Hankel and Modified Hankel Transforms, as per Sneddon PhL 7.27.10

Resuming Sneddon on page 29: the Hankel stuff

1. The Regular Hankel transform and its usual matrix operator notation.

In my transforms.doc, I write it this way:

$$F_\nu(\mu) = \int_0^\infty dx\, xJ_\nu(\mu x)\, f(x) \qquad f(x) = \int_0^\infty d\mu\, \mu J_\nu(\mu x)\, F_\nu(\mu)$$

One can define a kernel C_ν that works in both directions like so: [note that ν is just a fixed label. I put it on C because I have put it on F and want to be consistent.]

$$F_\nu(\mu) = \int_0^\infty dx\, xJ_\nu(\mu x)\, f(x) \qquad f(x) = \int_0^\infty d\mu\, \mu J_\nu(\mu x)\, F_\nu(\mu)$$

$$F_\nu(\mu) = \int_0^\infty dx\, C_\nu(\mu,x)\, f(x) \qquad f(x) = \int_0^\infty d\mu\, C_\nu(x,\mu)\, F_\nu(\mu)$$

$$C_\nu(\mu,x) = xJ_\nu(\mu x) \qquad C_\nu(x,\mu) = \mu J_\nu(\mu x)$$

$$F_\nu = C_\nu f \qquad f = C_\nu F_\nu$$

$$C_\nu^2 = 1 \qquad C_\nu = C_\nu^{-1}$$

The operator C_ν has lots of nice properties as you see, but it is missing one property, namely

$$C_\nu \neq C_\nu^T \qquad \text{ie} \qquad C_\nu(\mu,x) = xJ_\nu(\mu x) \neq \mu J_\nu(\mu x) = C_\nu(x,\mu)$$

which is to say, this kernel is not symmetric. I know from Stak Ch 3 that you WANT kernels to be symmetric if possible because then lots of "theory" becomes nice. Recall from Chap 3 meta meta:

3.3 The Spectrum of a self-adjoint H-S operator. (p 212)

If the kernel k(x,y) is symmetric, then the operator K is self-adjoint (as well as symmetric). In this case the following facts are true: (but the last two items also require that K be HS.)

(1) The eigenvalues of K are all real
(2) the EF's of different EV's are orthogonal
(3) the multiplicity of any *non-zero* EV μ is finite.
(4) if number of EV's is ∞, then μ=0 is the only possible limit point.

These "facts" refer to the spectrum of the operator K being "nice" when K is symmetric. In a later section below we will come up with a "symmetric version" of our Hankel Transform, but first we want to work on notation which is a little treacherous.

2. The Regular Hankel using {...} notation; why this notation is useful.

We could use the following alternate notation for the above transform:

$F_\nu = C_\nu f$ $\qquad$ $f = C_\nu F_\nu$

$F_\nu(\mu) = \mathcal{C}_\nu\{f(x); \mu\}$ $\qquad$ $f(x) = \mathcal{C}_\nu\{F_\nu(\mu); x\}$

The first line shows the usual matrix notation where we think of F_ν and f as "vectors" and we think of the operator C_ν as a "matrix", both of infinite dimension. Remember ν is not an index on a vector, it is just a label, and we could write it as $F^{(\nu)}(\mu)$ perhaps to better show this, but that is too much work.

The second line needs some comment. The curly bracket has two arguments. The first argument is a function. The argument of this function is a *dummy argument*! The second argument in {...} is the variable of the function we are talking about, which appears on the LHS.

The purpose of the dummy argument is to allow us to write out a function that does not have a name. Suppose we had instead this notation

$$F_\nu(\mu) = \mathcal{C}_\nu\{f; \mu\}$$

and suppose $f(x) = x^2$. The function "x^2" has no name, so our notation cannot handle it! We could have to write something like

$$F_\nu(\mu) = \mathcal{C}_\nu\{\text{the squared function}; \mu\}$$

But using the dummy argument notation we can say

$$F_\nu(\mu) = \mathcal{C}_\nu\{f(x); \mu\} = \mathcal{C}_\nu\{f(s); \mu\} = \mathcal{C}_\nu\{f(z); \mu\}$$

$$F_\nu(\mu) = \mathcal{C}_\nu\{x^2; \mu\} = \mathcal{C}_\nu\{s^2; \mu\} = \mathcal{C}_\nu\{z^2; \mu\}$$

where we emphasize that the name given to the dummy argument is irrelevant. We just have to have *some* name for this variable so we can express a nameless function!

How do you handle a product of two functions?

So we restate from above

$F_\nu = C_\nu f$ $\qquad$ $f = C_\nu F_\nu$

$F_\nu(\mu) = \mathcal{C}_\nu\{f(x); \mu\}$ $\qquad$ $f(x) = \mathcal{C}_\nu\{F_\nu(\mu); x\}$

Now suppose we have f(x) = a(x) b(x). How would we write things out in our purely matrix notation? We cannot say f = ab where a,b,f are all vectors because such a notation has no meaning in a matrix and vectors sense. Thus you cannot say

$F_\nu = C_\nu ab$ $\qquad$ $ab = C_\nu F_\nu$

because that means nothing. We could perhaps *invent* a notation f = a#b => f(x) = a(x) b(x) and then say

$F_\nu = C_\nu a\#b$ $\qquad\qquad$ $a\#b = C_\nu F_\nu$

We hesitate to use the notation a*b because this usually means a convolution thing. Recall from Stak Ch 5 notes

> Using symbolic functions, we can say
>
> $(f_1 * f_2)(x) = h(x) = \int f_1(x-y) f_2(y)\, dy = \int f_2(x-y) f_1(y)\, dy$
>
> where we see the usual "convolution integral" $h(g_1) = \int dg\, f_1(g)\, f_2(g_1 g^{-1})$ in group notation.

The notation $a_x b$ is really reserved to mean a direct product thing. That is to say, this notion of f(x) = a(x) b(x) is more f = a ⊗ b, a direct product thing, where $f_{xy} = a_x b_y$ and then $f_{xx} = a_x b_x = a(x)b(x)$. But "f" is then really a matrix, and the function f(x) is the diagonal of the matrix "f" .

Sometimes people might say [ab](x) = [a](b(x)) = a(b(x)), but that is certainly not what we are dealing with here.

So we might consider plowing ahead with our invented notation f = a#b to mean f(x) = a(x) b(x) . But then suppose one of our two product functions has no name. Suppose f(x) = x^2 b(x). Then we are back to saying f = (the squared function) # b which is pretty unsightly.

On the other hand, f(x) = a(x) b(x) is a perfectly fine function, so there is no confusion about the matrix meaning of something like $F_\nu = C_\nu f$. So our problem here is that our compact *notation* simply cannot handle the situation! This is where the {...} notation does well. We can simply write

$F_\nu(\mu) = \mathcal{C}_\nu\{f(s);\, \mu\}$ $\qquad$ $f(x) = \mathcal{C}_\nu\{F_\nu(r); x\}$

$F_\nu(\mu) = \mathcal{C}_\nu\{a(s)b(s);\, \mu\}$ $\qquad$ $a(x)b(x) = \mathcal{C}_\nu\{F_\nu(r); x\}$

where here I make sure to display oddball dummy parameters like r and s which have nothing at all to do with the arguments μ and x.

If one (or both) of the functions has no name, we again have no problem with the {...} notation

$F_\nu(\mu) = \mathcal{C}_\nu\{s^2 b(s);\, \mu\}$ $\qquad$ $x^2 b(x) = \mathcal{C}_\nu\{F_\nu(r); x\}$

$F_\nu(\mu) = \mathcal{C}_\nu\{s^2 \ln(s);\, \mu\}$ $\qquad$ $x^2 \ln(x) = \mathcal{C}_\nu\{F_\nu(r); x\}$

How does the {...} notation deal with matrix products?

So we like this {...} notation for the above applications. But then how do we express something like matrix multiplication (product of operators) using the {...} notation. Recall for example that we had

$C_\nu^2 = 1$ $\qquad$ $C_\nu = C_\nu^{-1}$

Let's start with $C_\nu^2 = 1$ and apply this to a vector

$C_\nu^2 f = 1f = f$

$\mathcal{C}_\nu^2\{f(s);\, \mu\} = f(\mu)$

But we might write things like this

$C_\nu[C_\nu f] = f$

$\mathcal{C}_\nu\{\, [C_\nu f](s)\, ;\, \mu\} = f(\mu)$ // where $[C_\nu f](s)$ is a function of s, call it g(s)

But we know that

$g(s) = [C_\nu f](s) = \mathcal{C}_\nu\{\, f(r);\, s\,\}$

so we can combine these to get

$\mathcal{C}_\nu\{\, \mathcal{C}_\nu\{\, f(r);\, s\,\}\, ;\, \mu\} = f(\mu)$

Now both r and s are dummy variables and it helps I think to have them be different to avoid confusion. So we have now done this elaboration:

$C_\nu^2 = 1$

$C_\nu^2 f = f$

$\mathcal{C}_\nu\{\, \mathcal{C}_\nu\{\, f(r);\, s\,\}\, ;\, \mu\} = f(\mu)$

and we can go on to write these using $C_\nu = C_\nu^{-1}$ in the following manners

$\mathcal{C}_\nu^{-1}\{\, \mathcal{C}_\nu\{\, f(r);\, s\,\}\, ;\, \mu\} = f(\mu)$ $C_\nu^{-1} C_\nu = 1$

$\mathcal{C}_\nu\{\, \mathcal{C}_\nu^{-1}\{\, f(r);\, s\,\}\, ;\, \mu\} = f(\mu)$

and we could invent an identity operator which says

$\mathcal{I}\,\{\, f(r);\, s\,\} = f(s)$

More generally, here is how you "compose" matrix products in the {...} notation:

A f = BC f
$\mathcal{A}\{\, f(r);\, \mu\,\} = \mathcal{B}\{\, \mathcal{C}\,\{\, f(r);\, s\,\}\, ;\, \mu\}$

A f = BCD f
$\mathcal{A}\{\, f(r);\, \mu\,\} = \mathcal{B}\{\, \mathcal{C}\,\{\, \mathcal{D}\,\{\, f(t);\, r\,\};\, s\,\}\, ;\, \mu\}$

where t,r,s are all dummy variables. More systematically we might write

$$B\,f = A_1 A_2 A_3\,f$$

$$\mathcal{B}\{\, f(r);\, \mu \,\} = \mathcal{A}_1\{\, \mathcal{A}_2\, \{\, \mathcal{A}_3\, \{\, f(x_3);\, x_2 \,\};\, x_1 \,\} \,;\, \mu\}$$

where now the x_i are the dummy variables. It seems clear how to extend this to an arbitrary product.

Now as a quick *exercise*, let's apply $\mathcal{A}_1^{-1}$ to the last equation:

$$A_1^{-1} B\, f = A_1^{-1} A_1 A_2 A_3\, f = A_2 A_3\, f$$

$$\mathcal{A}_1^{-1}\{\, \mathcal{B}\{\, f(r);\, s \,\} \,;\, \mu \,\} = \mathcal{A}_1^{-1}\{\mathcal{A}_1\{\, \mathcal{A}_2\, \{\, \mathcal{A}_3\, \{\, f(x_3);\, x_2 \,\};\, x_1 \,\} \,;\, s\};\, \mu\}$$

$$= \mathcal{A}_1^{-1}\{\mathcal{A}_1\{q(s) \,;\, \mu\} \qquad \text{where } q(s) = \mathcal{A}_2\, \{\, \mathcal{A}_3\, \{\, f(x_3);\, x_2 \,\};\, x_1 \,\} \,;\, s\}$$

$$= q(\mu) \qquad \text{// from above that } \mathcal{C}_v^{-1}\{\, \mathcal{C}_v\{\, f(r);\, s \,\} \,;\, \mu\} = f(\mu)$$

$$= \mathcal{A}_2\, \{\, \mathcal{A}_3\, \{\, f(x_3);\, x_2 \,\};\, x_1 \,\} \,;\, \mu\}$$

So to summarize,

$$A_1^{-1} B\, f = A_2 A_3\, f$$

$$\mathcal{A}_1^{-1}\{\, \mathcal{B}\{\, f(r);\, s \,\} \,;\, \mu \,\} = \mathcal{A}_2\, \{\, \mathcal{A}_3\, \{\, f(x_3);\, x_2 \,\};\, x_1 \,\} \,;\, \mu\}$$

Finally, we would like to have some definition of or rule for the product of script operators. Suppose we have

$$ABf = g$$

$$\mathcal{A}\{\, \mathcal{B}\{\, f(r);\, s \,\} \,;\, \mu\} = g(\mu)$$

Let's now *define* the product of script operators like this:

$$\mathcal{A}\mathcal{B}\, \{\, f(r);\, \mu\} \equiv \mathcal{A}\{\, \mathcal{B}\{\, f(r);\, s \,\} \,;\, \mu\}$$

$$(AB)f = A\,(Bf)$$

Then for example we would have

$$\mathcal{A}\{\, \mathcal{B}\{\, f(r);\, s \,\} \,;\, \mu\} = \mathcal{A}\mathcal{B}\, \{\, f(r);\, \mu\}$$

$$\mathcal{A}\{\, \mathcal{A}^{-1}\{\, f(r);\, s \,\} \,;\, \mu\} = \mathcal{A}\,\mathcal{A}^{-1}\, \{\, f(r);\, \mu\} = \mathcal{I}\, \{\, f(r);\, \mu \,\} = f(\mu)$$

So now we have a perfectly streetable meaning for both these products

$\mathcal{C} = \mathcal{AB}$

C = AB

3. The *Symmetric* Regular Hankel Transform

In my transforms.doc, I write it this way:

$$F_\nu(\mu) = \int_0^\infty dx\, xJ_\nu(\mu x)\, f(x) \qquad f(x) = \int_0^\infty d\mu\, \mu J_\nu(\mu x)\, F_\nu(\mu)$$

But recall that you can rescale things so this transform looks instead like this:

$$G_\nu(\mu) = \int_0^\infty dx\, (x\mu)^{1/2} J_\nu(\mu x) g(x) \qquad g(x) = \int_0^\infty d\mu\, (x\mu)^{1/2} J_\nu(\mu x)\, G_\nu(\mu)$$

2.1.23

The required scaling is seen to be

$$x\, f(x) = x^{1/2} g(x) \qquad \text{and} \qquad G_\nu(\mu) = \mu^{1/2}\, F_\nu(\mu)$$

which is to say

$$g(x) = x^{1/2}\, f(x)$$
$$G_\nu(\mu) = \mu^{1/2}\, F_\nu(\mu)$$

Let's verify this to make sure no errors. Start with the second pair and insert:

$$G_\nu(\mu) = \int_0^\infty dx\, (x\mu)^{1/2} J_\nu(\mu x) g(x) \qquad g(x) = \int_0^\infty d\mu\, (x\mu)^{1/2} J_\nu(\mu x)\, G_\nu(\mu)$$

$$\mu^{1/2}\, F_\nu(\mu) = \int_0^\infty dx\, (x\mu)^{1/2} J_\nu(\mu x)\, x^{1/2}\, f(x) \qquad x^{1/2}\, f(x) = \int_0^\infty d\mu\, (x\mu)^{1/2} J_\nu(\mu x)\, \mu^{1/2}\, F_\nu(\mu)$$

$$F_\nu(\mu) = \int_0^\infty dx\, (x)^{1/2} J_\nu(\mu x)\, x^{1/2}\, f(x) \qquad f(x) = \int_0^\infty d\mu\, (\mu)^{1/2} J_\nu(\mu x)\, \mu^{1/2}\, F_\nu(\mu)$$

$$F_\nu(\mu) = \int_0^\infty dx\, x\, J_\nu(\mu x)\, f(x) \qquad f(x) = \int_0^\infty d\mu\, \mu\, J_\nu(\mu x) F_\nu(\mu) \qquad \text{QED}$$

Now that we have done this, let's restate the symmetric Hankel transform using f instead of g stuff, with the understanding that these are not the same f and F which appeared above:

$F_\nu(\mu) = \int_0^\infty dx\ (x\mu)^{1/2} J_\nu(\mu x) f(x)$ $\qquad$ $f(x) = \int_0^\infty d\mu\ (x\mu)^{1/2} J_\nu(\mu x)\ F_\nu(\mu)$

Why would you want to write the thing with $(x\mu)^{1/2}$ inside? The reason is that this causes the kernel to be symmetric . We can write, as we did above (none of these lines involves the symmetry of K)

$F_\nu(\mu) = \int_0^\infty dx\ (x\mu)^{1/2} J_\nu(\mu x) f(x)$ $\qquad$ $f(x) = \int_0^\infty d\mu\ (x\mu)^{1/2} J_\nu(\mu x)\ F_\nu(\mu)$

$F_\nu(\mu) = \int_0^\infty dx\ K_\nu(\mu,x)\ f(x)$ $\qquad$ $f(x) = \int_0^\infty d\mu\ K_\nu(x,\mu) F_\nu(\mu)$

$K_\nu(\mu,x) = (x\mu)^{1/2} J_\nu(\mu x)$ $\qquad$ $K_\nu(x,\mu) = (x\mu)^{1/2} J_\nu(\mu x)$

$F_\nu = K_\nu\ f$ $\qquad$ $f = K\nu\ G_\nu$

and now we have the new fact that

$K_\nu(\mu,x) = K_\nu(x,\mu)$

Thus, all our rules above for matrix C_ν apply for K_ν except K_ν is also symmetric. So

$K_\nu^2 = 1$ $\qquad$ $K_\nu = K_\nu^{-1}$ $\qquad$ and now $\qquad$ $K_\nu = K_\nu^{T}$

This extra last fact does not affect anything we did in the previous section with C, so we can just quote a few items from above and change from C to K where appropriate:

$F_\nu = K_\nu f$ $\qquad$ $f = K_\nu F_\nu$

$F_\nu(\mu) = \mathcal{K}_\nu\{f(x); \mu\}$ $\qquad$ $f(x) = \mathcal{K}_\nu\{F_\nu(\mu); x\}$

2.1.23 left $\qquad$ 2.1.24 $\qquad$ // I use $\mathcal{K}$ in place of Sned's $\mathcal{H}$

$F_\nu(\mu) = \mathcal{K}_\nu\{a(s)b(s); \mu\}$ $\qquad$ $a(x)b(x) = \mathcal{K}_\nu\{F_\nu(r); x\}$

$\mathcal{K}_\nu\{\ \mathcal{K}_\nu\{\ f(r); s\}\ ; \mu\} = f(\mu)$
$\mathcal{K}_\nu^{-1}\{\ \mathcal{K}_\nu\{\ f(r); s\}\ ; \mu\} = f(\mu)$ $\qquad$ $C_\nu^{-1} C_\nu = 1$
$\mathcal{K}_\nu\{\ \mathcal{K}_\nu^{-1}\{\ f(r); s\}\ ; \mu\} = f(\mu)$

$\mathcal{I}\ \{\ f(r); s\} = f(s)$

A f = BC f
$\mathcal{A}\{\ f(r); \mu\} = \mathcal{B}\{\ \mathcal{C}\ \{\ f(r); s\}\ ; \mu\}$

A f = BCD f
$\mathcal{A}\{\ f(r); \mu\} = \mathcal{B}\{\ \mathcal{C}\ \{\ \mathcal{D}\ \{\ f(t); r\}; s\}\ ; \mu\}$

(AB)f $\qquad$ = A (Bf)

$$\mathcal{AB}\ \{\ f(r);\ \mu\}\ \equiv\ \mathcal{A}\{\ \mathcal{B}\{\ f(r);\ s\ \};\ \mu\}$$

We have already learned about the product of script operators and the notion of the dummy variables and the usefulness of the curly bracket notation, so there is not much else to be said.

4. The Modified Hankel Transform: $S_{\eta,\alpha}\ f(\mu)$

[30] Now he talks about a ***modified*** **Hankel** which has a different power and J order. This thing is not exactly self-reciprocal, but has a similar result. Here is this new animal (second line)

$$F_\nu(\mu) \quad = \quad \int_0^\infty dx\ x \quad J_\nu(\mu x) \quad f(x) \qquad\qquad \text{// my original Hankel form}$$

$$F_{\eta,\alpha}(\mu) \equiv\ 2^\alpha\ \mu^{-\alpha}\ \int_0^\infty dx\ x^{1-\alpha}\ J_{2\eta+\alpha}(\mu x)\ f(x) \qquad \equiv\ S_{\eta,\alpha}\ f(\mu) \qquad \text{// = modified Hankel 2.1.25}$$

where we have now defined the integral operator $S_{\eta,\alpha}$ in this way. Notice that this is more a matrix notation than the operator notation we used above. We could write (hard to see script on S)

$$S_{\eta,\alpha}\ f(\mu)\ =\ [S_{\eta,\alpha}\ f](\mu)\ =[S_{\eta,\alpha}\ f]_\mu\ = \mathcal{S}_{\eta,\alpha}\ \{\ f(s);\ \mu\ \}$$

The power used to be x, it is now modified to be 1-α, and instead of J_ν we have $J_{2\eta+\alpha}$. You recover the original if $\alpha = 0$ and $2\eta = \nu$.

[I am back filling in now on 7.26.10]

Start with the original *symmetric* Hankel form above, then set $\nu = 2\eta+\alpha$,

$$G_\nu(\mu) = \int_0^\infty dx\ (x\mu)^{1/2} J_\nu(\mu x) g(x)\ =\ \mathcal{H}_\nu\{\ g(x);\ \mu\ \}$$

$$G_{2\eta+\alpha}(\mu) = \int_0^\infty dx\ (x\mu)^{1/2} J_{2\eta+\alpha}(\mu x) g(x)\ \ = \mu^{1/2}\ \int_0^\infty dx\ x^{1/2}\ J_{2\eta+\alpha}(\mu x) g(x)$$

Next, define f(x) by

$$x^{1/2} g(x)\ =\ x^{1-\alpha}\ f(x) \qquad\qquad \Rightarrow \qquad\qquad g(x) = x^{1/2-\alpha}\ f(x) \qquad f(x) = x^{-1/2+\alpha}\ g(x)$$

so we then have

$$G_{2\eta+\alpha}(\mu) = \mu^{1/2}\ \int_0^\infty dx\ x^{1/2}\ J_{2\eta+\alpha}(\mu x)[\ x^{1/2-\alpha}\ f(x)]$$

$$= \mu^{1/2}\ \int_0^\infty dx\ x^{1-\alpha}\ J_{2\eta+\alpha}(\mu x) f(x)$$

Now multiply both sides by $2^\alpha\ \mu^{-1/2-\alpha}$ to get

$$2^{\alpha}\,\mu^{-1/2-\alpha}\,G_{2\eta+\alpha}(\mu) \;=\; 2^{\alpha}\,\mu^{-\alpha}\;\int_0^{\infty} dx\; x^{1-\alpha}\;J_{2\eta+\alpha}(\mu x) f(x) \;\equiv\; S_{\eta,\alpha}\,f(x)$$

where we have now recognized our $S_{\eta,\alpha}$ operator object. The LHS is now

$$2^{\alpha}\,\mu^{-1/2-\alpha}\,G_{2\eta+\alpha}(\mu) \;=\; 2^{\alpha}\,\mu^{-1/2-\alpha}\,\mathcal{H}_{2\eta+\alpha}\{\, g(x);\,\mu\,\} \;=\; 2^{\alpha}\,\mu^{-1/2-\alpha}\,\mathcal{H}_{2\eta+\alpha}\{\, x^{1/2-\alpha} f(x);\,\mu\,\}$$

So we have shown that

$$S_{\eta,\alpha}\,f(x) \;=\; 2^{\alpha}\,\mu^{-1/2-\alpha}\,\mathcal{H}_{2\eta+\alpha}\{\, x^{1/2-\alpha} f(x);\,\mu\,\} \qquad \text{// which is 2.1.26}$$

or using the same notation on both sides:

$$\mathcal{S}_{\eta,\alpha}\,\{\, f(s);\,\mu\,\} \;=\; 2^{\alpha}\,\mu^{-1/2-\alpha}\,\mathcal{H}_{2\eta+\alpha}\{\, s^{1/2-\alpha} f(s);\,\mu\,\}$$

or

$$\mathcal{H}_{2\eta+\alpha}\{\, s^{1/2-\alpha} f(s);\,\mu\,\} \;= 2^{-\alpha}\,\mu^{1/2+\alpha}\,\mathcal{S}_{\eta,\alpha}\,\{\, f(s);\,\mu\,\}$$

Here we are just stating the connection between the modified and regular symmetric Hankel transforms. It is just a matter of scaling of the functions, as when we went from original to symmetric Hankel.

Now we could apply $\mathcal{H}_{2\eta+\alpha}^{-1} = \mathcal{H}_{2\eta+\alpha}$ to both sides to get

$$x^{1/2-\alpha} f(x) \;=\; \mathcal{H}_{2\eta+\alpha}^{-1}\,\{\; 2^{-\alpha}\,\mu^{1/2+\alpha}\,\mathcal{S}_{\eta,\alpha}\,\{\, f(s);\,\mu\,\};\; x\}$$

or

$$x^{-\alpha} f(x) \;=\; 2^{-\alpha}\,x^{-1/2}\mathcal{H}_{2\eta+\alpha}^{-1}\,\{\; \mu^{1/2+\alpha}\,\mathcal{S}_{\eta,\alpha}\,\{\, f(s);\,\mu\,\};\; x\} \qquad \text{// which is p30A}$$

or

$$f(x) \;=\; 2^{-\alpha}\,x^{-1/2+\alpha}\mathcal{H}_{2\eta+\alpha}\,\{\; \mu^{1/2+\alpha}\,\mathcal{S}_{\eta,\alpha}\,\{\, f(s);\,\mu\,\};\; x\} \qquad (*)$$

Now recall our S definition from above,

$$\mathcal{S}_{\eta',\alpha'}\,\{\, g(s);\, x\,\} \;=\; 2^{\alpha'}\,x^{-1/2-\alpha'}\,\mathcal{H}_{2\eta'+\alpha'}\{\, \mu^{1/2-\alpha'}\,g(\mu);\, x\,\}$$

Make the replacements $\eta' = \eta+\alpha$ and $\alpha' = -\alpha$. to get

$$\mathcal{S}_{\eta+\alpha,-\alpha}\,\{\, g(s);\, x\,\} \;=\; 2^{-\alpha}\,x^{-1/2+\alpha}\,\mathcal{H}_{2\eta+\alpha}\{\, \mu^{1/2+\alpha}\,g(\mu);\, x\,\}$$

Now for function g use this

$$g(\mu) = \mathcal{S}_{\eta,\alpha}\,\{\, f(s);\,\mu\,\}$$

Then we have

$$\mathcal{S}_{\eta+\alpha,-\alpha}\,\{\, \mathcal{S}_{\eta,\alpha}\,\{\, f(r);\, s\,\};\, x\,\} \;=\; 2^{-\alpha}\,x^{-1/2+\alpha}\,\mathcal{H}_{2\eta+\alpha}\{\, \mu^{1/2+\alpha}\,\mathcal{S}_{\eta,\alpha}\,\{\, f(s);\,\mu\,\};\, x\,\}$$

But the RHS of this equation is the same as the RHS of (*), so we then have shown that

$$f(x) = \mathcal{S}_{\eta+\alpha,-\alpha}\{\mathcal{S}_{\eta,\alpha}\{f(r); s\}; x\} = \mathcal{S}_{\eta+\alpha,-\alpha}\{S_{\eta,\alpha}\, f(s); x\}$$ // which is p30B

where after the last = we have revered to Sned's earlier non-script notation. Let's now ignore this and go back to

$$f(x) = \mathcal{S}_{\eta+\alpha,-\alpha}\{\mathcal{S}_{\eta,\alpha}\{f(r); s\}; x\}$$

Recall our rule for multiplying script operators,

$$(AB)f = A(Bf)$$
$$\mathcal{AB}\{f(r); \mu\} \equiv \mathcal{A}\{\mathcal{B}\{f(r); s\}; \mu\}$$

so we then have

$$f(x) = \mathcal{S}_{\eta+\alpha,-\alpha}\{\mathcal{S}_{\eta,\alpha}\{f(r); s\}; x\} = \mathcal{S}_{\eta+\alpha,-\alpha}\,\mathcal{S}_{\eta,\alpha}\{f(r); x\}$$

or

$$\mathcal{1}\{f(r); x\} = \mathcal{S}_{\eta+\alpha,-\alpha}\,\mathcal{S}_{\eta,\alpha}\{f(r); x\}$$

or

$$\mathcal{1} = \mathcal{S}_{\eta+\alpha,-\alpha}\,\mathcal{S}_{\eta,\alpha}$$

where all three operators are script. Back in our matrix notation this would be

$$1 = S_{\eta+\alpha,-\alpha}\, S_{\eta,\alpha}$$ // which is p30C

which we can compare to the non-modified Hankel result which was

$$1 = K_\nu\, K_\nu$$

Inverting these last two we get

$$S_{\eta,\alpha}^{-1} = S_{\eta+\alpha,-\alpha}$$ // which is 2.1.27

$$K_\nu^{-1} = K_\nu$$

Remember all the time that things like ν, η, α are all just labels, not vector or matrix indices.

Now let's take this 2.1.27 result and write it out a bit

$$S_{\eta,\alpha}\, g(\mu) = 2^\alpha\, \mu^{-\alpha} \int_0^\infty dx\, x^{1-\alpha}\, J_{2\eta+\alpha}(\mu x)\, g(x)$$

$$S_{\eta,\alpha}^{-1}\, g(\mu) = S_{\eta+\alpha,-\alpha}\, g(\mu) = 2^{-\alpha}\, \mu^\alpha \int_0^\infty dx\, x^{1+\alpha}\, J_{2\eta+\alpha}(\mu x)\, g(x)$$

Suppose we have

$$F_{\eta,\alpha}(\mu) = S_{\eta,\alpha}\ f(\mu)$$

We can then invert to get

$$f(\mu) = S_{\eta,\alpha}^{-1} F_{\eta,\alpha}(\mu) = S_{\eta+\alpha,-\alpha} F_{\eta,\alpha}(\mu) = 2^{-\alpha}\ \mu^{\alpha} \int_0^{\infty} dx\ x^{1+\alpha}\ J_{2\eta+\alpha}(\mu x)\ F_{\eta,\alpha}(x)$$

so here then is our "modified Hankel transform pair"

$$F_{\eta,\alpha}(\mu) = 2^{\alpha}\ \mu^{-\alpha} \int_0^{\infty} dx\ x^{1-\alpha}\ J_{2\eta+\alpha}(\mu x)\ f(x) \quad = \mathcal{S}_{\eta,\alpha}\ \{\ f(r);\ \mu\ \} = S_{\eta,\alpha}\ f(u)$$

$$f(x) = 2^{-\alpha}\ x^{\alpha} \int_0^{\infty} d\mu\ \mu^{1+\alpha}\ J_{2\eta+\alpha}(\mu x)\ F_{\eta,\alpha}(\mu) \quad = \mathcal{S}_{\eta+\alpha,-\alpha}\ \{\ F_{\eta,\alpha}(r);\ x\ \} = S_{\eta+\alpha,-\alpha}\ F_{\eta,\alpha}(x)$$

where I have tried to show all the different notations.

5. Comments:

(1) all we have done is a little rescaling of the functions in the regular Hankel transform. And we had a chance to exercise our various notations.

(2) We have two labels η and α instead of the usual one label ν. Thus, we have the possibility of keeping J_ν with a constant ν, but adjusting the power inside the integral.

(3) This "modified Hankel transform" is not some highly standardized thing you find in books on integral transforms. It is a specific version used by Erdelyi and Kober in 1940 and it will have some usefulness no doubt down our roadway.

(4) This modified Hankel thing is just one example of an "operator",

$$S_{\eta,\alpha}\ g(\mu) = 2^{\alpha}\ \mu^{-\alpha} \int_0^{\infty} dx\ x^{1-\alpha}\ J_{2\eta+\alpha}(\mu x)\ g(x)$$

$$s_{\eta,\alpha}(\mu,x) = 2^{\alpha}\ \mu^{-\alpha}\ x^{1-\alpha}\ J_{2\eta+\alpha}(\mu x) = 2^{\alpha}\ x\ (x\mu)^{-\alpha}\ J_{2\eta+\alpha}(\mu x)$$

Each kernel function you can think of course implies an operator, and I would normally use the same letter for the kernel function as for the operator (Stak used k and K). This particular kernel is not symmetric, and it can be related to that symmetric Hankel operator. We shall soon see other operators with other kernels which are algebraic and power functions rather than Bessel functions, such as the upcoming Erdelyi-Kober operators I and K. Once we have mastered the notation, I don't think any such operator should be "rocket science".

6. Application to a Smythian Form for the potential. [wrong, see section 6A for right]

Here is one way to write one of my "Smythian forms" in cylindricals, where z is non-osc:

$$V(\rho,\theta,z) = \Sigma_{n=0}^{\infty} \cos(n\theta + \varepsilon_n) \int_0^{\infty} dk\, A_n(k)\, J_n(k\rho)\, e^{-kz}$$

$$= \Sigma_{n=0}^{\infty} \cos(n\theta + \varepsilon_n) \int_0^{\infty} dk\, k^{1-2n} [\, k^{2n-1} A_n(k)]\, J_n(k\rho)\, e^{-kz}$$

$$= \Sigma_{n=0}^{\infty} \cos(n\theta + \varepsilon_n) \int_0^{\infty} dk\, k^{1-2n} \psi_n(k)\, J_n(k\rho)\, e^{-kz} \qquad \text{// which is p 31 A}$$

where we just change the definition of our "coefficient" to $\psi_n(k) = t^{2n-1} A_n(k)$. Now keep processing this thing a bit, while keeping this in mind:

$$S_{\eta,\alpha}\, g(\mu) = 2^{\alpha}\, \mu^{-\alpha} \int_0^{\infty} dx\, x^{1-\alpha}\, J_{2\eta+\alpha}(\mu x)\, g(x)$$

$$S_{\eta,2\alpha}\, g(\mu) = 2^{2\alpha}\, \mu^{-2\alpha} \int_0^{\infty} dx\, x^{1-2\alpha}\, J_{2\eta+2\alpha}(\mu x)\, g(x)$$

We want $2\eta + 2\alpha = n$ which says $\eta+\alpha = n/2$ and $\eta = n/2-\alpha$. Then we have

$$S_{n/2-\alpha,2\alpha}\, g(\mu) = 2^{2\alpha}\, \mu^{-2\alpha} \int_0^{\infty} dx\, x^{1-2\alpha}\, J_n(\mu x)\, g(x)$$

Now go back to our potential in the third line above,

$$V(\rho,\theta,z) = \Sigma_{n=0}^{\infty} \cos(n\theta + \varepsilon_n) \int_0^{\infty} dk\, k^{1-2n}\, J_n(k\rho)\, \psi_n(k)\, e^{-kz}$$

At this point, despite Sned, we have to set $\alpha = n$ so we get the correct power. Then we have

$$S_{n/2-\alpha,2\alpha}\, g(\mu) = 2^{2\alpha}\, \mu^{-2\alpha} \int_0^{\infty} dx\, x^{1-2\alpha}\, J_n(\mu x)\, g(x)$$

$$S_{-n/2,2n}\, g(\rho) = 2^{2n}\, \rho^{-2n} \int_0^{\infty} dk\, k^{1-2n}\, J_n(\rho k)\, g(k)$$

Our potential is then

$$V(\rho,\theta,z) = \Sigma_{n=0}^{\infty} \cos(n\theta + \varepsilon_n) \{ \int_0^{\infty} dk\, k^{1-2n}\, J_n(k\rho)\, \psi_n(k)\, e^{-kz} \}$$

$$= \Sigma_{n=0}^{\infty} \cos(n\theta + \varepsilon_n)\, 2^{-2n}\, \mu^{2n} \{ 2^{2n}\, \mu^{-2n} \int_0^{\infty} dk\, k^{1-2n}\, J_n(k\rho)\, \psi_n(k)\, e^{-kz} \}$$

$$= \Sigma_{n=0}^{\infty} \cos(n\theta + \varepsilon_n)\, 2^{-2n}\, \rho^{2n}\, S_{-n/2,2n}\{\psi_n(k)\, e^{-kz}\, ;\, \rho \}$$

so I think 2.1.35 is wrong! This is the first real problem I have found. It makes no sense to have some α just floating around in this 2.1.35, it has no meaning. Once I set α = n, you cannot then remove it from the sum as he has done.

In my corrected form, we are still expressing the potential in terms of the modified Hankel transform with certain labels, so that fact is still true.

Idea: Perhaps he has a typo in p 31A . So I will now redo this section assuming so!

6A. Application to a Smythian Form for the potential.

Here is one way to write one of my "Smythian forms" in cylindricals, where z is non-osc:

$$V(\rho,\theta,z) = \Sigma_{n=0}^{\infty} \cos(n\theta + \varepsilon_n) \int_0^{\infty} dk\, A_n(k)\, J_n(k\rho)\, e^{-kz}$$

$$= \Sigma_{n=0}^{\infty} \cos(n\theta + \varepsilon_n) \int_0^{\infty} dk\, k^{1-2\alpha} [\, k^{2n-1} A_n(k)]\, J_n(k\rho)\, e^{-kz}$$

$$= \Sigma_{n=0}^{\infty} \cos(n\theta + \varepsilon_n) \int_0^{\infty} dk\, k^{1-2\alpha} \psi_n(k)\, J_n(k\rho)\, e^{-kz} \qquad \text{// which is p 31 A}$$

where we just change the definition of our "coefficient" to $\psi_n(k) = t^{2\alpha-1} A_n(k)$, where α is just some fixed number we use . Now keep processing this thing a bit, while keeping this in mind:

$$S_{\eta,\alpha}\, g(\mu) = 2^{\alpha}\, \mu^{-\alpha} \int_0^{\infty} dx\, x^{1-\alpha}\, J_{2\eta+\alpha}(\mu x)\, g(x)$$

$$S_{\eta,2\alpha}\, g(\mu) = 2^{2\alpha}\, \mu^{-2\alpha} \int_0^{\infty} dx\, x^{1-2\alpha}\, J_{2\eta+2\alpha}(\mu x)\, g(x)$$

We want $2\eta + 2\alpha = n$ which says $\eta+\alpha = n/2$ and $\eta = n/2-\alpha$. Then we have

$$S_{n/2-\alpha,2\alpha}\, g(\mu) = 2^{2\alpha}\, \mu^{-2\alpha} \int_0^{\infty} dx\, x^{1-2\alpha}\, J_n(\mu x)\, g(x)$$

$$S_{n/2-\alpha,2\alpha}\, g(\rho) = 2^{2\alpha}\, \rho^{-2\alpha} \int_0^{\infty} dk\, k^{1-2\alpha}\, J_n(\rho k)\, g(k)$$

Now go back to our potential in the third line above,

$$V(\rho,\theta,z) = \Sigma_{n=0}^{\infty} \cos(n\theta + \varepsilon_n) \{ \int_0^{\infty} dk\, k^{1-2\alpha} J_n(k\rho)\, \psi_n(k)\, e^{-kz} \}$$

$$= \Sigma_{n=0}^{\infty} \cos(n\theta + \varepsilon_n)\, 2^{-2\alpha}\, \rho^{2\alpha} (\; 2^{2\alpha}\, \rho^{-2\alpha} \int_0^{\infty} dk\, k^{1-2\alpha} J_n(k\rho)\, \psi_n(k)\, e^{-kz} \;)$$

$$= \Sigma_{n=0}^{\infty} \cos(n\theta + \varepsilon_n)\, 2^{-2\alpha}\, \rho^{2\alpha}\, S_{n/2-\alpha,2\alpha} \{ \psi_n(k)\, e^{-kz} ; \rho \}$$

$$= 2^{-2\alpha}\, \rho^{2\alpha} \Sigma_{n=0}^{\infty} \cos(n\theta + \varepsilon_n)\, S_{n/2-\alpha,2\alpha} \{ \psi_n(k)\, e^{-kz} ; \rho \} \qquad \text{// which is 2.1.35}$$

Now if we do this all over again with $\partial_z V(\rho,\theta,z)$, everything is the same except our power of k increases by 1, and we have an overall minus sign. We absorb the power increase by saying

$$k^{1-2\alpha}\, k = k^{1-(2\alpha-1)}$$

so all we have to do is replace $2\alpha \rightarrow 2\alpha - 1$ in our result and add the minus, so $[\ \alpha \rightarrow \alpha - 1/2\ ]$

$$\partial_z V(\rho,\theta,z) = 2^{-2\alpha+1}\, \rho^{2\alpha-1}\ \Sigma_{n=0}^{\infty} \cos(n\theta + \varepsilon_n)\, S_{n/2-\alpha+1/2,2\alpha-1}\ \{\ \psi_n(k)\, e^{-kz}\ ;\ \rho\ \}$$

and if we then set $z = 0$, this gives 2.1.37.