The Shankar Connection PhL 1.18.11

Summary: In Sec 1 I review how the problem arises, Ku=μu. In Sec 2 I repeat the Rothwell expansion idea and conclude there is no $L_{xy}(1/r) = \delta$ solution for L which matches at each point in k_t space. I can only come up with non-linear L that sort of work, but non-linear is no good. In Sec 3 I show explicitly that constant coefficient L_{xy} cannot work due to "analyticity". In Sec 4 I repeat Rothwell for $\delta(\mathbf{r-r'})$ and get same no-go conclusion. In Sec 5 I report on my 1/4/11 emails with Shankar in which he tersely claims no solution because |q| is not analytic, which I interpreted to mean $|\mathbf{k_t}|$ of my Rothwell thing. I reply that I think he is only ruling out constant coefficient solutions but he does not respond to that. In Sec 6 I note that Jim later in fact found two solutions, and they do not have constant coefficients. In Sec 7 I ask why the Shankar/Rothwell method misses these solutions, and the answer is clear: In Sec 8 I take one of Jim's solutions and show that it in fact solves the Rothwell expansion, but only *globally* where you include the d^2k integral, there is not a match locally at each point in k space. Sec 9 states my **conclusions**.

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1. Why was I interested in L_{xy} [1/r] = $\delta^{(2)}$(r).

In the v3 doc I am wondering about this integral equation

$$\int\int_S dx'dy'\ 1/(\sqrt{(x-x')^2+(y-y')^2})\ u(x',y') = \mu\ u(x,y) \qquad Ku = \mu u \qquad (0)$$

though I don't seem to write this anywhere in v3 in Cartesian coordinates (but I do in my doc to R Price later on). Looking at the above, I am wondering if there is some operator L_{xy} such that

$$L_{xy}\ [\ 1/(\sqrt{(x-x')^2+(y-y')^2}\ ]\ = \delta(x-x')\delta(y-y') \qquad (1)$$

If there does exist such an L_{xy} , then the following would have to also be true

$$L_{xy}\ [\ 1/r]\ = \delta(x)\delta(y) = \delta^{(2)}(\mathbf{r}) \qquad (2)$$

Probably when I first did this, I thought (2) $\Rightarrow$ (1) but in fact that is not true. But (1) $\Rightarrow$ (2) and therefore we know that !(2) $\Rightarrow$!(1). So if (1) was to be true, then (2) would have to be true, and so I focused at this point only on (2).

2. Using the Rothwell/Cloud expansion for 1/r : no solution seems possible.

In Section 3 of the v3 doc I remembered a fact from the Rothwell/Cloud Cartesian work I did where I showed that (either sign works)

$$1/r = /\sqrt{x^2+y^2} = (1/\pi) \int d^2k_t \exp(\pm i\mathbf{k_t} \bullet \mathbf{r}) /(2k_t) \qquad (3)$$

So I thought maybe I could expand the LHS and RHS of (2) to see what happens. That gave

$$\int d^2k_t \, (1/2k_t) \, L_{xy} \, [\exp(\pm i\mathbf{k_t} \bullet \mathbf{r})] = \int d^2k_t \exp(\pm i\mathbf{k_t} \bullet \mathbf{r}) \qquad (4)$$

At this point, you cannot remove the d^2k_t integral from both sides because you don't have a clean set of basis functions because you don't know what L_{xy} [exp($\pm i\mathbf{k_t} \bullet \mathbf{r}$)] is going to do. BUT, IF you could find an L_x such that

$$L_{xy} \, [\exp(\pm i\mathbf{k_t} \bullet \mathbf{r})] = 2k_t \exp(\pm i\mathbf{k_t} \bullet \mathbf{r}) \qquad (5)$$

then you *would* have a complete set, and then this equation would be justified. I came up with some nonlinear operators that would have this behavior

$$\begin{aligned} &L_{xy} \, [f] = 2f \, | \, \nabla f \, | \qquad 1 \\ &L_{xy} \, [f] = 2\sqrt{-f \, \nabla^2 f} \qquad 2 \qquad (6) \\ &L_{xy} \, [f] = 2\sqrt{-\nabla^2} \; f \qquad 3 \qquad = 2 \, (- \partial_x^2 - \partial_y^2)^{1/2} \, f \end{aligned}$$

Operators 1 and 2 are I nonlinear operators, while operator 3 does not exist in any sense that I know about. If you try a binomial expansion, you get negative powers of the derivative. You could somehow try continuing off the set of positive integral exponents to the value (1/2). There may be some sense to this in the world of fractional calculus, but that is not what I am looking for here. So I dismiss the last operator 3 as making no sense at all, and I have to dismiss the first two since they are non-linear. This is because, if you want to solve (0) by using (1). L_{xy} has to pass through a sum = integral and thus must be linear for things to work. So I have no solution!

3. Trying for L_{xy} with constant coefficients: no solution is possible.

I then asked this sort of related question:

"Is there an L_{xy} with constant coefficients that solves (5)? "

I answered this question on the whiteboard which I now transcribe. We would have to have

$$\{\Sigma_{n=0}^{\infty}\Sigma_{m=0}^{\infty} a_{nm}\ \partial_x^n \partial_y^m\}\ [\exp(-i\mathbf{k_t}\bullet \mathbf{r})] = 2k_t \exp(-i\mathbf{k_t}\bullet \mathbf{r})$$

which says

$$\{\Sigma_{n=0}^{\infty}\Sigma_{m=0}^{\infty} a_{nm}\ (-ik_x)^n (-ik_y)^m\}\ [\exp(-i\mathbf{k_t}\bullet \mathbf{r})] = 2\sqrt{k_x^2 + k_y^2}\ \exp(-i\mathbf{k_t}\bullet \mathbf{r})$$

which says

$$\Sigma_{n=0}^{\infty}\Sigma_{m=0}^{\infty} a_{nm}\ (-ik_x)^n (-ik_y)^m = 2\sqrt{k_x^2 + k_y^2} = 2\,|\mathbf{k_t}| = 2k_t \qquad (7)$$

But there is no solution to this equation, and there are several ways to explain why:

• if you use the binomial expansion for the RHS, you get negative powers of one or the other k_i terms

$$(a+b)^n = a^n + n\,a^{n-1}b + (n,2)\,a^{n-2}b^2 + \ldots\ldots \qquad \text{use with } n = 1/2$$

but the left hand side has only positive powers, so no go. In other words, there is no solution to this

$$\Sigma_{n=0}^{\infty}\Sigma_{m=0}^{\infty} a_{nm}\, x^n y^m = \sqrt{x^2 + y^2}$$

• Due to the range of our Fourier expansions, we have to have (7) be true for k_x,k_y in the full ranges $(-\infty,\infty)$, so if a double series existed, it would have to converge for all k_x,k_y. The series on the LHS of (7), for fixed y, must then be analytic in the entire k_x complex plane. But the series on the RHS clearly has branch points at $k_x = \pm i\,k_y$ and so is not analytic in the k_x complex plane.

So the bottom line is that I have no solution to (5) and probably none exists.

4. Rothwell/Cloud also gives no solution to $L_{xy}(1/R) = 2\pi\delta^{(2)}(\mathbf{r}-\mathbf{r}')$

I think I can redo the above analysis more generally this way. I write

$$1/R = 1/\sqrt{(x-x')^2+(y-y')^2} = (1/\pi)\int d^2k_t \exp(\pm i\mathbf{k_t}\bullet(\mathbf{r}-\mathbf{r}'))/(2k_t) \qquad (3)'$$

where I have just replaced **r** by **r**-**r'** on both sides. Then the next equation becomes

$$\int d^2k_t\,(1/2k_t)\,L_{xy}\,[\exp(\pm i\mathbf{k_t}\bullet(\mathbf{r}-\mathbf{r}'))] = \int d^2k_t \exp(\pm i\mathbf{k_t}\bullet(\mathbf{r}-\mathbf{r}')) \qquad (4)'$$

And now we seek a solution which says

$$L_{xy}\,[\exp(\pm i\mathbf{k_t}\bullet(\mathbf{r}-\mathbf{r}')] = 2k_t \exp(\pm i\mathbf{k_t}\bullet(\mathbf{r}-\mathbf{r}')) \qquad (5)'$$

But now we just divide both sides by $\exp(\pm i\mathbf{k_t}\bullet\mathbf{(-r')})$ and we are back to equation (5) having to be true. So my inability to find L_{xy} in this context applies to the more general case. The constant coefficients argument goes through as well.

5. Report of my exchange with Shankar (sent 1/5/11)

> Hey Shankar,
>
> Sorry to bother you so often with my stupid questions, but do you happen to know the solution to the problem pasted below? It has been bugging me; I'm sure the answer is trivial but I just don't see it.
>
> Thanks,
>
> -Phil Lucht
> Salt Lake City
>
> **What 2D differential operator L makes this be true:**
>
> $$L(1/r) = 4\pi\,\delta^2(\mathbf{r}) \quad \text{where} \quad \mathbf{r} = (x,y), \qquad r = \sqrt{x^2+y^2}$$

His reply is this

> In momentum space such an operator would have to be |q| which, because it is not analytic cannot be written in terms of any simple differential operator.
> S

This agrees I think with my general analysis above, where I think I have made clearer what variable we are talking about for analyticity. I then replied

> Shankar,
>
> Thanks for your quick response.
>
> 1. If I understand you correctly, I think you are ruling out for L any simple differential operator with constant coefficients since in momentum space such an L would have to be a polynomial in the momentum components, whereas for my equation if you expand 1/r you need momentum space L = 2 sqrt(qx^2+qy^2) which is not a polynomial (not analytic) in qx,qy. I suppose this would still allow some obscure L with more general function coefficients? Or is that ruled out as well by your argument?

and then issued an open Torrey invitation to which he replied he would try to exploit that invitation.

So that then is the Shankar connection.

6. But despite the Shankar method failure, Jim finds two solutions.

I then entered my little project with Jim, and we in fact *found* two solutions to the equation:

$$L(1/r) = 2\pi\,\delta^{(2)}(\mathbf{r})$$

Here are the solutions.

$$L_{xy}^{(3)}([1/r]) = (1/r)\partial_r(r\,[1/r])$$

$$= (1/\sqrt{x^2+y^2})\;[\,x\partial_x + y\partial_y + 1]\,(\sqrt{x^2+y^2})\,[1/\sqrt{x^2+y^2}]$$

$$L_{xy}^{(1)}([1/r]) = -\partial_r^2(r[1/r])$$

$$= -(1/\sqrt{x^2+y^2})\;[\,x^2\,\partial_x^2 + y^2\partial_y^2 + 2xy\partial_x\partial_y + 2x\,\partial_x + 2y\partial_y\,]\,[1/\sqrt{x^2+y^2}]$$

These are NOT solutions to

$$L_{xy}(1/R) = 2\pi\,\delta^{(2)}(\mathbf{r}-\mathbf{r}')$$

but OK.

7. Why did these solutions not show up in the Shankar analysis?

They don't have constant coefficients, so that part is OK. But what about

$$L_{xy}\,[\exp(-i\mathbf{k}_t\bullet\,\mathbf{r})] = 2k_t\exp(-i\mathbf{k}_t\bullet\,\mathbf{r}) \qquad \text{(5 with -)}$$

This came from an integral which swept over the entire $\mathbf{k}_t$ space and we can regard $\mathbf{r}$ as fixed. For a particular value of $\mathbf{k}_t$ there will be some angle θ such that $\mathbf{k}_t\bullet\,\mathbf{r} = k_t r\cos\theta$. Then we have

$$L_{xy}\,[\exp(-irk_t\cos\theta)] = 2k_t\exp(-irk_t\cos\theta)) \qquad \text{(5 with -)}$$

Now lets define $k_t = k$ and $a = -i\cos\theta$ so we want to consider

$$L_{xy}\,e^{\alpha kr} = 2k\,e^{akr}$$

and we now try one of our known solutions

$$-\partial_r^2(r\,e^{akr}) = 2k\,e^{akr} \qquad ?$$

But Maple says

$$-\partial_r^2(r\,e^{akr}) = [\,-2ak - ra^2k^2]\,e^{akr} = -ak(2+akr)\,e^{akr}$$

so obviously this is not going to work. In the Shankar thing, we looked for a solution which made the equality in (4)

$$\int d^2k_t\ (1/2k_t)\ L_{xy}\ [\exp(\pm i\mathbf{k_t}\bullet \mathbf{r})] \ = \ \int d^2k_t\ \exp(\pm i\mathbf{k_t}\bullet \mathbf{r}) \qquad (4)$$

such that there was equality at every point in k-space. This is a special kind of solution and it seems that the solutions we found are more global in nature. In fact, this must be true

$$\int d^2k_t\ (1/2k_t)\ \{-\partial_r^2(r\ [\exp(-i\mathbf{k_t}\bullet \mathbf{r})])\} = (2\pi)^2\ \delta^{(2)}(\mathbf{r})$$

8. Brute force calculation of the Rothwell integral for one of the Jim solutions.

Let's now work on the LHS:

$$\text{LHS} = \int d^2k_t\ (1/2k)\ \{\ -ak(2+akr)\ e^{akr}\ \} \qquad \text{where} \quad a = -i\cos\theta$$

Now write

$$\int d^2k_t = \int k\ dk \int d\theta$$

so we then have

$$\text{LHS} = \int k\ dk \int d\theta\ (1/2k)\ \{\ -ak(2+akr)\ e^{akr}\ \}$$

$$= (1/2) \int dk \int d\theta\ \{\ -(-i\cos\theta)k(2-i\cos\theta kr)\ e^{-i\cos\theta kr}\ \}$$

$$= (i/2) \int dk\ k \int_{-\pi}^{\pi} d\theta\ \{\ \cos\theta\ (2-i\cos\theta kr)\ e^{-i\cos\theta kr}\ \}$$

$$= i \int dk\ k \int_{0}^{\pi} d\theta\ \{\ \cos\theta\ (2-i\cos\theta kr)\ e^{-i\cos\theta kr}\ \}$$

Let's define $z = \cos\theta$ so this becomes $dz = -\sin\theta d\theta = -\sqrt{1-z^2}\ d\theta$

$$= i \int dk\ k \int_{-1}^{1} dz/\sqrt{1-z^2}\ \ \{z\ (2-izkr)\ e^{-izkr}\ \} \qquad \textbf{(*)}$$

There are two terms. The first is proportional to

$$\int_{-1}^{1} dz/\sqrt{1-z^2} * z\ e^{-izkr} \ = \ \int_{-1}^{1} dz/\sqrt{1-z^2} * z\ (\ -i\ \sin(krz)\ )$$

$= -i \int_{-1}^{1} dz/\sqrt{1-z^2} * z \sin(krz) = -2i \int_{0}^{1} dz/\sqrt{1-z^2} * z \sin(krz) = (-2i)(\pi/2)J_1(kr)$

and we looked up to find in GR7

5. $\int_0^1 \frac{x \sin(ax)}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2} J_1(a)$ $[a > 0]$ WA 30(6)

The second term is proportional to

$\int_{-1}^{1} dz/\sqrt{1-z^2} * z^2 e^{-izkr} = \int_{-1}^{1} dz/\sqrt{1-z^2} * z^2 (\cos(krz)) = 2 \int_{0}^{1} dz/\sqrt{1-z^2} * z^2 \cos(krz)$

$= 2 \int_{0}^{1} dz/\sqrt{1-z^2} * z^2 (-1)(1/rz)^2 \partial_k^2 \cos(krz) = -2/(r)^2 * \partial_k^2 \int_{0}^{1} dz/\sqrt{1-z^2} * \cos(krz)$

$= -2/(r)^2 * (\pi/2)\, \partial_k^2 J_0(kr) = -(\pi/r^2)\, \partial_k^2 J_0(kr)$

and we looked up to find in GR7

2. $\int_0^1 \frac{\cos(ax)\,dx}{\sqrt{1-x^2}} = \frac{\pi}{2} J_0(a)$ WA 30(7)a

What a nightmare! So we now have for our LHS = LHS1 + LHS2,

LHS1 = $i \int dk\, k\, (2)(-2i)(\pi/2)J_1(kr) = i\,(2)(-2i)(\pi/2) \int dk\, k\, J_1(kr)$

$= i\,(-2i)(\pi)(1/r^2)$ // Maple says the integral is just $1/r^2$

$= +(2\pi/r^2)$

LHS2 = $i \int dk\, k\, (-ikr)(-\pi/r^2)\, \partial_k^2 J_0(kr)$

$= i\,(-\pi/r^2))(-ir) \int dk\, k^2\, \partial_k^2 J_0(kr)$

$= -\pi/(r) \int dk\, k^2\, \partial_k^2 J_0(kr)$

Wow. Maple says that

$\partial_k^2 J_0(kr) = -r^2 [\, J_0(kr) - (1/rk)J_1(rk)\,]$

so we then have

$$\text{LHS2} = -\pi/(r)\,(-r^2) \int dk\, k^2\ [\, J_0(kr) - (1/rk) J_1(rk)\,]$$

But we just saw from above that

$$\int dk\, k\, J_1(rk) = (1/r^2)$$

so writing LHS2 = LHS2a + LHS2b we have

$$\text{LHS2b} = -\pi/(r)\,(-r^2)\,(-1/r) \int dk\, k\, J_1(rk) = -\pi/(r)\,(-r^2)\,(-1/r)\,(1/r^2)$$

$$= -\pi/(r)\,(1/r) = -\pi/r^2$$

$$\text{LHS2a} = -\pi/(r)\,(-r^2) \int dk\, k^2\ J_0(kr) = -\pi/(r)\,(-r^2)\,(-1/r^3) = -\pi/(r^2) \qquad \text{// Maple int}$$

So barring errors, I have shown that

$$\text{LHS} = +(2\pi/r^2)\ -\pi/r^2\ -\pi/(r^2) = 0 \qquad \text{Hurray!!!}$$

All the steps above are certainly valid for r > 0 and this is just the result we wanted to get in this case. But the δ(r) is somehow hidden. We would have to do everything with regulation, but I am ready to stop now.

9. Conclusions

(1) The Jim solutions did not show up in the Shankar analysis because the latter required equality at each point in k-space, $L_{xy}\,[\exp(\pm i\mathbf{k}_t \bullet \mathbf{r})] = 2k_t \exp(\pm i\mathbf{k}_t \bullet \mathbf{r})$, and doing it that way, we get no solutions.

(2) The Jim solutions involve the global k-space integration.

(3) For r > 0, I explicitly computed $\int d^2k_t\ (1/2k_t)\ L_{xy}\,[\exp(\pm i\mathbf{k}_t \bullet \mathbf{r})]$ for one of Jim's solutions, and rather amazingly, you really do get 0, which is consistent with the required result $(2\pi)^2\ \delta^{(2)}(\mathbf{r})$. But things are extremely tangled in this approach and it would be pretty hard I think to ferret out the δ(r) part, but I know it must work because the Jim solutions really are solutions.

(4) Maybe I will send Shankar these results.