A little math and maple problem PhL 6.19.13

This relates do stuff after (1.2.7) and then later after (1.2.9).

I want Maple to perform the infinite polynomial division in line with my examples here from scrambler doc

$$
\begin{array}{l}
z+1 \mid z^2+1 \\
\quad \underline{z^2+z} \\
\qquad -z+1 \\
\qquad \underline{-z-1} \\
\qquad\quad 2
\end{array}
\qquad \Rightarrow \qquad \frac{z^2+1}{z+1} = (z-1) + \frac{2}{z+1} \qquad \text{// the usual stopping point}
$$

$$
\begin{array}{l}
\quad \underline{z-1+2z^{-1}} \\
z+1 \mid z^2+1 \\
\quad \underline{z^2+z} \\
\qquad -z+1 \\
\qquad \underline{-z-1} \\
\qquad\quad 2 \\
\qquad\quad \underline{2+2z^{-1}} \\
\qquad\qquad -2z^{-1}
\end{array}
\qquad \Rightarrow \qquad \frac{z^2+1}{z+1} = (z-1+2z^{-1}) - \frac{2z^{-1}}{z+1}
$$

$$
\begin{array}{l}
\quad \underline{z-1+2z^{-1}-2z^{-2}} \\
z+1 \mid z^2+1 \\
\quad \underline{z^2+z} \\
\qquad -z+1 \\
\qquad \underline{-z-1} \\
\qquad\quad 2 \\
\qquad\quad \underline{2+2z^{-1}} \\
\qquad\qquad -2z^{-1} \\
\qquad\qquad \underline{-2z^{-1}-2z^{-2}} \\
\qquad\qquad\quad 2z^{-2}
\end{array}
\qquad \Rightarrow \qquad \frac{z^2+1}{z+1} = (z-1+2z^{-1}-2z^{-2}) + \frac{2z^{-2}}{z+1}
$$

I conjecture (with Maple proof later) that the powers just alternate forever. Then we have

$$\frac{z^2+1}{z+1} = (z-1+2z^{-1}-2z^{-2}+\ldots..) = (z-1) + 2\,(z^{-1}-z^{-2}+z^{-3}-z^{-4}+\ldots..)$$

$$= (z-1) + 2z^{-1}\,[1+(-z)^{-1}+(-z)^{-2}+(-z)^{-3}+\ldots..] = (z-1) + 2z^{-1}\ [\,1/(1+z^{-1})\,]$$

$=(z-1) + 2\ [\ 1/(z+1)\ ] \quad = [\ z^2 -1 + 2\ ]/(z+1) = \dfrac{z^2+1}{z+1}$ so it works!

****

Consider this arbitrary polynomial division with positive and negative powers:

$$O(z) = I(z) / H(z) \quad . \tag{1.2.5}$$

In the time domain we can write this as

$$i_n = \sum_{j=-\infty}^{\infty} h_j\, o_{n+j} \quad . \tag{1.9.5}$$

The Maple problem is to find the coefficients o_j. Let's start with the simple example above

$I(z) = z^2 + 1$ highest power is z^2 with coefficient i_{-2}
$H(z) = z + 1$ highest power is z with coefficient h_1

Recall this from scrambler, where in our current example k = 1, max power of h(z):

$$[o_k z^{-k} + o_{k+1} z^{-k-1} + o_{k+2} z^{-k-2} + \ .....]$$

$$= [i_0 + i_1 z^{-1} + i_2 z^{-2} + ... + i_r z^{-r}] / [h_k z^k + h_{k-1} z^{k-1} + ... + h_1 z + h_0] \ .$$

If we back up 2 clocks so the first i_j is i_{-2}, then the first o_j will not be o_1, it will be o_{-1}. So

$$i_n = \sum_{j=0}^{1} h_j\, o_{n+j} = h_0 o_n + h_1 o_{n+1} \tag{1.9.5}$$

$I(z) = i_{-2} z^2 + i_{-1} z + i_0 + i_1 z^{-1} + i_2 z^{-2} + ... + i_r z^{-r}$. $i_{-2} = 1$ and $i_0 = 1$

$$O(z) = o_{-1} z + o_0 + o_1 z^{-1} + o_2 z^{-2} + o_3 z^{-3} + \ .....$$

This of course matches the long divisions shown above.

This seems to be an iteration of this form

$$h_1 o_{n+1} = - h_0\ o_n + i_n \qquad (*)$$

Where do we want to start this thing? Maybe here with n = -2 :

$h_1 o_{-1} = - h_0\ o_{-2} + i_{-2} = i_{-2} = 1$ which is correct since it says $o_{-1} = 1/h_1 = 1$ in my example.

If we go to large negative n, we have both o_j and $i_j = 0$ in that regime. Then 0 = 0 + 0 at each clock in the past.

$h_1 o_{-4} = - h_0\ o_{-5} + i_{-5}$ $\quad 0 = 0+0$
$h_1 o_{-3} = - h_0\ o_{-4} + i_{-4}$ $\quad 0 = 0+0$
$h_1 o_{-2} = - h_0\ o_{-3} + i_{-3}$ $\quad 0 = 0+0$
$h_1 o_{-1} = - h_0\ o_{-2} + i_{-2}$ $\quad$ says $h_1 o_{-1} = i_{-2}$

****

This seems to be an iteration of this form

$$h_1 o_{n+1} = - h_0\ o_n + i_n$$

$$o_{n+1} = - (h_0/h_1)\ o_n + (1/h_1)\ i_n$$

I know that $o_0 = 1$ so we can take it from there:

$o_1 = - (h_0/h_1)\ o_0 + (1/h_1)\ i_0$
$o_2 = - (h_0/h_1)\ o_1 + (1/h_1)\ i_1 = - (h_0/h_1)\ [- (h_0/h_1)\ o_0 + (1/h_1)\ i_0] + (1/h_1)\ i_1$
$o_3 = - (h_0/h_1)\ o_2 + (1/h_1)\ i_2 = - (h_0/h_1) [- (h_0/h_1)\ [- (h_0/h_1)\ o_0 + (1/h_1)\ i_0] + (1/h_1)\ i_1] + (1/h_1)\ i_2$

OK, lets make some abbreviations for our starting equation

$$o_{n+1} = - (h_0/h_1)\ o_n + (1/h_1)\ i_n$$

$$o_{n+1} = - a\ o_n + b\ i_n$$

Now have Maple just solve this baby. Here it is, and I have now added this as a second rsolve example in the Maple user guide. First, here is the general method of solution and functionalization of the result

```
f := rsolve({o(n+1) = -a*o(n) + b*i[n],o(0) = c}, o);
```

$$f := c\,(-a)^{n} + \left(\sum_{n_0 = 1}^{n} (-a)^{(n - n_0)}\, b\, i_{n_0 - 1} \right)$$

```
f := subs(n[0] = j,f);
```

$$f := c\,(-a)^{n} + \left(\sum_{j = 1}^{n} (-a)^{(n - j)}\, b\, i_{j - 1} \right)$$

```
o := unapply(f,n);
```

$$o := n \rightarrow c\,(-a)^{n} + \left(\sum_{j = 1}^{n} (-a)^{(n - j)}\, b\, i_{j - 1} \right)$$

```
o(4);
```

$$c\,a^{4} + \left(\sum_{j = 1}^{4} (-a)^{(4 - j)}\, b\, i_{j - 1} \right)$$

```
value(%);
```

$$c\,a^{4} - a^{3}\, b\, i_{0} + a^{2}\, b\, i_{1} - a\, b\, i_{2} + b\, i_{3}$$

We know in this example that c = 1. Here then is code to show the first 10 of the o_n coefficients:

```
for n from 0 to 10 do
   value(o(n));
od;
```

$$1$$

$$-\frac{h_0}{h_1}+\frac{i_0}{h_1}$$

$$\frac{{h_0}^2}{{h_1}^2}-\frac{h_0 i_0}{{h_1}^2}+\frac{i_1}{h_1}$$

$$-\frac{{h_0}^3}{{h_1}^3}+\frac{{h_0}^2 i_0}{{h_1}^3}-\frac{h_0 i_1}{{h_1}^2}+\frac{i_2}{h_1}$$

$$\frac{{h_0}^4}{{h_1}^4}-\frac{{h_0}^3 i_0}{{h_1}^4}+\frac{{h_0}^2 i_1}{{h_1}^3}-\frac{h_0 i_2}{{h_1}^2}+\frac{i_3}{h_1}$$

$$-\frac{{h_0}^5}{{h_1}^5}+\frac{{h_0}^4 i_0}{{h_1}^5}-\frac{{h_0}^3 i_1}{{h_1}^4}+\frac{{h_0}^2 i_2}{{h_1}^3}-\frac{h_0 i_3}{{h_1}^2}+\frac{i_4}{h_1}$$

$$\frac{{h_0}^6}{{h_1}^6}-\frac{{h_0}^5 i_0}{{h_1}^6}+\frac{{h_0}^4 i_1}{{h_1}^5}-\frac{{h_0}^3 i_2}{{h_1}^4}+\frac{{h_0}^2 i_3}{{h_1}^3}-\frac{h_0 i_4}{{h_1}^2}+\frac{i_5}{h_1}$$

$$-\frac{{h_0}^7}{{h_1}^7}+\frac{{h_0}^6 i_0}{{h_1}^7}-\frac{{h_0}^5 i_1}{{h_1}^6}+\frac{{h_0}^4 i_2}{{h_1}^5}-\frac{{h_0}^3 i_3}{{h_1}^4}+\frac{{h_0}^2 i_4}{{h_1}^3}-\frac{h_0 i_5}{{h_1}^2}+\frac{i_6}{h_1}$$

$$\frac{{h_0}^8}{{h_1}^8}-\frac{{h_0}^7 i_0}{{h_1}^8}+\frac{{h_0}^6 i_1}{{h_1}^7}-\frac{{h_0}^5 i_2}{{h_1}^6}+\frac{{h_0}^4 i_3}{{h_1}^5}-\frac{{h_0}^3 i_4}{{h_1}^4}+\frac{{h_0}^2 i_5}{{h_1}^3}-\frac{h_0 i_6}{{h_1}^2}+\frac{i_7}{h_1}$$

$$-\frac{{h_0}^9}{{h_1}^9}+\frac{{h_0}^8 i_0}{{h_1}^9}-\frac{{h_0}^7 i_1}{{h_1}^8}+\frac{{h_0}^6 i_2}{{h_1}^7}-\frac{{h_0}^5 i_3}{{h_1}^6}+\frac{{h_0}^4 i_4}{{h_1}^5}-\frac{{h_0}^3 i_5}{{h_1}^4}+\frac{{h_0}^2 i_6}{{h_1}^3}-\frac{h_0 i_7}{{h_1}^2}+\frac{i_8}{h_1}$$

$$\frac{{h_0}^{10}}{{h_1}^{10}}-\frac{{h_0}^9 i_0}{{h_1}^{10}}+\frac{{h_0}^8 i_1}{{h_1}^9}-\frac{{h_0}^7 i_2}{{h_1}^8}+\frac{{h_0}^6 i_3}{{h_1}^7}-\frac{{h_0}^5 i_4}{{h_1}^6}+\frac{{h_0}^4 i_5}{{h_1}^5}-\frac{{h_0}^3 i_6}{{h_1}^4}+\frac{{h_0}^2 i_7}{{h_1}^3}-\frac{h_0 i_8}{{h_1}^2}+\frac{i_9}{h_1}$$

Our only requirement is that $(1/h_1)$ must exist. If $h_1 = 1$ in an example, this is no problem. Then the implied • and + operations in the above expressions can be either real operators or Mod(10) operators.

Now finally back to our example in which has $h_0 = h_1 = 1$. I then unprotect O and get

```
for n from 0 to 10 do
   O[n] := value(o(n));
od;
```

$$O_0 := 1$$
$$O_1 := -1 + i_0$$
$$O_2 := 1 - i_0 + i_1$$
$$O_3 := -1 + i_0 - i_1 + i_2$$
$$O_4 := 1 - i_0 + i_1 - i_2 + i_3$$
$$O_5 := -1 + i_0 - i_1 + i_2 - i_3 + i_4$$
$$O_6 := 1 - i_0 + i_1 - i_2 + i_3 - i_4 + i_5$$
$$O_7 := -1 + i_0 - i_1 + i_2 - i_3 + i_4 - i_5 + i_6$$
$$O_8 := 1 - i_0 + i_1 - i_2 + i_3 - i_4 + i_5 - i_6 + i_7$$
$$O_9 := -1 + i_0 - i_1 + i_2 - i_3 + i_4 - i_5 + i_6 - i_7 + i_8$$
$$O_{10} := 1 - i_0 + i_1 - i_2 + i_3 - i_4 + i_5 - i_6 + i_7 - i_8 + i_9$$

Now finally I can set in a specific input sequence:

$I(z) = z^2 + 1$

So now only $i_2 = i_0 = 1$. We then get

```
for n from 0 to 10 do
   O[n] := value(o(n)):
od;
```

$$O_0 := 1$$
$$O_1 := 0$$
$$O_2 := 0$$
$$O_3 := 1$$
$$O_4 := -1$$
$$O_5 := 1$$
$$O_6 := -1$$
$$O_7 := 1$$
$$O_8 := -1$$
$$O_9 := 1$$
$$O_{10} := -1$$