A 2D integral EV problem v1 (9 pages) PhL 12.17.10

Summary of v1: This v1 doc has an Overview but no Contents.. This is my first shot (done pre Cape Cod) on the question of converting Sneddon's p 203 Ku=μu or Kσ=f into a corresponding PDE using some kind of Lk=δ operator L. I know L = $-\nabla^2_{3D}$ is wrong, but I use it anyway to get a 2D Helmholtz equation $(\nabla^2+k^2)u(r,\theta) = 0$ ($k^2 = 1/\mu$) which I then explicitly solve for atoms of the form $J_m(rk)\sin(m\theta)$. I obtain two important angular integrals which are added to a doc elsewhere. Finally, I consider the Dirichlet equation Kσ = f in polar coordinates, expand σ and f on these atoms, and try to thereby invert the equation to obtain σ, but I do not succeed. I end up with a dimensional error (also present in the Helmholtz), but just let things be and move on to the v2 document.

Overview (1.5 pages, written 1.4.11)

This doc was motivated by Sneddon page 203. I was in a bit of a rush on 12/17 because I was going to Cape Cod on 12/19. So this is all Pre-Cod.

In **Section 0** I state the "problem of the day": solving a certain Ku = μu integral EV problem.

In **Section 1** I complain that Stak did give me no 2D integral equation EV problem examples.

In **Section 2** I review a certain Stak 1D integral EV equation example where the interval (a,b) is regarded as part of the definition of the kernel. So in 2D, the shape of the surface (disk) is part of K's definition.

In **Section 3** I correctly write out Ku = μu in polar coordinates. I tend to think of u as existing only on the disk and having no meaning outside.

In **Section 4** I note that there should be a "corresponding PDE EV problem" Lu = λu where λ = 1/μ. The connection between the 2DIE and PDE is Lk = δ. I note that in 3D, we think of $L = -\nabla^2$ since in that case with no BC's we know that Lk = δ gives us k = 1/R as the Green's function (ignoring 4π factors here). If we pretend that $L = -\nabla^2$ were the correct L for our problem at hand Ku = μu, then Lu = λu could be written $-\nabla^2 u = k^2 u$ where we define k by $k^2 = \lambda$. I note that this is the Helmholtz equation. I express various confusions about BC's and other things.

In **Section 4a** which I added today, I point out that this $L = -\nabla^2$ *cannot possibly be correct* because it has the wrong dimensions! We have dim(K) = L^1 but dim(L) = L^{-2} but we require KL = 1, so to speak. Another way to see this contradiction: we write $-\nabla^2 u = k^2 u$ which says dim(k) = L^{-1} as usual. We then have $\lambda = k^2$ so dim(λ) = L^{-2}. We also have $\mu = 1/\lambda$ so dim(μ) = L^2 . But in Ku = μu, we have a contradiction since dim(K) = L^1 and dim(μ) = L^2.

In **Section 5** I was detecting something must be wrong, but I did not know what.

In **Section 6** I solve the Helmholtz equation $(\nabla^2+k^2)u(r,\theta) = 0$ where r,θ are cylindrical coordinates and we assume no z-dependence in u. This section is I think all correct, and I find $J_m(rk)\sin(m\theta)$ type solutions

and verify this against Moon and Spencer. Never mind that $L = -\nabla^2$ is not right. Just take this as a correct standalone modular section, good to have it in the record.

In **Section 7** I still assume $L = -\nabla^2$ is OK and conclude that the EF's of $Ku = \mu u$ are $J_m(rk)\sin(m\theta)$ and that spectrum $\mu = 1/k^2$ are all positive real values with no quantization. At this time I did not see the inconsistency in dimensions of things.

In **Section 8** I logically forge ahead following Sneddon's suggestion. I assume that $J_m(rk)\sin(m\theta)$ are the eigenfunctions of $Ku = \mu u$, so I try to solve the Dirichlet problem expanding σ and f on these EF's. I use sin and cos separately, and have A,B as coefficients for σ, and C,D for f. We want to find A,B.

In **Section 9** I quote my two angular integrals from my integrals doc, and I end up with

$$\int_0^a dr'\, r'\, 2Q_{m-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'} \;*\; \int_0^\infty dk\, J_m(r'k)\, A_m(k) \;=\; \int_0^\infty dk\, J_m(rk)\, C_m(k)$$

I then quote my triple relation between the Q object, the JJ object, and the min object. I do manipulations using the JJ object in place of the Q object, but I arrive at a dead end because of endpoint a.

In **Section 10** I try to verify that my assumed eigenfunctions in fact solve $Ku = \mu u$. I insert $u(r,\theta) = J_m(rk)\sin(m\theta)$ into the integral, do the angular integral to get the Q object, replace that with the JJ object, and I end up with

$$2\pi \int_0^a dr'\, r'\, J_m(r'k) \int_0^\infty dx\; J_m(rx)J_m(r'x) \;=\; (1/k^2)\, J_m(rk) \qquad \text{// dim wrong! } L = L^2$$

and I think even then I realized I had some kind of dimension problem. If $a = \infty$, this becomes $2\pi\; J_m(rk)/k = (1/k^2)\, J_m(rk)$ which is obviously not right. The document ends at this point, and I decided to start a whole new document and take a fresh start on this problem.

Too bad, I ended up in a ditch instead of a good finish before my Cod trip. This is the way research goes, you cannot plan a good ending, whatever wants to happen happens. I will try to detangle this stuff when and if I get back.

(0) I have been flailing on this "problem" now for about 8 hours with nothing written down, trying to do it by lying down, no progress.

The problem arises on page 203 of Sneddon where we have a tie-in to the Stakgold world. If we write the usual $V = \int dA\sigma/R$ integral equation for a flat localized piece of metal, let's assume its a disk, then the Dirichlet equation is $\int dx'dy'\, \sigma(x',y')/R = f(x,y)$ where R is the distance between two points on the z=0 plane and we regard this as a kernel, so of course $k(x,y;\, x',y') = 1/R = [\,(x-x')^2+(y-y')^2]^{-1/2} = k(x,x')$ so to speak. So let's associate this kernel with an integral operator K.

Here is the problem: what are the eigenvalues and the eigenfunctions of this operator K ?

Here are some of my confusions:

(1) Stak does not seem to provide even a single example of a 2D integral equation like this one, so I cannot study any super simple cases.

(2) From his 1D examples, such as p 196 Example 1, I get the impression that the eigenvalues depend on the interval (a,b), which in this example is (0,1), but not on any "boundary conditions". The "eigenvalue equation" is just Ku = μu, and in this 1D example where K(x,x') = xx', he finds μ = 0 and μ = 1/3, and he gives the eigenfunction in each case. The interval enters the problem because it is part of the definition of the K operator.

So you imagine that in the 2D problem listed above, the shape of the 2D flat object (perhaps a disk) is part of the operator K, as well as its functional form.

(3) In polar coordinates my 2D eigenvalue problem looks like this: (for a disk of radius a)

$$\int_0^a dr'\, r' \int_0^{2\pi} d\theta'\, u(r',\theta') \,/\, \sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')} \;=\; \mu\; u(r,\theta) \qquad\qquad Ku = \mu u$$

At least I can write it down. I am pretty sure I don't know how to solve this equation. The function u(r,θ) here only "exists" for r,θ on the disk. Once we find solutions, I suppose we could continue them to r > a. The disk is the region R for this integral equation, like the interval (a,b) in a 1D problem.

(4) The eigenvalue equation Ku = μu is supposed to have a corresponding ODE eigenvalue equation, which I will write as Lu = λu where λ = 1/μ and where the eigenfunctions u are the same. Is this always true? If so, then what is the operator L for the above situation? The formal connection is supposed to be this: the kernel k(x;x') is the Green's Function of Lk = δ. In our case, the kernel is 1/R for two points on the z=0 plane (on the disk in fact), and we *usually* think of 1/R as the Green's Function of $-\nabla^2 k = \delta$. So this would argue that in fact, $L = -\nabla^2$. *If this is true*, then I should be able to find the eigenfunctions by examining the equation $-\nabla^2 u = \lambda u$. If we let $\lambda = k^2$ we get $(\nabla^2+k^2)u = 0$ as our eigenvalue equation. But this is just the Helmholtz equation which no doubt has a solution for any k^2 you want, which would say that any real k^2 value is an eigenvalue. [NB: Helmholtz arises here if you write $\lambda = k^2$ in your EV problem, and if you assume $L = -\nabla^2$. We are vague about whether this is a 2D or 3D ∇^2.]

But it seems odd that the Ku=μu problem needs to know the shape of the 2D object (disk), whereas the Lu = λu problem does not seem to need this information. I am completely stumped by this fact. The Helmholtz equation is going to allow a whole world of solutions. [Well, since Lu=λu is a PDE, quantization of λ will be affected by the BC which in turn is related to the shape of the 2D object, eg, maybe u = 0 on the perimeter of a disk.]

A subconfusion here is that u is a function of only 2 variables, not 3. If we take these to be cylindrical coordinates, then I guess $\partial_z u = 0$? Remember that in our $V = \int dA\sigma/R$ equation, the function going up against the kernel, σ which I have been calling u, is not a potential, it is a charge density. If you have some σ(x,y) lying on an x-y plane, what would say about $\partial_z\, \sigma(x,y)$?? [perhaps has a δ(z) thing]

Another subconfusion: when we say 1/R is the Green's Function, we imply there are no boundary conditions, there is no surface on which 1/R must vanish other than the Great Sphere.

(4a). Noted added 1.4.11 on Dimensions: In the equation $\int dx'dy'\ \sigma(x',y')/R = f(x,y)$, dimensions are all as usual. But when I write $\int dx'dy'\ u(x',y')/R = \mu u(x,y)$, it must be that μ has dimensions L. But that conflicts with $\mu = 1/k^2$ as I write in the next section! What is going on here? When we write $Ku = \mu u$, it is very clear that K has dimensions L^1. Therefore, my operator L *cannot be* ∇^2 because L is supposed to be K^{-1}. If we had a 3D situation, we would have $\int dx'dy'dz'\ u(x',y',z')/R = \mu u(x,y)$ and then $\dim K = L^2$ and all is well. Later in my third doc I conclude that $L[f] = 2\sqrt{-f\,\nabla^2 f}$ and this L does in fact have $\dim L = L^{-1}$.

(5) Probably something is wrong with the last subsection. If we really had a 3D 1/R with z and z' included, then I think it would be true and ∇^2 would be the 3D Laplacian as usual. [correct, something is wrong with the last subsection!]

(6) Let's blindly try to solve $(\nabla^2+k^2)u(r,\theta) = 0$ assuming things above. TK tells us that

$$\nabla^2 u(r,\theta) = r^{-1}\partial_r(r\,\partial_r\,u) + r^{-2}\partial_\theta^2 u$$

So our PDE is this

$$r^{-1}\partial_r(r\,\partial_r\,u) + r^{-2}\partial_\theta^2 u + k^2 u = 0$$

We try separation of variables $u = \Theta(\theta)R(r)$ and we get

$$\Theta(\theta)r^{-1}\partial_r(r\,\partial_r\,R(r)) + r^{-2}\,R(r)\,\partial_\theta^2\,\Theta(\theta) + k^2\,\Theta(\theta)R(r) = 0$$

$$r^{-1}\partial_r(r\,\partial_r\,R(r))/R(r) + r^{-2}\,[\partial_\theta^2\Theta(\theta)]/\Theta(\theta) + k^2 = 0$$

$$r\partial_r(r\,\partial_r\,R(r))/R(r) + [\partial_\theta^2\Theta(\theta)]/\Theta(\theta) + r^2k^2 = 0$$

$$r\partial_r(r\,\partial_r\,R(r))/R(r) + r^2k^2 \;=\; -[\partial_\theta^2\Theta(\theta)]/\Theta(\theta) \;= m^2$$

We thus get a separation constant I will call m. The θ equation is

$$\Theta(\theta) = \sin(m\theta) \qquad \partial_\theta^2\Theta = -m^2\sin(m\theta) \qquad [\partial_\theta^2\Theta(\theta)]/\Theta(\theta) = -m^2$$ so checks out

We get quantization of m since $u(r,\theta)$ is single valued on the disk. So our radial equation is this

$$r\partial_r(r\,\partial_r\,R(r))/R(r) + r^2k^2 - m^2 = 0$$

$$r\partial_r(r\,\partial_r\,R(r)) + (\,r^2k^2 - m^2)R(r) = 0$$

$$r\,[\,r\,R'' + 1\,R'] + (\,r^2k^2 - m^2)R \;\; = 0$$

$$r^2\partial_r{}^2R(r) + r\partial_rR(r) + (\,r^2k^2 - m^2)R(r) \;\; = 0$$

Change variables now to x = rk. We then have $r^2\partial_r{}^2 = x^2\partial_x{}^2$ and $r\partial_r = x\partial_x$ and R(r) = R(x/k) ≡ S(x), then we have

$$x^2\partial_x{}^2\,S(x) + x\partial_x\,S(x) + (x^2 - m^2)\,S(x) \;\; = 0$$

From Schaum p 136 this is the Bessel equation with solution $J_m(x) = J_m(rk)$, for example. Then one solution to our "eigenvalue equation" $(\nabla^2+k^2)u(r,\theta)$ is $J_m(rk)\sin(m\theta)$ for m = 0,1,2... and k = real. I see no reason why we should require that $u(r,\theta) = 0$ at r =a, do you? [$J_m(rk)\sin(m\theta)$ agrees with Moon and Spencer page 16 "for φ independent of z"]

(7) Suppose this is correct. Then we are claiming that the eigenfunctions of $Lu = k^2u$ are these functions with a mix of sine and cosine. So these must be the eigenfunctions of $Ku = \mu u$ and $\mu = 1/k^2$ can have any real positive value. [And, if we require u = 0 on the disk boundary, then k = the usual k_n's .]

(8) Now the Sneddon idea is to go back to our Stak Dirichlet integral equation

$$\int_0^a dr'\,r' \int_0^{2\pi} d\theta'\;\sigma(r',\theta') \;/\sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')} \;\;\; = f(r,\theta)$$

We are supposed to then expand σ and f on our newly found eigenfunctions: ($A_m = A_m(k)$ really.)

$$\sigma(r',\theta') = \Sigma_{m=0}{}^{\infty} \;\int_0^{\infty} dk\; J_m(r'k)\,[\,A_m\sin(m\theta') + B_m\cos(m\theta')]$$

$$f(r',\theta') = \Sigma_{m=0}{}^{\infty} \;\int_0^{\infty} dk\; J_m('rk)\,[\,C_m\sin(m\theta') + D_m\cos(m\theta')]$$

Installing we get

$$\int_0^a dr'\,r' \int_0^{2\pi} d\theta'\;\{\,\Sigma_{m=0}{}^{\infty} \;\int_0^{\infty} dk\; J_m(r'k)\,[\,A_m\sin(m\theta') + B_m\cos(m\theta')]\,\} \;/\sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')}$$

$$= \Sigma_{m=0}{}^{\infty} \;\int_0^{\infty} dk\; J_m(rk)\,[\,C_m\sin(m\theta) + D_m\cos(m\theta)]$$

Our goal is then to determine the A_m and B_m coefficients.

(9) I really don't know what I would do next with the above equation. I do have these facts from New 2 and New 3 in my doc on this stuff,

$$\int_0^{2\pi} d\theta' \sin(m\theta') / \sqrt{r^2 + r'^2 - 2rr' \cos(\theta-\theta')} \;=\; 2\;\sin(m\theta)\, Q_{m-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'}$$

$$\int_0^{2\pi} d\theta' \cos(m\theta') / \sqrt{r^2 + r'^2 - 2rr' \cos(\theta-\theta')} \;= 2\cos(m\theta)\, Q_{m-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'}$$

I could do out these integrals and that would give

$$\int_0^{a} dr'\, r' \{ \Sigma_{m=0}^{\infty} \int_0^{\infty} dk\, J_m(r'k) [A_m(k)\sin(m\theta) + B_m(k)\cos(m\theta)] \} 2Q_{m-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'}$$

$$= \Sigma_{m=0}^{\infty} \int_0^{\infty} dk\, J_m(rk) [C_m(k)\sin(m\theta) + D_m(k)\cos(m\theta)]$$

or

$$\Sigma_{m=0}^{\infty} \int_0^{a} dr'\, r' 2Q_{m-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'} \;*\; \{ \int_0^{\infty} dk\, J_m(r'k) [A_m(k)\sin(m\theta) + B_m(k)\cos(m\theta)] \}$$

$$= \Sigma_{m=0}^{\infty} \int_0^{\infty} dk\, J_m(rk) [C_m(k)\sin(m\theta) + D_m(k)\cos(m\theta)]$$

We can now match the θ terms and end up with two equations: (they are identical)

$$\int_0^{a} dr'\, r' 2Q_{m-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'} \;*\; \int_0^{\infty} dk\, J_m(r'k)\, A_m(k) \;=\; \int_0^{\infty} dk\, J_m(rk)\, C_m(k)$$

$$\int_0^{a} dr'\, r' 2Q_{m-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'} \;*\; \int_0^{\infty} dk\, J_m(r'k)\, B_m(k) \;=\; \int_0^{\infty} dk\, J_m(rk)\, D_m(k)$$

I don't know what to do next. Cannot do a Hankel transform to pry free $A_m(k)$ because both r' and k inside the J are already tied down to integration variables. I do have these extra facts ready to go,

$$\int_0^{\infty} J_m(rx) J_m(r'x)\, dx \;=\; (2/\pi)\,(rr')^{-m} \int_0^{\mathbf{min(r,r')}} ds\, s^{2m} / [\, \sqrt{r^2-s^2}\, \sqrt{r'^2-s^2}\,]$$

$$= (1/\pi)\,(rr')^{-1/2}\, Q_{m-1/2}[\,(r^2+r'^2)/(2rr')]$$

Here then is one way we could write the first equation above

$$\pi \int_0^{a} dr'\, r' \int_0^{\infty} dx\, 2\, J_m(rx)\, J_m(r'x) \int_0^{\infty} dk\, J_m(r'k)\, A_m(k) \;=\; \int_0^{\infty} dk\, J_m(rk)\, C_m(k)$$

Now we have a "free" Hankel parameter r, so apply $\int_0^{\infty} dr\, r\, J_m(ry)$ to both sides to get

$$2\pi \int_0^{a} dr'\, r' \int_0^{\infty} dx\, [\, \int_0^{\infty} dr\, r\, J_m(ry) J_m(rx)]\, J_m(r'x) \int_0^{\infty} dk\, J_m(r'k)\, A_m(k)$$

$$= \int_0^{\infty} dk\, [\, \int_0^{\infty} dr\, r\, J_m(ry)\, J_m(rk)]\, C_m(k)$$

$$2\pi \int_0^a dr'\, r' \int_0^\infty dx\, [\delta(y-x)/y]\, J_m(r'x) \int_0^\infty dk\, J_m(r'k)\, A_m(k)$$

$$= \int_0^\infty dk\, [\delta(y-k)/k]\, C_m(k)$$

$$2\pi \int_0^a dr'\, r'\, J_m(r'y) \int_0^\infty dk\, J_m(r'k)\, A_m(k) = C_m(y)$$

Find and good, but I really want to find $A_m(k)$. Suppose we now apply $\int_0^\infty dy\, y\, J_m(sy)$ to both sides:

$$2\pi \int_0^a dr'\, r' \left[\int_0^\infty dy\, y\, J_m(sy)\, J_m(r'y) \right] \int_0^\infty dk\, J_m(r'k)\, A_m(k) = \int_0^\infty dy\, y\, J_m(sy) C_m(y)$$

Then we get to do another Hankel

$$2\pi \int_0^a dr'\, r' [\delta(s-r')/s] \int_0^\infty dk\, J_m(r'k)\, A_m(k) = \int_0^\infty dy\, y\, J_m(sy) C_m(y)$$

$$2\pi\, \theta(a-s) * \int_0^\infty dk\, J_m(sk)\, A_m(k) = \int_0^\infty dy\, y\, J_m(sy) C_m(y)$$

$$\theta(a-s) \int_0^\infty dk\, J_m(sk)\, A_m(k) = (1/2\pi) \int_0^\infty dy\, y\, J_m(sy) C_m(y)$$

Now apply $\int_0^\infty ds\, s\, J_m(sk')$ to both sides to get

$$\int_0^\infty dk \left[\int_0^a ds\, s\, J_m(sk')\, J_m(sk) \right] A_m(k) = \int_0^\infty ds\, s\, J_m(sk') \left[(1/2\pi) \int_0^\infty dy\, y\, J_m(sy) C_m(y) \right]$$

So because of the θ function, we fail to "get a Hankel" on the LHS. So can go no farther.

(10). Getting back to the eigenfunctions of K, can I show that the eigenfunctions I arrived at from the corresponding PDE actually work in the integral equation? I would have to show this:

$Ku = \mu u$

$$\int_0^a dr'\, r' \int_0^{2\pi} d\theta'\, u(r',\theta') \,/\, \sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')} = \mu\, u(r,\theta) \qquad \text{// LHS} = L^1 \;\; \text{RHS} = L^2$$

$u(r,\theta) = J_m(rk) \sin(m\theta)$ = supposed eigenfunction // ignore the $\cos(m\theta)$ term for now

So we would then have

$$\text{LHS} = \int_0^a dr'\, r' \int_0^{2\pi} d\theta'\, J_m(r'k) \sin(m\theta') \;/\; \sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')}$$

I then do my New 3 trick already used above which was this

$$\int_0^{2\pi} d\theta' \sin(m\theta') / \sqrt{r^2 + r'^2 - 2rr' \cos(\theta-\theta')} \;=\; 2 \sin(m\theta)\, Q_{m-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'}$$

$$\int_0^{\infty} J_m(rx) J_m(r'x)\, dx \;=\; (1/\pi)\, (rr')^{-1/2}\, Q_{m-1/2}[\,(r^2+r'^2)/(2rr')]$$

so

$$\int_0^{2\pi} d\theta' \sin(m\theta') / \sqrt{r^2 + r'^2 - 2rr' \cos(\theta-\theta')} \;=\; 2\pi \sin(m\theta) \int_0^{\infty} J_m(rx) J_m(r'x)\, dx$$

so I could rewrite the LHS as

$$\text{LHS} = \int_0^a dr'\, r'\, J_m(r'k)\, 2\pi \sin(m\theta) \int_0^{\infty} dx\, J_m(rx) J_m(r'x)$$

$$= 2\pi \sin(m\theta) \int_0^a dr'\, r'\, J_m(r'k) \int_0^{\infty} dx\, J_m(rx) J_m(r'x) \qquad \text{// dim} = L$$

If things are to work out right, this has to equal the RHS which is this

$$\text{RHS} = (1/k^2)\, J_m(rk) \sin(m\theta) \qquad \text{// dim} = L^2 \text{ ???}$$

But that would then require this:

$$2\pi \int_0^a dr'\, r'\, J_m(r'k) \int_0^{\infty} dx\; J_m(rx) J_m(r'x) \;=\; (1/k^2)\, J_m(rk) \qquad \text{// dim wrong! } L = L^2$$

but it is not working right, something is wrong, we don't get a Hankel when we should. If we had $a = \infty$, then things might work:

$$2\pi \int_0^{\infty} dx\, J_m(rx) \int_0^a dr'\, r'\, J_m(r'k)\, J_m(r'x) = (1/k^2)\, J_m(rk)$$

$$2\pi \int_0^{\infty} dx\; J_m(rx)\, \delta(k-x)/k = (1/k^2)\, J_m(rk)$$

$$2\pi\; J_m(rk)\, /k = (1/k^2)\, J_m(rk)$$

So if $a = \infty$, we get "close".