty}^{\infty} \mathrm{sinc}^2(m\pi/P)\ \delta(\omega - m\omega_1/P)$$

$$= \frac{1}{P}\ (1/4)(1+1/P) \sum_{m=-\infty}^{\infty} \mathrm{sinc}^2(m\pi/P)\ \delta(\omega - m\omega_1/P)$$

The white limit is ill formed

$$= \frac{1}{P}\ (1/4) \sum_{m=-\infty}^{\infty} 1\ \delta(\omega)$$

giving a crummy DC line with no amplitude. So certainly things have gone awry!

This was just a shot in the dark idea. Bad convergence in the z-plane, lots of trouble. Let's go back now to

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\ \frac{1}{P^2}\ |\ C(z)\ |^2\ \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/P) \tag{34.31}$$

$$|\ C(z)\ |^2 = (\Sigma_{j=0}^{P} a_j\ z^{-j})\ (\Sigma_{i=0}^{P} a_i\ z^{-i}) = \Sigma_{i,j}\ a_j a_i\ z^{-(i+j)} = \Sigma_{i,n}\ a_{n-i} a_i\ z^{-n}$$

Again, I am not sure my repeated-sequence formula is correct. Rewrite

$$|\ C(z)\ |^2 / P^2 = (1/P)\ \Sigma_n\ \{\ (1/P)\ \Sigma_i\ a_{n-i} a_i\ \}\ z^{-n}$$

but sum endpoints are not simple. But do it again with implied θ controls, then we get

$$| C(z) |^2 = \Sigma_{j=-\infty}^{\infty} a_j z^{-j} \Sigma_{i=-\infty}^{\infty} a_i z^{-i} = \Sigma_{i,j} a_j a_i z^{-(i+j)} = \Sigma_{i,n} a_{n-i} a_i z^{-n}$$

$$= (1/P) \Sigma_{n=-\infty}^{\infty} \{ (1/P) \Sigma_{i=-\infty}^{\infty} a_{n-i} a_i \} z^{-n}$$

$$= (1/P) \Sigma_{n=-\infty}^{\infty} \langle a_{n-i} a_i \rangle z^{-n}$$

But I showed that

$$\alpha = \langle a_m a_n \rangle = (1/4)(1+1/P) = Q$$

so then I get

$$| C(z) |^2 = (1/P) \Sigma_{n=-\infty}^{\infty} Q z^{-n} = (1/P)Q \Sigma_{n=-\infty}^{\infty} z^{-n}$$

This sum I don't like much! I have lost my θ restrictors! Suppose I put them back in this way

$$\Sigma_{n=-\infty}^{\infty} \langle a_{n-i} a_i \rangle z^{-n}$$

I think I might use the cyclic nature of the a_i here.

$$| C(z) |^2 = (\Sigma_{j=0}^{P} a_j z^{-j}) (\Sigma_{i=0}^{P} a_i z^{-i}) = \Sigma_{j=0}^{P} \Sigma_{i=0}^{P} a_j a_i z^{-(i+j)}$$

Try to use Galois Appendix C:

$$a(x) b(x) = (\Sigma_{i=0}^{A} a_i x^i) (\Sigma_{j=0}^{B} b_j x^j) = \Sigma_{s=0}^{A+B} c_s x^s$$

where

$$c_s = \sum_{j = \max(0,s-A)}^{\min(s, B)} (a_{s-j} \otimes b_j) \quad . \qquad \text{(C.11)}$$

Application would say

$$| C(z) |^2 = \Sigma_{s=0}^{2P} c_s z^{-s} \qquad c_s = \sum_{j = \max(0,s-P)}^{\min(s, P)} a_{s-j} a_j$$

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New idea. Go back to "the product of two shifted...", That is what we have there, it is an MLS and a shifted copy of itself. No ensembles here, just original + one shift. Let's now review carefully what is going on in that section. I claim to calculate the four n_i. I did show back in that an MLS sequence has a

weight of 2^{k-1} for GF(2), and this is exact, not some average. Now back to those n_i. The three sums for pairs of n_i are EXACT for any single MLS sequence. How long is an MLS sequence? It is a code word and code words have n coefficients I thought but as poly c(x) has degree n-1. Where do I even mention cyclic codes in scrambler?? Appendix B ! So yes, there are n coefficients of c(x) and degree is n-1. So we identify P = n = 2^k - 1. Fine. Back again now to the four n_i. I just verified the pair sum rules as exact. The four sum rule is also exact, sum of all cases is P. Thus, the counts IO show for the four n_i in the shifted column apply to an MLS and any one non-trivial shift of itself. You don't need any kind of statistical ensemble here. So the point of this paragraph is that the n_i are exact for MLS + any shift.

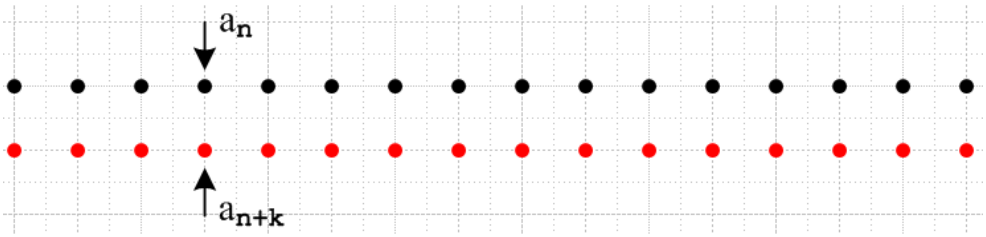
Now jump down below the xpulse drawings again and ponder Case 1. What is the *meaning* of $\langle a_m a_n \rangle$ in this picture? I think this is the Big Question. We do a symbol-wise multiplication of MLS and a single shifted MLS. If we pick some particular a_m in the original MLS, it abuts against some a_{m+k} in that shifted sequence. The product is either 0 or 1. If we do this for all a_m (P of them), what is the average value we get for the product $a_m a_{m+k}$? Out of P terms to consider in the average, only n_4 are 1, so only they count, and that is why I write $\langle a_m a_n \rangle = n_4 / P$. This result is not a statistical thing, it is EXACT for a single MLS and one shifted by k. Clearer maybe to write this as $\langle a_m a_{m+k} \rangle$.

Conclusion: $\langle a_n a_{n+k} \rangle$ defined as this average for a single specific MLS and its k-shifted partner sequence is independent of k. So here we have a case that is uncorrelated, but still this fact is true. I never considered that possibility in FT !!

Now when we do our "average" to get $\langle a_n a_{n+k} \rangle$, we are doing *exactly this*

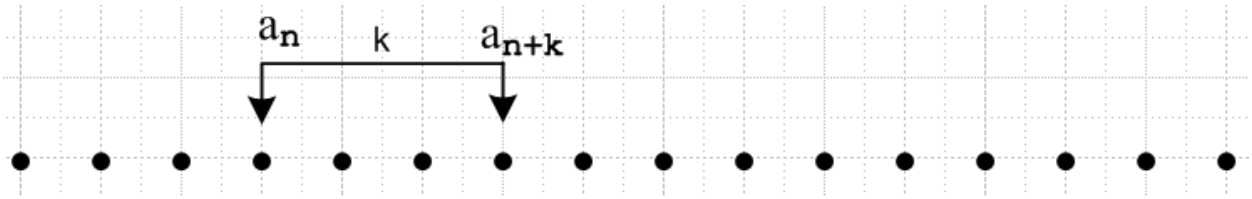
$$\langle a_n a_{n+k} \rangle_1 = (1/P) \sum_{n=1}^{P} a_n a_{n+k} = (1/4)(1+1/P) = \text{constant}$$

I now put subscript 1 to define a particular kind of average. Here is how that average is computed: We have our two sequences abutting, and we are doing this sum over abutting symbols down the entire MLS sequence. Here is a suggestive picture for how this average is computed,]



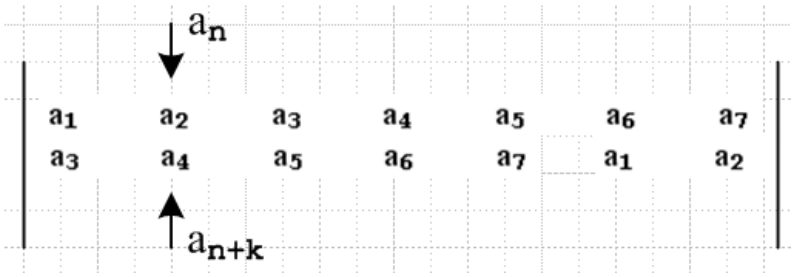
We sum over all P different pairs of arrows by sliding the arrow pair to the right. The red dots on the lower line are shifted k from those of the upper row of dots.

Now I claim we can produce this same average by working just with the black MLS sequence this way



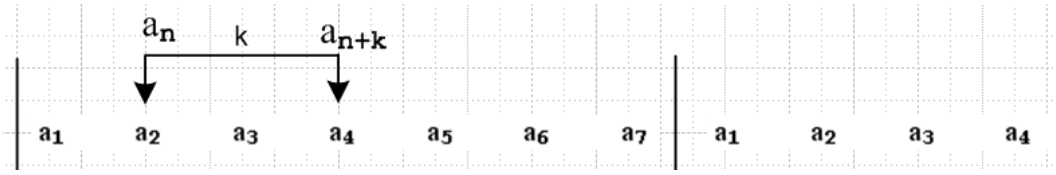
where as now slide the rigidly connected pair of arrows horizontally such that the left arrow covers exactly one period of the MLS sequence. $<a_n a_{n+k}>_1$ is the same computed in either picture.

STOP, I already have better pictures. We start with this (drawn with $P = 2^3-1 = 7$ and $k = 2$)



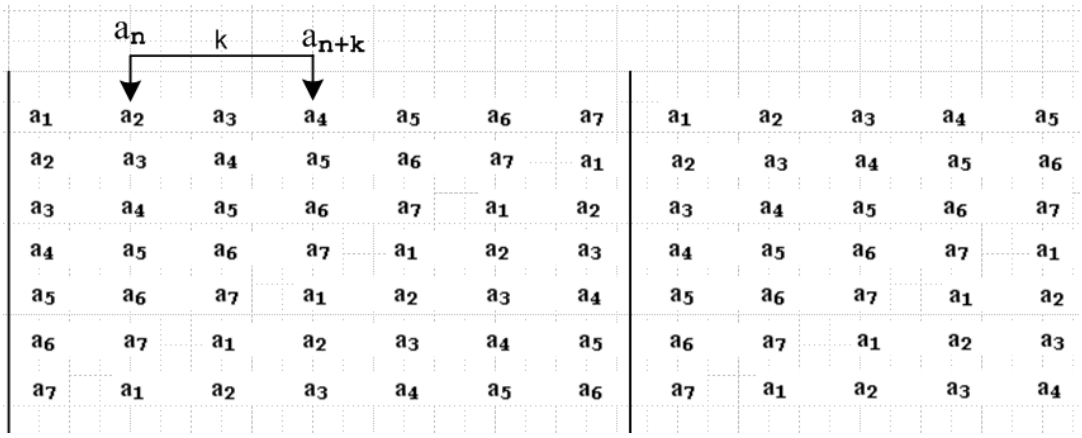
To get the sum, as slide this pair of arrows over the range of P positions.

Now we get the same sum this way



where now we have a single MLS sequence shown, and we have our rigidly connected pair of arrows that slide horizontally on the fixed MLS sequence to generate the same sum!

Now here is a third way. Instead of sliding the arrow pair, we keep it fixed and sum over a set of P different MLS sequences, each shifted one symbol relative to the previous one.



Think of this then as an "ensemble" of P different pulse trains and the positions n and n+k are fixed. The ensemble average over this set of pulse trains would be

$$\langle a_n a_{n+k} \rangle_2 \equiv (1/P)\,(1/P) \sum_{n=1}^{P} a_n a_{n+k}$$

But this is the same average as $\langle a_n a_{n+k} \rangle_1$ shown above!!!

Comments: This is what I wanted. In order to have these two averages be the same, you have to have all P pulse trains in the ensemble, you cannot just have two of them.

Now look at FT for a while. We get down with no assumptions to this point:

$$\mathcal{P}(\omega) \equiv \frac{\mathcal{E}(\omega)}{T} = (1/2\pi T)\,|\,X_{pulse}(\omega)\,|^2 \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} y_m^*\, y_n\, e^{i\omega(m-n)T_1} \qquad (34.12)$$

We now average over the above ensemble to get

$$\langle \mathcal{P}(\omega) \rangle = T_1\, \mathcal{P}_{pulse}(\omega)\,(1/T) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \langle y_m^*\, y_n \rangle\, e^{i\omega(m-n)T_1} \qquad (35.2)$$

But since each pulse train is really the same pulse train just shifted, we know that each pulse train in the ensemble has the same power spectrum, so $\langle \mathcal{P}(\omega) \rangle = \mathcal{P}(\omega)$ and we then have

$$\mathcal{P}(\omega) = T_1\, \mathcal{P}_{pulse}(\omega)\,(1/T) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \langle y_m^*\, y_n \rangle\, e^{i\omega(m-n)T_1}$$

We just showed that $\langle y_m^*\, y_n \rangle$ is independent of m and n since

$$\langle a_n a_{n+k} \rangle_1 = (1/P) \sum_{n=1}^{P} a_n a_{n+k} = (1/4)(1+1/P) = \text{constant} \qquad k \neq 0$$

Thus, we may refer to $\langle y_m^* y_n \rangle$ by the name α and extract it from the double sum. We are able to do this even through the MLS sequence is correlated!

Similarly, we have

$$\langle a_n^2 \rangle_1 = (1/P) \Sigma_{n=1}^{P} a_n^2 = (1/2)(1+1/P) = \text{constant} \qquad k = 0$$

which we call β.

Now we can trace through the FT math from this point and we get to this point:

$$\langle \mathcal{P}(\omega) \rangle = T_1 \mathcal{P}_{pulse}(\omega) (1/T) \{ \alpha \sum_{n=-\infty}^{\infty} \sum_{m \neq n} [e^{i\omega(m-n)T_1}] + \beta \sum_{n=-\infty}^{\infty} \sum_{m=n} [1] \} . \qquad (35.7)$$

We cannot associate these α and β with μ^2 and σ^2 as we do in the uncorrelated case because for example we *don't* have $\alpha = \langle a_n a_m \rangle = \langle a_n \rangle \langle a_m \rangle = \mu\,\mu = \mu^2$. Nevertheless, (35.7) above is correct as it stands with our special α and β coefficients.

Note: This is an excellent justification for my using α and β in the first place in FT!

I then carry through all the math and end up then with this result for the PSD of an MLS sequence.

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \{ (\beta-\alpha) + \alpha \sum_{m=-\infty}^{\infty} 2\pi \delta(\omega T_1 - 2\pi m) \} \qquad (35.11)$$

In the Case 1 situation we have

$$\alpha = (1/4)(1+1/P)$$
$$\beta = (1/2)(1+1/P) = 2\alpha$$
$$(\beta - \alpha) = 2\alpha - \alpha = \alpha$$

so the formula then says

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\, \alpha \{ 1 + \sum_{m=-\infty}^{\infty} 2\pi \delta(\omega T_1 - 2\pi m) \} \qquad (35.11)$$

Now we come along with our NRZ square pulse of with T_1 and amplitude 1

$$X_{pulse}(\omega) = (T_1) \operatorname{sinc}(\omega T_1/2) \qquad (36.1)$$

$$\mathcal{P}_{pulse}(\omega) = \frac{|X_{pulse}(\omega)|^2}{2\pi T_1} = (T_1)^2 \operatorname{sinc}^2(\omega T_1/2)/(2\pi T_1) = (1/2\pi)\, T_1 \operatorname{sinc}^2(\omega T_1/2)$$

and our formula is then

$$\mathcal{P}(\omega) = (1/2\pi)\, T_1\, \mathrm{sinc}^2(\omega T_1/2)\, \alpha\, \{\, 1 + \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m)\, \} \qquad (35.11)$$

The second term here is involves

$$\sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m)\, \mathrm{sinc}^2(\omega T_1/2) = \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m)\, \mathrm{sinc}^2(\pi m) = 2\pi\, \delta(\omega T_1)$$

$$= (2\pi./T_1)\, \delta(\omega)$$

so our result is then

$$\mathcal{P}(\omega) = (1/2\pi)\, T_1\, \alpha\, \{\, \mathrm{sinc}^2(\omega T_1/2) + (2\pi./T_1)\, \delta(\omega)\, \}$$

$$= (1/\omega_1)\, \alpha\, \mathrm{sinc}^2(\omega T_1/2) + \alpha\, \delta(\omega) \qquad \alpha = (1/4)(1+1/P)$$

In the limit $P \to \infty$ this becomes $\alpha = 1/4$ so

$$\mathcal{P}(\omega) = (1/\omega_1)\, (1/4)\, \mathrm{sinc}^2(\omega T_1/2) + (1/4)\, \delta(\omega)$$

I think I need to "reorganize" Sections 2.5 and 2.6 of Scrambler now that I have seen the light.

Question: Does this have any impact on Chapter 3 ? Yes because I am going to say things about the output of a scrambler and its spectrum. But I think I just claim all is white.

Jumping now to Chapter 3: Consider $r_i = g_i i_{n-j}$ where g_i is an MLS sequence. Suppose i_n were a square wave and we are in GF(2). Then r_i is every other bit of an MLS sequence.

Question: the sequence formed from every other bit of a MLS sequence: is it pretty white?

We still have that $\langle a_n a_{n+k} \rangle = 1/4$ for large P. And still have $\langle a_m \rangle = 1/2$. These facts don't change if we omit every other bit. It just limits k in $\langle a_n a_{n+k} \rangle$ to k = even, and .. I could compute this I think, but let's just claim it is true without proof, because this is not my main interest in life!

********** added later **********

Start with

$$\mathcal{P}(\omega) = T_1\, \mathcal{P}_{pulse}(\omega)\, (1/T) \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_m\, a_n\, e^{i\omega(m-n)T_1} \qquad \text{FT (34.14)}$$

But now assume that a_m is periodic. It is easy to show that the above can be written

$$\mathcal{P}(\omega) = T_1\, \mathcal{P}_{pulse}(\omega)\, (1/T) \sum_{N=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} e^{i\omega P(M-N)T_1} \sum_{n=0}^{P-1} \sum_{m=0}^{P-1} a_m a_n e^{i\omega(m-n)T_1}$$

where we defined for example n = NP + n', m = MP +m', then in the end we rename n' and m' to be n and m. The double sum on the right seems more manageable since it involves a finite square summation region, and then the N and M sums handle the infinite number of these squares that paper the infinite plane we started with using the original n and m.

Notice that the N,M double sum is totally "decoupled"

$$\sum_{N=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} e^{i\omega P(M-N)T_1} = \left[\sum_{N=-\infty}^{\infty} e^{-i\omega PNT_1} \right] \left[\sum_{M=-\infty}^{\infty} e^{i\omega PMT_1} \right]$$

This is a very tricky animal but I am an expert at this stuff! Rewrite with I and J'

$$\left[\sum_{I=-\infty}^{\infty} e^{-i\omega PIT_1} \right] \left[\sum_{J=-\infty}^{\infty} e^{i\omega PJT_1} \right]$$

I know this is wrong, but suppose I try

$$\sum_{n=-\infty}^{\infty} e^{ink} = \sum_{m=-\infty}^{\infty} 2\pi\delta(k - 2\pi m) \qquad -\infty < k < \infty \quad . \tag{13.2}$$

Then take n = I, k = ωPT_1 so

$$\sum_{I=-\infty}^{\infty} e^{-i\omega PIT_1} = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega PT_1 - 2\pi m)$$

And then

$$\left[\sum_{I=-\infty}^{\infty} e^{-i\omega IT_1} \right] \left[\sum_{J=-\infty}^{\infty} e^{i\omega JT_1} \right] = \sum_{m=-\infty}^{\infty} 2\pi\delta(\omega PT_1 - 2\pi m) \sum_{n=-\infty}^{\infty} 2\pi\delta(\omega PT_1 - 2\pi n)$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} 2\pi\, \delta(\omega PT_1 - 2\pi m)\, 2\pi\delta(\omega PT_1 - 2\pi n)$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} 2\pi\, \delta(\omega PT_1 - 2\pi m)\, 2\pi\delta(2\pi m - 2\pi n)$$

$$= \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega PT_1 - 2\pi m) \sum_{n=-\infty}^{\infty} 2\pi\delta(2\pi m - 2\pi n)$$

$$= \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega P T_1 - 2\pi m)\; 2\pi\delta(0)$$

$$= [\,2\pi\delta(0)] \sum_{m=-\infty}^{\infty} 2\pi\,\delta(\omega P T_1 - 2\pi m)$$

$$= [\,2\pi\delta(0)] \sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega P T_1 - 2\pi k)$$

Then we end up with

$$\mathcal{P}(\omega) = T_1\,\mathcal{P}_{pulse}(\omega)\,(1/T) \sum_{N=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} e^{i\omega P (M-N) T_1} \sum_{n=0}^{P-1} \sum_{m=0}^{P-1} a_m\, a_n\, e^{i\omega (m-n) T_1}$$

$$= T_1\,\mathcal{P}_{pulse}(\omega)\,(1/T)\,[\,2\pi\delta(0)] \sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega P T_1 - 2\pi k) \sum_{n=0}^{P-1} \sum_{m=0}^{P-1} a_m\, a_n\, e^{i\omega (m-n) T_1}$$

And then I cancel T against $2\pi\delta(0)$ blindly to get

$$= T_1\,\mathcal{P}_{pulse}(\omega)\,[\sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega P T_1 - 2\pi k)\,]\,[\sum_{n=0}^{P-1} \sum_{m=0}^{P-1} a_m\, a_n\, e^{i\omega (m-n) T_1}\,]$$

$$= T_1\,\mathcal{P}_{pulse}(\omega)\,(1/P)\,[\sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega T_1 - 2\pi k/P)\,]\,[\sum_{n=0}^{P-1} \sum_{m=0}^{P-1} a_m\, a_n\, e^{i\omega (m-n) T_1}\,]$$

$$= T_1\,\mathcal{P}_{pulse}(\omega)\,(1/P)\,[\sum_{n=0}^{P-1} \sum_{m=0}^{P-1} a_m\, a_n\, e^{i\omega (m-n) T_1}\,]\,[\sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega T_1 - 2\pi k/P)\,]$$

$$= \mathcal{P}_{pulse}(\omega)\,(1/P)\,[\sum_{n=0}^{P-1} \sum_{m=0}^{P-1} a_m\, a_n\, e^{i\omega (m-n) T_1}\,]\,[\sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega - \omega_1 k/P)\,]$$

Now compare this with my conjectured result where M = P.

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega)\,\frac{1}{M^2}\,\Big|\sum_{k=0}^{M-1} A_k e^{-ik\omega T_1}\Big|^2\;\omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/M)\,. \qquad (34.30)$$

The only thing wrong is a combinatoric 1/P factor.

Go back and be practical and think what Maple would do

$$\mathcal{P}(\omega) = T_1\, \mathcal{P}_{pulse}(\omega)\, (1/T) \sum_{I=-\infty}^{\infty} \sum_{J=-\infty}^{\infty} e^{i\omega P(I-J)T_1} \sum_{n=0}^{P-1} \sum_{m=0}^{P-1} a_m\, a_n\, e^{i\omega(m-n)T_1}$$

I would limit my number of multiples of the MLS sequence to 2N+1 copies, so then

$$\mathcal{P}(\omega) = T_1\, \mathcal{P}_{pulse}(\omega)\, (1/T) \sum_{I=-N}^{N} \sum_{J=-N}^{N} e^{i\omega P(I-J)T_1} \sum_{n=0}^{P-1} \sum_{m=0}^{P-1} a_m\, a_n\, e^{i\omega(m-n)T_1}$$

$$T = (2N+1)PT_1$$

Then our sum of interest is $\sum_{I=-N}^{N} e^{i\omega PIT_1}$. Then use

$$\sum_{n=-N}^{N} e^{ink} = 2\pi \left\{ \frac{\sin[(N+1/2)k]}{2\pi \sin(k/2)} \right\} \equiv 2\pi\, \delta_5(k,N) \quad . \qquad -\infty < k < \infty \qquad (13.3)$$

so that

$$\sum_{I=-N}^{N} e^{i\omega PIT_1} = 2\pi\, \delta_5(\omega PT_1, N)$$

And then

$$\sum_{I=-N}^{N} \sum_{J=-N}^{N} e^{i\omega P(I-J)T_1} = [2\pi\, \delta_5(\omega PT_1, N)]^2$$

Then use

$$\delta_6(k,N) \equiv \frac{1}{2\pi} \frac{[2\pi\delta_5(k,N)]^2}{(2N+1)} = \frac{1}{2N+1} \frac{\sin^2[(N+1/2)k]}{2\pi \sin^2(k/2)} \quad . \qquad (A.20)$$

to write

$$[2\pi\, \delta_5(\omega PT_1, N)]^2 = (2N+1)\, 2\pi\delta_6(\omega PT_1, N)$$

Then we have

$$(1/T) \sum_{I=-N}^{N} \sum_{J=-N}^{N} e^{i\omega P(I-J)T_1} = (1/T)\, (2N+1)\, 2\pi\delta_6(\omega PT_1, N)$$

But $T = (2N+1)PT_1$ so then

$$(1/T) \sum_{I=-N}^{N} \sum_{J=-N}^{N} e^{i\omega P(I-J)T_1} = (1/PT_1)\, 2\pi\delta_6(\omega PT_1, N)$$

Then take $N \to \infty$ and make use of,

$$\lim_{N\to\infty} \delta_6(k,N) = \lim_{N\to\infty} \frac{\sin^2[(N+1/2)k]}{(2N+1)2\pi\,(k/2)^2} = \sum_{m=-\infty}^{\infty} \delta(k-2\pi m) \qquad (A.21)$$

so then

$$(1/T) \sum_{I=-N}^{N} \sum_{J=-N}^{N} e^{i\omega P(I-J)T_1} = (1/PT_1)\, 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega PT_1 - 2\pi m)$$

and our final result is then

$$\mathcal{P}(\omega) = T_1\, \mathcal{P}_{pulse}(\omega)\, (1/T) \sum_{I=-N}^{N} \sum_{J=-N}^{N} e^{i\omega P(I-J)T_1} \sum_{n=0}^{P-1} \sum_{m=0}^{P-1} a_m a_n e^{i\omega(m-n)T_1}$$

$$= T_1\, \mathcal{P}_{pulse}(\omega)\, (1/PT_1)\, 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega PT_1 - 2\pi m) \sum_{n=0}^{P-1} \sum_{m=0}^{P-1} a_m a_n e^{i\omega(m-n)T_1}$$

$$= \mathcal{P}_{pulse}(\omega)\, (1/P) \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega PT_1 - 2\pi m) \sum_{n=0}^{P-1} \sum_{m=0}^{P-1} a_m a_n e^{i\omega(m-n)T_1}$$

$$= \mathcal{P}_{pulse}(\omega)\, (1/P^2) \sum_{m=-\infty}^{\infty} 2\pi\, \delta(\omega T_1 - 2\pi m/P) \sum_{n=0}^{P-1} \sum_{m=0}^{P-1} a_m a_n e^{i\omega(m-n)T_1}$$

$$= \mathcal{P}_{pulse}(\omega)\, (1/P^2) \sum_{m=-\infty}^{\infty} 2\pi/T_1\, \delta(\omega - \omega_1 m/P) \sum_{n=0}^{P-1} \sum_{m=0}^{P-1} a_m a_n e^{i\omega(m-n)T_1}$$

which compare to

$$\mathcal{P}(\omega) = \mathcal{P}_{pulse}(\omega) \frac{1}{M^2} \left| \sum_{k=0}^{M-1} A_k e^{-ik\omega T_1} \right|^2 \omega_1 \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_1/M)\,. \qquad (34.30)$$

and finally I have this result.

I think I will just add Appendix F to FT which has this exact derivation.