A 2D integral EV problem v2 (14 pages) PhL 12.17.10

Summary of v2: This builds on the v1 work, does not replace it. Sections 1-9 are pre Cape Cod, while Sec 10 was written Dec 29-31. I review the complete Stak 1D string example of how an integral equation has an associated ODE, including BC issues. I then review how one derives $-\nabla^2(1/r) = 4\pi\delta^3(\mathbf{r})$. I review how one inverts the 3D Poisson equation $Lu=\rho$ for u, and this then gives a 3D example of how an integral equation has an associated PDE. I show that K=1/R (2D) is not Hilbert-Schmidt, but Sned says it is still OK. Section 9 states Dec 19 status pretty well. Issue of having a kink at r=a is mentioned and what should radial range be, (0,a) or (0,∞)? In Sec 10, as in v1, I then blindly try expanding everything in Helmholtz solution atoms $J_m(rk)\sin(m\theta)$ to solve the EV problem $Ku=\mu u$. This leads to the following unsolvable double integral equation for u(r,θ)'s coefficients: (my dimensional error is still present)

$$2\pi \int_0^\infty dx\, J_n(rx) \int_0^a dr'\, r' \{ a_n(r')\, J_n(r'x) \} = \mu\, a_n(r) \qquad n = 0,1,2...$$

where a is the disk radius. Even with a = ∞ which allows use of the Hankel transform, I cannot solve the above EV problem but I might invert the $K\sigma=f$ problem in the a=∞ case. I then move on to the v3 doc.

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Overview (2 pages, written 1.5.11)

In **Section 1** I **state the problem** which is Ku = μu with a certain simple kernel.

In **Section 2** I **review a simple Stakgold 1D example** showing the relation between an integral EV problem and the corresponding ODE EV problem -- the string. One finds that for the ODE, solutions are too general and get narrowed down to correct EF solutions by the BCs, whereas for the integral problem, there are no solutions until the BC's are imposed. I am presuming that a similar thing might happen for a 2D situation, but Stak gives no 2D examples.

In **Section 3** I **review** the known 3D situation that $-\nabla^2(1/r) = 4\pi\delta^3(\mathbf{r})$ and how it is derived.

In **Section 4** I look at the **formal inversion of the Poisson Equation** $Lu=\rho$ to get $u=G\rho$ where $G = L^{-1}$. I then set $\rho = \lambda u$ to get $Lu=\lambda u$ and $\mu u=Gu$ where $\lambda=1/\mu$ as the differential and integral EV problems. I wonder if a solution to one is necessarily a solution to the other. I don't mention BC's here at all. **I conjecture** that somehow with reasonable BC's, the EV's and EF's of one are those of the other as well. Perhaps this requires that G be Hilbert-Schmidt or nearly so.

In **Section 5** I review the solution to the **Helmholtz equation** $(\nabla^2 + k^2)u(r,\theta) = 0$ which equation is the differential EV problem for the 3D Laplace operator $L = -\nabla^2$ and $\lambda = k^2$. I think of my 2D region being a disk of radius a, and I use ∇^2 as written in cylindrical coordinates, but z drops out since u does not depend on z. The solutions have the form $u \sim J_m(rk) \sin(m\theta)$. If we add the BC that u = 0 on the disk boundary, then of course we have $k = k_n$ where $J_m(ak_n) = 0$.

I then wrongly suggest that these must be the solution to my $Ku = \mu u$ problem thinking that this L is the one corresponding to my integral operator K. At this point I realized I was on a wrong path.

In **Section 6** I "start over" and **restate the problem** at hand. This section makes me think I have to perhaps think of $\rho(x,y,z) = \sigma(x,y)\delta(z)$ and then R is the real 3D R. I will pursue this idea soon.

In **Section 7** I show that the integral operator K of my problem is **NOT** in fact **Hilbert-Schmidt**, but I note that Sneddon claims its EF's still form a complete set since the singularity is "weak".

In aborted short **Section 8,** I just write $Ku = \mu u$ in polar coordinates for the first time in this doc.

In **Section 9** I makes some **"pre Cape Cod" comments** which are pretty good. (1) For the first time, I state clearly that I don't know what 2D L satisfies $L(1/R) = \delta^{(2)}(\mathbf{r})$ and this is what I need to know. I claim that whatever this L is, it is not ∇^2_{3D} and it is not ∇^2_{2D}. (2) (2) I observe that if you were to expand the charged disk potential V in eigenfunctions that vanish at the edge of your disk. you would get V=0 at the edge which is wrong. If you expand in functions whose derivatives vanish at disk edge, at least you get the right answer as $r \to a$ from inside for the charged disk, but it seems unlikely that V' = 0 at the edge for a general disk $f(r,\theta)$. (3) I note that there will likely be a "kink" in V at the edge, and the only way you can "handle a kink" is to have it be somewhere in the middle of an "interval" on which you have a complete set of eigenfunctions. I suggest that the interval should be $(0,\infty)$ instead of (0,a) and that I should be doing Hankel stuff, but I note that in doc 1 the finite endpoint a caused trouble.

In **Section 10,** written post Cape Cod, I resolve to **"try out" the Hankel approach** mentioned in the third comment of Section 9 above.

In Plan A, I do a simple partial wave expansion of $u(r,\theta)$ and quickly end up with

$$2\pi \int_0^\infty dx\, J_n(rx) \int_0^a dr'\, r' \{ a_n(r') J_n(r'x) \} = \mu\, a_n(r) \qquad n = 0,1,2...$$

as my integral equation of interest. Since I am "thinking Hankel", I use JJ for the Q object. At this point, I don't know what to do because $a \neq \infty$. I then try the case $a = \infty$ and define $A_n(x)$ as the Hankel transform of $a_n(r)$. The above then becomes

$$2\pi \int_0^\infty dx\ x\ J_n(rx)\ [A_n(x)/x] = \mu\ a_n(r) \qquad \text{or} \qquad 2\pi\ B_n(r) = \mu\ a_n(r)$$

but then I am at a dead end since I don't know a or B.

In Plan B I insert the "min form" for the Q object instead of the JJ form. This gives

$$\int_0^a dr'\ r'\ a_n(r')\ 4(rr')^{-n} \int_0^{\min(r,r')} ds\ s^{2n} / [\ \sqrt{r^2-s^2}\ \sqrt{r'^2-s^2}\ ] = \mu\ a_n(r)$$

But then again I don't know what to do. I know I can invert this thing using the Copson double-Abel method, but that does not solve the problem. I suppose I could play with doing just one Abel and see if I could not get something solvable, but I did not do that, and here the doc just ends.

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1. The problem arises on page 203 of Sneddon where we have a tie-in to the Stakgold world. If we write the usual $V = \int dA\sigma/R$ integral equation for a flat localized piece of metal, let's assume it's a disk, then the Dirichlet equation is $\int dx'dy'\ \sigma(x',y')/R = f(x,y)$ where R is the distance between two points on the z=0 plane and we regard this as a kernel, so of course $k(x,y;\ x',y') = 1/R = [\ (x-x')^2+(y-y')^2]^{-1/2} = k(x,x')$ so to speak. So let's associate this kernel with an integral operator K.

Here is the problem: what are the eigenvalues and the eigenfunctions of this operator K ?

2. Review of the relationship between integral and differential eigenvalue equations in 1D.

Let's look at Stak V1 p 199 Example 5. The ODE is just the string eigenvalue problem Lu=λu. The interval is (0,1), and the BC's are u=0 at both ends. We know all about this string eigenvalue problem. We have $\lambda_n = (n\pi)^2$ and $\varphi_n = \sin(n\pi x)$. If we don't apply the BC's, we get sin(kx) and cos(kx) as two independent solutions of the ODE where $k^2 = \lambda$. The solution is narrowed down by the BC's.

We can convert this to an integral equation of the form Ku = μu where the kernel is the Green's Function which has the linear form shown p199A. We *expect* to have the same sin(nπx) solutions and to have eigenvalues $\mu_n = 1/\lambda_n$. Here is the integral eigenvalue equation:

$$\int_0^1 d\xi\ u(\xi)\ [\ \theta(x<\xi)x(1-\xi) + \theta(x>\xi)\xi(1-x)\ ] = \mu\ u(x) \qquad Ku = \mu u$$

$$\int_x^1 d\xi\ u(\xi)\ x(1-\xi) + \int_0^x d\xi\ \xi(1-x) = \mu\ u(x)$$

$$x \int_x^1 d\xi\ u(\xi)\ (1-\xi) + (1-x) \int_0^x d\xi\ u(\xi)\xi = \mu\ u(x)$$

Staring at this integral equation, it is surely not very obvious what the solution is. But let's *try* a solution of the form sin(kx) and see what happens:

$$LHS1 = x\ [\ \int_x^1 d\xi\ \sin(k\xi) - \int_x^1 d\xi\ \xi\ \sin(k\xi)\ ]$$

LHS2 $= (1-x)\left[\int_0^x d\xi\, \xi \sin(k\xi)\right]$

Maple shows that we get

$$LHS = LHS1 + LHS2 = -(1/k^2)\left[x \sin(k) - \sin(kx)\right]$$

For general real k, our function sin(kx) is NOT an eigenfunction of Ku = μu! Unlike the ODE case, here we don't get a solution which is too broad and needs to be narrowed with some BC's. In the ODE, if we put in sin(kx) we find that sin(kx) is an eigenfunction for *all* k where $\lambda = k^2$ if we ignore the BC's. But in the integral equation, if we put in sin(kx), we find it is NOT a solution for all k. The problem here is not to "narrow" the solution to meet the BC's. The problem is that we need the BC's to make it *be* a solution. If we put in sin(kx) and get out the above, we might be inclined to say k = nπ is required, then we get

$$LHS = (1/k^2)\sin(kx) = (1/n\pi)^2 \sin(n\pi x) = \mu_n \sin(n\pi x)$$

3. Review of the no-BC Green's function in 3D

Let's start with the ODE Lg = δ where $L = -\nabla^2$ and no BC's. We know that g = 1/4πR (I am using Stakgold definition of g).

Digression: How do we know this, you ask? That is to say, how do we know that $-\nabla_{\mathbf{r}}^2(1/|\mathbf{r}-\mathbf{r}'|) = 4\pi\delta(\mathbf{r}-\mathbf{r}')$? All we really need to show is that $-\nabla_{\mathbf{x}}^2(1/|\mathbf{x}|) = 4\pi\delta(\mathbf{x})$ then we set $\mathbf{x} = \mathbf{r}-\mathbf{r}'$. But change variable name for convenience so we then have $-\nabla^2(1/r) = 4\pi\delta(\mathbf{r})$ to show. For r > 0 we can compute

$$\nabla^2(1/r) = r^{-2}\partial_r(r^2\partial_r r^{-1}) = -r^{-2}\partial_r(r^2 r^{-2}) = -r^{-2}\partial_r(1) = 0$$

so we have to show what happens at r = 0. The divergence theorem says

$$\int_V dV\, \nabla\bullet\mathbf{A} = \int_S d\mathbf{S}\bullet\mathbf{A} \qquad \text{// divergence theorem = Gauss's theorem} \qquad (1)\ \text{Schaum 22.59}$$

$$\int_V dV\, \nabla^2\varphi = \int_S d\mathbf{S}\bullet\nabla\varphi = \int_S dS\, \partial_n\varphi \qquad \text{// application of the above with } \mathbf{A} = \nabla\varphi$$

We apply the second line with φ = 1/r for S = a sphere of radius a. $\partial_n = \partial_r$ so

$$\partial_n\varphi = \partial_r(r^{-1}) = -r^{-2} \qquad \Rightarrow \qquad \int_S dS\, \partial_n\varphi = \int a^2 d\Omega\,(-a^2) = -4\pi$$

So we have shown that

$$\int_V dV\, \nabla^2(1/r) = -4\pi$$

where our volume is a sphere of radius a. As we take a → 0, the result remains true. According to distribution theory, it must then be that $\nabla^2(1/r) = -4\pi\,\delta(\mathbf{r})$ since we know that $\nabla^2(1/r) = 0$ for any finite r. A more rigorous proof would be to show that, for any test function $\psi(\mathbf{r})$,

$$\int_V dV\, \nabla^2(1/r)\, \psi(\mathbf{r}) = -4\pi\psi(\mathbf{0})$$

but I don't want to stop and do that now. The fact that $\nabla^2(1/r) = -4\pi\,\delta(\mathbf{r})$ is listed on my TK page of vector identities.

So resuming where we were, we know that the solution to $Lg=\delta$ when $L = -\nabla^2$ is $g = 1/(4\pi R)$. We might just think of there in fact being some BC's, and the BC's are this: g decays at ∞.

4. Developing the Laplace Eigenvalue Equation and its associated Integral Form

Suppose we know that $-\nabla^2 u = \rho$ (Poisson's equation, Stak convention, Jackson would have $4\pi\rho$) We know that we can do this symbolic set of operations

$$Lu = \rho$$
$$GLu = G\rho$$
$$u = G\rho$$

where $G = L^{-1}$ and G is the integral operator having the Green's function of L (with only the ∞ BC). So our potential due to some charges ρ is given by this familiar formula

$$u = \int dV\, \rho / 4\pi R$$

Suppose we somewhat strangely want to consider the case where $\rho = \lambda u$. Then we have

$$-\nabla^2 u = \lambda u \qquad\qquad Lu = \lambda u$$

$$u = \int dV\, \lambda u / 4\pi R \quad \Rightarrow \quad \int dV\, u / 4\pi R = \mu u \qquad \mu = 1/\lambda \qquad Ku = \mu u$$

Here we have made an *association* between two equations

$$Lu = \lambda u \qquad\qquad Ku = \mu u$$

The first, just staring at its form, can be regarded as a differential eigenvalue equation, while the second can be regarded as an integral eigenvalue equation. Since we forced $\rho = \lambda u$, it seems quite likely that there may be no solution to our resulting eigenvalue equation, whereas we know there is always a solution to the Poisson equation.

Question: Suppose we can find a solution u to the left equation $Lu = \lambda u$. Does that tell us that this same u is a solution to $Ku = \mu u$? I think the answer is no. Technically, it might have to do with the fact that for any solution u to $Lu = \lambda u$, it might happen that Lu is not "in the domain of operator G", and therefore we cannot apply G and get $u = \lambda Gu$. Putting this theory stuff aside for now, we have seen pretty clearly in our 1D example above that we can have a solution to $Lu = \lambda u$ which is NOT a solution of $Ku = \mu u$, so we certainly have a precedent for a negative answer to our question.

I am going to bypass the proper theory (which I could probably support eventually with hours of labor) and I am going to conjecture the following:

Theorem: " In the association shown above between our two equations $u = \lambda u$ and $Ku = \mu u$, if we have some "reasonable" boundary conditions added to the problem, then somehow a solution to one equation is a solution to the other equation".

5. Solving the Laplace eigenvalue problem on a disk.

So consider our Laplace **eigenvalue problem**:

$$-\nabla^2 u = \lambda u$$

If we set $\lambda = k^2$ (just a new name), this eigenvalue equation can be written as

$$(\nabla^2 + k^2)u = 0$$

This is the Helmholtz equation, fancy cousin to the Laplace equation. I show in Appendix A below [there is no such Appendix, see previous doc in this series!], and confirm in Moon and Spencer, that the solutions to this equation in *cylindrical* coordinates, where we know for some reason that u does not depend on z, are

$$u_{k,m}(r,\theta) = [\, J_m(rk), N_m(rk) \,]\, [\, \sin(m\theta), \cos(m\theta) \,] \qquad \lambda = k^2$$

where we use θ for our azimuth. The Appendix shows how m gets quantized to integers, but k is not quantized and is a continuous real parameter. So these $u_{k,m}$ are our "eigenfunctions".

If we were to insist that $u_{k,m} = 0$ on the boundary of a disk of radius a, then our solutions and eigenvalues would be these

$$u_{k,m}(r,\theta) = J_m(rk_{m,i}/a)[\, \sin(m\theta), \cos(m\theta) \,] \qquad \lambda_{m,i} = k_{m,i}^2$$

where $J_m(k_{m,i}) = 0$, so $k_{m,i}$ for i = 1,2,3... are the zeros of J_m. So this boundary condition causes quantization of k as well as m. We then have a discrete set of eigenvalues $\lambda_{m,i} = k_{m,i}^2$.

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What then is the associated *integral* eigenvalue equation? From the previous section, it is this

$$\int\int_S dx'dy' \; u(x',y') \; 1/(4\pi\sqrt{(x-x')^2+(y-y')^2}) = \mu \, u(x,y) \qquad\qquad Ku = \mu u$$

It is this because $1/4\pi R$ is the Green's Function (and thus the kernel of K) for $L = -\nabla^2$, where the boundary conditions are just decay at ∞. [but that is not correct!!!]

STOP: This makes no sense to me. The real associated equation is this

$$\int\int_V dx'dy'dz' \; u(x',y',z') \; 1/(4\pi\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}) = \mu \, u(x,y,z) \qquad\qquad Ku = \mu u$$

where recall we had $\rho = \lambda u$. We might *conjecture* that the above equation has a solution of the form $u(x,y,z) = u(x,y)\delta(z)$ $\theta(x,y \text{ in } S)$. If this conjecture could be justified, ...

STOP again. This again does not work, I don't want a $\delta(z)$ sitting on the RHS.

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6. Start Over. Someone comes along and hands us this integral statement

$$V = \int dA \; \sigma/(4\pi R)$$

where we include in σ all surface charges in a problem, and we assume there are no ρ charges, and then V will be the potential. If we apply = $L = -\nabla^2$ to both sides, we get some version of Poisson's equation when you have surface charges. But we want instead to assume we have a flat piece of metal in the $z = 0$ plane (a disk say) with σ on it, and we have a prescribed potential f on this piece of metal. Thus

$$f(x,y) = \int\int_S dx'dy' \; \sigma(x',y') \; 1/(4\pi\sqrt{(x-x')^2+(y-y')^2}) \qquad R = \sqrt{(x-x')^2+(y-y')^2} \qquad (*)$$

where S is our flat piece of metal lying in the x,y plane. Notice that f(x,y) is defined only on S, and that the integration is only over S, and we are going to care about our solution σ only on S. So think of S as our "region of interest". So, this thing is "a 2D integral equation for σ". We would like to solve it somehow for σ.

Sneddon suggests that as a tool to doing this, we might want to find the eigenfunctions of $Ku = \mu u$ where K implies the kernel $g(x,y;x',y') = 1/(4\pi\sqrt{(x-x')^2+(y-y')^2})$. Sneddon implies that these eigenfunctions are defined on the region S, but he does not say it. Here is our eigenvalue integral equation:

$$\int\int_S dx'dy' \; u(x',y') \; 1/(4\pi\sqrt{(x-x')^2+(y-y')^2}) = \mu \, u(x,y)$$

We suspect that the kernel is quite well-behaved (probably H-S) over region S, so the eigenfunctions of this equation will form a complete set $\varphi_n(x,y)$ on S which we can then try to use to expand both f and σ in our integral equation (*) above.

7. Is our kernel Hilbert-Schmidt? According to Stak Vol II p 135, a kernel is Hilbert-Schmidt if its square is integrable on the domain. From Wiki we have this:

Hilbert-Schmidt integral operator

From Wikipedia, the free encyclopedia

In mathematics, a **Hilbert-Schmidt integral operator** is a type of integral transform. Specifically, given a domain (an open and connected set) Ω in *n*-dimensional Euclidean space $\mathbf{R}^n$, a **Hilbert-Schmidt kernel** is a function $k : \Omega \times \Omega \to \mathbf{C}$ with

$$\int_\Omega \int_\Omega |k(x,y)|^2 \, dx \, dy < \infty$$

(that is, the $L^2(\Omega\times\Omega; \mathbf{C})$ norm of k is finite), and the associated **Hilbert-Schmidt integral operator** is the operator $K : L^2(\Omega; \mathbf{C}) \to L^2(\Omega; \mathbf{C})$ given by

$$(Ku)(x) = \int_\Omega k(x,y)u(y) \, dy.$$

Then K is a Hilbert-Schmidt operator with Hilbert-Schmidt norm

$$\|K\|_{\mathrm{HS}} = \|k\|_{L^2}.$$

Hilbert-Schmidt integral operators are both continuous (and hence bounded) and compact (as with all Hilbert-Schmidt operators).

So I think this is the integral of interest in our case

$$\int dx\, dy \int dx'\, dy' \;\; 1/[\,(x-x')^2 + (y-y')^2] \;\; < \infty \quad ??$$

where we integrate over the same region in both integrals. In polar this would be

$$\int dr\, r\, d\theta \int dr'\, r'\, d\theta' \;\; 1/[r^2 + r'^2 - 2rr'\cos(\theta-\theta')] \;\; < \infty \quad ??$$

The region of concern only occurs when r is near r' and θ near θ' .

But go back to Cartesian. Suppose we have

$$\int_0^1 dx \int_0^1 dy \int_0^1 dX \int_0^1 dY \;\; 1/[\,(x-X)^2 + (y-Y)^2]$$

Our concern is in any region where x ≈ x' and y ≈ y', only there does 1/R blow up. Maple cannot do this quad integral, it can only do three of the four.

Back to polar coordinates. I know that

$$\int_0^{2\pi} d\theta/[r^2 + r'^2 - 2rr'\cos(\theta-\theta')] = \pi/\sqrt{a^2-b^2} \qquad a > |b| > 0 \qquad a = r^2 + r'^2 \quad b = 2rr'$$

and the condition is met. Now $a^2-b^2 = (a+b)(a-b) = (r+r')^2(r-r')^2$ so $\sqrt{a^2-b^2} = (r+r')\,|r-r'|$ so

$$\int_0^{2\pi} d\theta/[r^2 + r'^2 - 2rr' \cos(\theta-\theta')] = \pi / [(r+r') |r-r'|]$$

We can then do the dθ' integral to get another 2π so we have

$$\int dr\, r\, d\theta \int dr'\, r'\, d\theta' \; 1/[r^2 + r'^2 - 2rr' \cos(\theta-\theta')]$$

$$= 2\pi^2 \int_0^1 dr\, r \int_0^1 dr'\, r' / [(r+r') |r-r'|]$$

where I now pick some endpoints. Consider the second integral where r' is near r. We then have

$$\int_0^1 dr'\, r' / [(r+r') |r-r'| \approx (r/2r) \int_0^1 dr' / |r-r'|$$

This integral is log divergent! So my conclusion is that this kernel is NOT Hilbert-Schmidt and that may be why I am having so much trouble with this problem. Sneddon remarks on this but claims that a certain "Hilbert Schmidt Theorem" is still valid despite this "weak singularity". That theorem is that the eigenfunctions of the kernel form a complete set.

Here is another way to think of this divergent integral. Start with this

$$\int dx\, dy \int dx'\, dy' \; 1/[(x-x')^2 + (y-y')^2]$$

Replace **r'** by $\boldsymbol{\xi} = \mathbf{r}' - \mathbf{r}$ for the inner integration,

$$\int dx\, dy \int d\xi_x'\, d\xi_y' \; 1/[\xi_x^2 + \xi_y^2] = \int dx\, dy \int d\Omega_\xi \int \xi\, d\xi \; 1/\xi^2 = \int dx\, dy \int d\Omega_\xi \int d\xi/\xi$$

$$= 4\pi \int dx\, dy \int d\xi/\xi = 4\pi \int dx\, dy \, [\ln\varepsilon - \ln 0] \sim 4\pi A \ln(0)$$

where for the inside integration we focus on a little disk of radius ε about the point ξ = 0, as in the Stak boundary layer discussion. You see the ln(0) divergence which is why we are not H-S. The divergence arises because we have to do $\int dA dA' k^2$. In the actual equation, we have $\int dA\sigma/R$ and of we define ξ in this case, we get (as Stak showed) $\int d\xi\, \xi\, 1/\xi = \int d\xi = \varepsilon$, so there is no divergence. That is to say, the LHS of Ku = μu has a convergent integral even when **r** lies on the disk where **r'** is.

8. Consider now our 2D eigenvalue equation from Sneddon.

In cylindrical coordinates we are faced with this eigenvalue integral equation:

$$\int_0^a dr'\, r' \int_0^{2\pi} d\theta'\, u(r',\theta') \,/\, \sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')} \;=\; \mu\; u(r,\theta) \qquad\qquad Ku = \mu u$$

But symbolically we can write this as

$$\int dA\; u\,/R \;= \mu\, u$$

The 1/R is the G

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9. Quick Entry 12.19.10 before Cod departure.

• Given the 2D kernel 1/R shown above, I don't really know what the corresponding 2D differential operator L is which would cause $L(1/R) = \delta^{(2)}$. I know what L is *not*: (1) it is not ∇^2_{2D} because we know for that case, we have $L(\ln(1/R)) = \delta^{(2)}$. (2) We do know that $\nabla^2_{3D}\,(1/R) = \delta^{(3)}$. We could set $\mathbf{r}'$ = (x',y',0) in there, but that does not seem to generate $\delta^{(2)}$. We cannot also set z = 0 IMHO.

• Going back to the problem of finding a complete set on the circle of radius a: I know two sets of basis functions that would *not* work. (1) First, the set with discrete $k_{m,i}$ which are the zeros of the J's. This is no good because any potential expanded in such would be V=0 at the edge of the disk. Even in our simple charged disk case where Dirichlet says $V = V_0$ on the disk, we know we don't have V=0 at the edge of the disk. (2) Second, similarly, if we used functions involving the zeros of the derivatives of Bessel J functions (as Stak did somewhere), for any potential expanded in such, we would have dV/dr = 0 at the edge of the disk, and we know this is not true. As I recall, V(r) has a kink at the disk edge, so $E_r = 0$ on the disk and then $E_r = \infty$ just outside the edge. (3) In general, for any complete set where you have r = a be the end of the range, I don't think you can arrange to have a kink there, at least not in the above methods. Moreover, we really want some functions in our complete set that let us get outside r = a since the potential continues there.

• So my last idea of the day is to go ahead and use the full Hankel transform "expansion" for both the potential and for the charge density. The potential in general will be non-zero on the disk and outside it. The σ vanishes outside the disk, and thus has a kink (I'll say) at the edge, but we know we can have kinks in mid range in a complete set expansion (whole doc on that subject). So why not go ahead and try to use the Smythian form $J_m(k\rho)$ [cos(mφ),sin(mφ)] with continuous k for both the σ and the prescribed potential on the disk f(ρ,φ). I found in my original doc that this thing does not "diagonalize" very well because we have integral 0 to a. Maybe I can somehow work around that in some new way. Maybe throw in a theta function for σ.

To be continued hopefully Dec 29.

10. Continuing Dec 31.

I will try to do the full Hankel as suggested in the last bullet above. Our starting point is this 2D eigenvalue equation (I take the simple disk case for S)

$$\int_0^a dr'\, r' \int_0^{2\pi} d\theta'\, u(r',\theta') \,/\, \sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')} \;=\; \mu\; u(r,\theta) \qquad\qquad Ku = \mu u$$

This sure looks like an integral equation that Sneddon has already treated on page 70 in his review of Copson's 1947 solution to the disk problem. Copson was solving an integral equation for σ, but here I am trying to find eigenfunctions u(r,θ). Suppose instead of using the Copson parameterization, we do this

$$u(r,\theta) = \Sigma_{n=0}^{\infty} [\, a_n(r)\cos(n\theta) \;+ b_n(r)\sin(n\theta)] \qquad\qquad r \le a \;\text{ since we only care about on the disk}$$

The comment I think is important. Just as we work in 1D with an interval (a,b) and we care only about functions on that interval, here we care only about functions on the disk. The functions will exist off the disk, but our region is the disk, the analogy of our 1D interval. So insert and we get this for our eigenvalue problem:

$$\int_0^a dr'\, r' \int_0^{2\pi} d\theta'\, \{ \Sigma_{n=0}^{\infty} [\, a_n(r')\cos(n\theta') + b_n(r')\sin(n\theta')]\} \,/\, \sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')}$$

$$= \mu\, \Sigma_{n=0}^{\infty} [\, a_n(r)\cos(n\theta) + b_n(r)\sin(n\theta)] \qquad\qquad \text{LHS} = \text{RHS}$$

Now isolate the angular integrals on the LHS, so LHS =

$$\Sigma_{n=0}^{\infty} \int_0^a dr'\, r'$$

$$\{\, a_n(r') \int_0^{2\pi} d\theta'\, \cos(n\theta') / \sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')} \;+ b_n(r') \int_0^{2\pi} d\theta'\, \sin(n\theta') / \sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')}\, \}$$

We now call upon these results (see integrals folder), where I swap ψ and ψ' then they are θ and θ'

$$\int_0^{2\pi} d\psi'\, \cos(n\psi') / \sqrt{r^2 + r'^2 - 2rr'\cos(\psi-\psi')} \;= 2\cos(n\psi)\, Q_{n-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'}$$

$$= 2\cos(n\psi)\, \pi \int_0^{\infty} J_n(rx) J_n(r'x)\, dx = 4\cos(n\psi)\, (rr')^{-n} \int_0^{\min(r,r')} ds\, s^{2n} / [\, \sqrt{r^2-s^2}\, \sqrt{r'^2-s^2}\,]$$

$$\int_0^{2\pi} d\psi'\, \sin(n\psi') / \sqrt{r^2 + r'^2 - 2rr'\cos(\psi-\psi')} \;= 2\sin(n\psi)\, Q_{n-1/2}[(r^2+r'^2)/(2rr')] / \sqrt{rr'}$$

$$= 2\sin(n\psi)\, \pi \int_0^{\infty} J_n(rx) J_n(r'x)\, dx = 4\sin(n\psi)\, (rr')^{-n} \int_0^{\min(r,r')} ds\, s^{2n} / [\, \sqrt{r^2-s^2}\, \sqrt{r'^2-s^2}\,]$$

Plan A: I want first to try out the JJ forms and see what happens. Our new LHS becomes this

$$\Sigma_{n=0}^{\infty} \int_0^a dr'\, r'$$

$$\{ a_n(r') \, 2\cos(n\theta)\, \pi \int_0^\infty J_n(rx) J_n(r'x)\, dx \; + b_n(r') \, 2\sin(n\theta)\, \pi \int_0^\infty J_n(rx) J_n(r'x)\, dx \}$$

$$= 2\pi \, \Sigma_{n=0}^{\infty} \int_0^a dr' \, r' \, \{ a_n(r') \cos(n\theta) \int_0^\infty J_n(rx) J_n(r'x)\, dx \; + b_n(r') \sin(n\theta) \int_0^\infty J_n(rx) J_n(r'x)\, dx \}$$

Comparing this to the RHS , we have to have equality in each partial wave, so we have

$$2\pi \int_0^a dr' \, r' \, \{ a_n(r') \int_0^\infty J_n(rx) J_n(r'x)\, dx \} = \mu \, a_n(r)$$

$$2\pi \int_0^a dr' \, r' \, \{ b_n(r') \int_0^\infty J_n(rx) J_n(r'x)\, dx \} = \mu \, b_n(r)$$

But these two equations are the same equation! So fiddle with the first one which says

$$2\pi \int_0^a dr' \, r' \, \{ a_n(r') \int_0^\infty J_n(rx) J_n(r'x)\, dx \} = \mu \, a_n(r) \qquad n = 0,1,2...$$

$$2\pi \int_0^\infty dx \, J_n(rx) \int_0^a dr' \, r' \, \{ a_n(r') \, J_n(r'x) \} = \mu \, a_n(r)$$

Now I am unsure what to do next to solve this integral equation. The r' integral is neither a Hankel nor a Fourier-Bessel projection. Can I somehow "require" that $a_n(r') = 0$ outside the disk? If this were σ, that might be reasonable. I could just make that an ansatz and same for b and then u = 0 outside the disk. In that case, with that assumption, we have a full Hankel then since we can extend the dr' integral to $r' = \infty$. Then we roll out our Hankel equipment,

$$f(\rho) = \int_0^\infty dk \, k \, J_\nu(k\rho) \, F_\nu(k) \qquad \text{// expansion}$$

$$F_\nu(k) = \int_0^\infty d\rho \, \rho \, J_\nu(k\rho) \, f(\rho) \qquad \text{// projection}$$

$$\int_0^\infty d\rho \, \rho \, J_\nu(k\rho) \, J_\nu(k'\rho) = \delta(k-k')/k \qquad \text{// orthogonality}$$

$$\int_0^\infty dk \, k \, J_\nu(k\rho) \, J_\nu(k\rho') = \delta(\rho-\rho')/\rho \qquad \text{// completeness}$$

and then we must set ν = n and let's say ρ = r' and k = x and we get

$$2\pi \int_0^\infty dx \, J_n(rx) \int_0^\infty dr' \, r' \, \{ a_n(r') \, J_n(r'x) \} = \mu \, a_n(r)$$

$$2\pi \int_0^\infty dx \, J_n(rx) \, A_n(x) = \mu \, a_n(r)$$

where

$$A_n(x) = \int_0^\infty dr'\, r'\, J_n(r'x)\, a_n(r')$$

which has the Hankel inversion

$$a_n(r) = \int_0^\infty dx\, x\, J_n(xr)\, A_n(x)$$

Then we seem to have reduced our problem to this equation for $A_n(x)$

$$2\pi \int_0^\infty dx\, J_n(rx)\, A_n(x) = \mu \int_0^\infty dx\, x\, J_n(xr)\, A_n(x)$$

Now we have only a single integral on each side. Rewrite as

$$\int_0^\infty dx\, J_n(rx)\, A_n(x)\{\, 2\pi - \mu\, x\, \} = 0$$

But I have no idea how to solve an integral equation like this. Plus I have the horrible ansatz which is probably totally unjustified. So let's switch now to

Notes added 1.5.11. Let's try to continue on a bit here. We had a few lines above

$$2\pi \int_0^\infty dx\, J_n(rx)\, A_n(x) = \mu\, a_n(r)$$

which I now write as

$$2\pi \int_0^\infty dx\, x\, J_n(rx)\, [A_n(x)/x] = \mu\, a_n(r)$$

I will now define the integral shown to be $B_n(r)$, which is the Hankel transform of $[A_n(x)/x]$. We then obtain

$$2\pi\, B_n(r) = \mu\, a_n(r)$$

This result certainly is simple, but I don't know $a_n(r)$ and I don't know $B_n(r)$ so what would I do next? This approach would be useful perhaps for an a = ∞ Dirichlet problem where you would know the thing on the right.

Plan B: We back up and go back to this point above where we write LHS =

$$\Sigma_{n=0}^\infty \int_0^a dr'\, r'$$

$$\{\, a_n(r') \int_0^{2\pi} d\theta'\, \cos(n\theta') / \sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')} + b_n(r') \int_0^{2\pi} d\theta'\, \sin(n\theta') / \sqrt{r^2+r'^2-2rr'\cos(\theta-\theta')}\, \}$$

$$= \Sigma_{n=0}^{\infty} \int_0^a dr'\, r'$$

$$\{ a_n(r')\, 4\cos(n\theta)\, (rr')^{-n} \int_0^{\min(r,r')} ds\, s^{2n} / [\sqrt{r^2-s^2}\sqrt{r'^2-s^2}\,]$$

$$+ b_n(r')\, 4\sin(n\theta)\, (rr')^{-n} \int_0^{\min(r,r')} ds\, s^{2n} / [\sqrt{r^2-s^2}\sqrt{r'^2-s^2}\,] \}$$

$$= \mu \Sigma_{n=0}^{\infty} [\, a_n(r)\cos(n\theta) + b_n(r)\sin(n\theta)]$$

We again appeal to our partial waves in θ to say then that this requires

$$\int_0^a dr'\, r'\, a_n(r')\, 4(rr')^{-n} \int_0^{\min(r,r')} ds\, s^{2n} / [\sqrt{r^2-s^2}\sqrt{r'^2-s^2}\,] = \mu\, a_n(r)$$

$$\int_0^a dr'\, r'\, b_n(r')\, 4(rr')^{-n} \int_0^{\min(r,r')} ds\, s^{2n} / [\sqrt{r^2-s^2}\sqrt{r'^2-s^2}\,] = \mu\, b_n(r)$$

and again these two equations are the same equation, so let's fiddle only with the first one. We see our desirable "Abel form" here. BUT: the Abel form method is for solving an integral equation where the RHS is a known function, whereas here it is unknown because we are doing our eigenvalue problem.